A NECESSARY AND SUFFICIENT CONDITION FOR ARC-PANCYCLICITY OF TOURNAMENTS

Wu Zhengsheng* (吴正声), Zhang Kemin** (张克民)

AND Zou Yuan (邹园)
(Nanjing University)

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Abstract

In this paper, it is proved that a tournament $T$ with $p$ vertices has arc-pancyclicity, if and only if $T$ has both 3-arc-cyclicity and $p$-arc-cyclicity.

Let $T = (V, A)$ be a tournament with $p$ vertices. A tournament $T$ is called $k$-arc-cyclic if for any arc $(v_0, v_0) \in A$, there is a $(k - 1)$-path $\mu_{k-1} (v_1, v_0)$ from $v_1$ to $v_0$ in $T$, $k(3 \leq k \leq p)$ being an integer. A tournament $T$ is called arc-pancyclic if for any integer $k$ ($3 \leq k \leq p$), $T$ has $k$-arc-cyclicity.

In 1967, B. Alspach first studied the arc-pancyclicity of tournaments. A sufficient condition on arc-pancyclicity of tournaments was presented in [1]. In 1979, Zhu Yongjin and Tian Feng studied the problem again. And Zhu Yongjin put forth the following problem in [3]. "If the squared graph of a tournament $T$ is completely symmetrical, has $T$ arc-pancyclicity?" By Lemma 2 in [4], we know that this problem is equivalent to one asking whether or not the 3-arc-cyclicity of a tournament $T$ is the necessary and sufficient condition for its arc-pancyclicity. In another paper, we have given a negative answer to this problem in a more general case. We have proved: "For $p \geq 6$ and $p = 7.9$, there exists a tournament with $p$ vertices which is $k$-arc-cyclic ($k = 3, 4, \ldots, p - 1$) but not arc-pancyclic". On the other hand, in [2] it has been shown that "For $p = 2q > 4$, there exists an almost regular tournament which is $k$-arc-cyclic ($k = 4, 5, \ldots, p$) but not arc-pancyclic". So, it is very likely to suggest a conjecture as stated in the following Theorem 1. The main purpose of this paper is to prove this conjecture.

Theorem 1. A tournament $T = (V, A)$ with $p$ vertices has arc-pancyclicity if and only if $T$ has both 3-arc-cyclicity and $p$-arc-cyclicity.

The necessity is very clear. The sufficiency will be also clear after the following Theorem 2 is proved.

Theorem 2. Suppose a tournament $T = (V, A)$ with $p$ vertices has 3-arc-cyclicity...
ity. Let \((v_0, v_1) \in A\). If there is a \((p - 1)\)-path \(\mu_{p-1}(v_1, v_0)\) from \(v_1\) to \(v_0\) in \(T\) then there exists an \((h - 1)\)-path \(\mu_{h-1}(v_1, v_0)\) \((h = 4, 5, \ldots, p - 1)\) from \(v_1\) to \(v_0\) in \(T\).

Proof. By the above assumptions, it is evident that the Theorem is equivalent to the following proposition. If for any integer \(k(3 \leq k < p - 1)\), there is a \((k - 1)\)-path:

\[
\mu_{k-1}(v_1, v_0) = \{v_1, v_2, \ldots, v_k = v_0\},
\]

then there exists a \(k\)-path \(\mu_k(v_1, v_k)\) from \(v_1\) to \(v_k\).

Now let us prove this proposition.

For convenience, \(v_i(i = 1, 2, \ldots, k)\) will be simply denoted by \(i\), and let

\[
W = V - \mu_{k-1} = V - \{1, 2, \ldots, k\}.
\]

Since \(k < p - 1\), we have \(|W| > 1\), \(|W|\) being the number of the vertices of \(W\).

Now, for exhausting all possible cases, we shall prove that there always exists a path \(\mu_k(1, k)\).

(I) If \((l, v_0), (w_0, m) \in A\), where \(w_0 \in W\) and \(1 \leq l < m \leq k\), then there exist \((t, v_0)\) and \((w_0, t + 1) \in A\), where \(t \leq t < m\). Thus we obtain:

\[
\mu_k(1, k) = \{1, \ldots, t, w_0, t + 1, \ldots, k\}.
\]

So, in the following, we always assume that for any vertex \(w \in W\), there exists an integral index \(s(w)(1 \leq s(w) \leq k + 1)\) such that

\[
(w, 1), (w, 2), \ldots, (w, s(w) - 1), (s(w), w), (s(w) + 1, w), \ldots, (k, w) \in A^W.
\]

Let \(s_1 = \min_{w \in W} \{s(w)\}\), \(s_2 = \max_{w \in W} \{s(w)\}\).

(II) Suppose \(s_1 = 1\). Then there exists \(w_1 \in W\) such that \(s(w_1) = s_1 = 1\). Note that \(k \geq 3\) and \(3, w_1) \in A\). By the assumptions of the theorem, there exists a 2-path \(\{w_1, u, 3\}\). Clearly, \(u \in \mu_{k-1}\). Hence \(u \in W\), and thus we have

\[
\mu_k(1, k) = \{1, w_1, u, 3, \ldots, k\}.
\]

Similarly, we can prove that when \(s_2 = k + 1\), there also exists \(\mu_k(1, k)\).

Thus in the following, we always assume that \(2 \leq s_1 \leq s_2 \leq k\). In other words, for any vertex \(w \in W\), we have \((w, 1), (k, w) \in A\).

(III) Now, we shall prove \(s_1 = 2\). Otherwise, there exists \(w_1 \in W\) such that \(s(w_1) = s_1 = 2\). We consider the case of \((k, w_1) \in A\). By the assumptions of the theorem, there exists a 2-path \(\{w_1, u, k\}\). By the final assumption of (II), for any vertex \(w \in W\), we have \((k, w) \in A\). Thus by \((w, k) \in A\) we have \(u \in W\), i.e. \(u \in \mu_{k-1}\). And we have \(u = 1\) by \((w_1, u) \in A\). Hence, \((1, k) \in A\). It contradicts the assumption \((k, 1) = (0, 1) \in A\). Thus we have \(s_1 = 2\).

Similarly, we have \(s_2 = k\).

Therefore, in the following, we always assume \(3 \leq s_1 \leq s_2 \leq k - 1\) (thus \(k \geq 4\)).

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1) By \(s(w) = 1\), we mean that \((1, w), (2, w), \ldots, (k, w) \in A\), and by \(s(w) = k + 1\), we mean that \((w, 1), (w, 2), \ldots, (w, k) \in A\).
Then for any vertex \( w \in W \), we have \((w, 1), (w, 2), (k - 1, w), (k, w) \in A\).

(IV) Suppose \( s_1 \neq s_2 \) satisfying \( 3 \leq s_1 \leq s_2 \leq k - 1 \). Thus \( k \geq 5 \). Let \( s(w_1) = s_1, s(w_2) = s_2 \), where \( w_1, w_2 \in W \). We consider the case of \((k, w_1) \in A\). By the assumptions of the theorem, there exists a 2-path \( \{w_1, l, k\} \). Clearly, \( l \in W \) and \( 2 \leq l \leq s_1 - 1 \). Thus we have proved that there exists an integer \( l(2 \leq l \leq s_1 - 1) \) such that \((l, k) \in A\). Consider the case of \((w_2, 1) \in A\). Similarly, there exists an integer \( m(s_2 \leq m \leq k - 1), \) satisfying \((1, m) \in A\). Since \( s_1 < s_2 \), we have \( l + 1 \leq m - 1 \). Now, two cases may be considered separately.

1. When \( l + 1 = m - 1 \), from \( l + 1 \leq s_1 \leq s_2 - 1 \leq m - 1 \), we have \( l + 1 = s_1 = s_2 - 1 = m - 1 \). Since \(|W| > 1\), there are \( w', w'' \in W \). Without loss of generality, assume that \((w', w'') \in A\). By the final assumption of (III) there are \((k - 1, w')\) and \((w'', 2) \in A\). Thus we have

\[
\mu_k(1, k) = \{1, m = s_2, \ldots, k - 1, w', w'', 2, \ldots, s_2 - 2 = l, k\}.
\]

2. When \( l + 1 < m - 1 \), we consider two possible cases.

(1) For \( l + 2 < s_2 \), we have

\[
\mu_k(1, k) = \{1, m, \ldots, k - 1, w_2, l + 2, \ldots, m - 1, w_1, 2, \ldots, l, k\}.
\]

(2) For \( l + 2 \geq s_2 \), noting \( s_1 + 1 \geq l + 2 \geq s_2 \), we have \( s_1 + 1 = s_2 \). Thus \( m - 2 > l \geq s_2 - 2 = s_1 - 1 \) and we have

\[
\mu_k(1, k) = \{1, m, \ldots, k - 1, w_2, l + 1, \ldots, m - 2, w_1, 2, \ldots, l, k\}.
\]

Therefore, in the following, we always assume that \( s_1 = s_2 = s \) and \( 3 \leq s \leq k - 1 \), i.e. for any vertex \( w \in W \), we have \((w, 1), (w, 2), \ldots, (w, s - 1), (s, w), \ldots, (k - 1, w), (k, w) \in A\).

At this point, the following lemmas are valid.

**Lemma 1.** \( T[W] = (W, A_W) \), the subgraph of \( T \) induced by \( W \) is a 3-arc-cyclic tournament. Hence \(|W| \geq 3\), and \( T[W] \) is strongly connected.

**Proof.** Clearly, \( T[W] \) is a tournament. Note that \( T \) has 3-arc-cyclicicity. Thus for any \((w_1, w_2) \in A_W(\subseteq A)\), there must be a 2-path \( \{w_2, u, w_1\} \). Clearly \( u \in \mu_{k-1} \), so that \( u \in W \). Hence \( \{w_2, u, w_1\} \) is a 2-path in \( T[W] \). Q. E. D.

**Lemma 2.** If \((\alpha, \gamma), (\gamma - 1, \delta) \in A\), where \( 1 \leq \alpha \leq s - 2, \alpha + 1 < \gamma < \delta \leq k \) and \( s + 1 \leq \delta \), then there is a \( k \)-path \( \mu_k(1, k) \) from 1 to \( k \).

**Proof.** Taking \( w_0 \in W \), we have

\[
\mu_k(1, k) = \{1, \ldots, \alpha, \gamma, \ldots, \delta - 1, w_0, \alpha + 1, \ldots, \gamma - 1, \delta, \ldots, k\}.
\]

**Lemma 3.** If \((\alpha, \gamma), (\beta, \delta) \in A\), where \( 1 \leq \alpha < \beta < s - 1 \) and \( s < \gamma < \delta \leq k \), then there is a \( \mu_k \)-path \( \mu_k(1, k) \) from 1 to \( k \).

**Proof.** Two cases may be considered.

(1) When \( \gamma > s + 1 \), taking \( w_0, w_1 \in W \), we have

\[
\mu_k(1, k) = \{1, \ldots, \alpha, \gamma, \ldots, \delta - 1, w_1, \beta + 1, \ldots, \gamma - 2, w_2, \alpha + 1, \ldots, \beta, \delta, \ldots, k\}.
\]
(2) When \( \gamma = s + 1 \), noting \( \beta \leq s - 2 \), we examine two possible cases

(a) For \( \beta < s - 2 \), taking \( w_1, w_2 \in W \), we have

\[
\mu_k(1, k) = \{1, \ldots, \alpha, \gamma, \cdots, \delta - 1, \nu_1, \beta + 2, \cdots, \\
\gamma - 1, \nu_2, \alpha + 1, \cdots, \beta, \delta, \cdots, k\}.
\]

(b) For \( \beta = s - 2 \), by Lemma 1, there is a 2-path \( \{w_1, w_2, w_3\} \) in \( T[W] \). Then we have

\[
\mu_k(1, k) = \{1, \cdots, \alpha, \gamma = s + 1, \cdots, \delta - 1, \nu_1, \nu_2, \nu_3, \\
\alpha + 1, \cdots, s - 2 = \beta, \delta, \cdots, k\}.
\]

The proof of the lemma is completed.

**Lemma 4.** Let

\[
R(l) = \{i \mid (i, i) \in A, s \leq i \leq k\} \quad (1 \leq l \leq s - 1),
\]

and

\[
L(r) = \{i \mid (i, r) \in A, 1 \leq i \leq s - 1\} \quad (s \leq r \leq k).
\]

Then for any integer \( l (1 \leq l \leq s - 1) \), we have \( R(l) \cong \phi \), and \( k \in R(1) \). And for any integer \( r (s \leq r \leq k) \), we have \( L(r) \cong \phi \), and \( 1 \in L(k) \).

**Proof.** Taking \( w_0 \in W \), we have \( (w_0, l) \in A \). By the assumptions of the theorem, there is a 2-path \( \{l, u, w_0\} \) in \( T \), where \( u \in V \). Clearly \( u \in W \cup \{1, 2, \cdots, s - 1\} \). Hence \( u \in \{s, s + 1, \cdots, k\} \) and \( (l, u) \in A \), so that \( u \in R(l) \), \( R(l) \cong \phi \). Since \( (1, k) \in A \), we have \( k \in R(1) \). Similarly, we can prove the rest of the lemma. Q. E. D.

Let

\[
\phi(l) = \max R(l), \quad \phi_1(l) = \min R(l), \quad (1 \leq l \leq s - 1);
\]

and

\[
\varphi(r) = \min L(r), \quad \varphi_1(r) = \max L(r), \quad (s \leq r \leq k).
\]

Thus \( (l, \phi(l)), (l, \phi_1(l)), (r, \varphi(r)), (r, \varphi_1(r)) \in A \), and

\[
\begin{align*}
& s \leq \phi(l) \leq \phi_1(l) \leq k, \quad (2 \leq l \leq s - 1), \\
& s \leq \phi_1(1) \leq \varphi_1(1) \leq k - 1; \\
& 1 \leq \varphi(r) \leq \varphi_1(r) \leq s - 1, \quad (s \leq r \leq k - 1), \\
& 2 \leq \varphi(k) \leq \varphi_1(k) \leq s - 1.
\end{align*}
\]

(V) Let \( 4 \leq s \leq k - 2 \). Thus \( k \geq 6 \). Two cases may be considered.

1. When \( 2 \leq \varphi(k) < s - 1 \), we examine two possible cases.

   (1) For \( s < \varphi(1) \leq k - 1 \), letting \( \alpha = 1 \), \( \beta = \varphi(k) \), \( \gamma = \varphi(1) \) and \( \delta = k \), we have \( \mu_k(1, k) \) by Lemma 3.

   (2) For \( \varphi(1) = s \), by the following 8 steps, we shall prove that there always exists \( \mu_k(1, k) \).

      1) If \( \varphi_1(s + 1) = 1 \), letting \( \alpha = \varphi_1(s + 1) = 1 \), \( \beta = \varphi_1(k) \), \( \gamma = s + 1 \) and \( \delta = k \), we have \( \mu_k(1, k) \) by Lemma 3.

      If \( \varphi_1(s + 1) = s - 1 \), letting \( \alpha = 1 \), \( \gamma = \varphi(1) = s \), and \( \delta = s + 1 \), we have \( \mu_k(1, k) \) by Lemma 2.

      If \( \varphi_1(s + 1) = s - 2 \), taking \( w_1, w_2 \in W \), and assuming \( (w_1, w_2) \in A \), we have
\[ \mu_k(1, k) = \{1, \phi(1) = s, w_1, w_2, \ldots, s - 2 = q_1(s + 1), s + 1, \ldots, k\}. \]

Thus, in the following, we shall assume that \(2 \leq q_1(s + 1) \leq s - 3\).

2) If \(s < \phi(s - 1) \leq k\), letting \(\alpha = 1, \gamma = \phi(1) = s\), and \(\delta = \phi(s - 1)\), we have \(\mu_k(1, k)\) by Lemma 2.

Thus, in the following, we shall assume that \(\phi(s - 1) = s\), i.e. \((s - 1, s) \in A\) and \((i, s - 1) \in A\), where \(s < i \leq k\).

3) If \((s, k) \in A\), letting \(\alpha = q_1(s + 1), \gamma = s + 1\), and \(\delta = k\), we have \(\mu_k(1, k)\) by Lemma 2.

Thus, in the following, we shall assume that \((k, s) \in A\).

4) By the final assumption of 2) we have \((k, s - 1) \in A\). Thus by the assumptions of the theorem, there is a 2-path \((s - 1, q, k)\), where \(q \in V\). Clearly, \(q \in W\). Thus by the final assumptions of 2) and 3) once again, and \((k, 1) \in A\), we have \(2 \leq q \leq s - 2\).

If \(q_1(s + 1) < q \leq s - 2\), letting \(\alpha = q_1(s + 1), \beta = q, \gamma = s + 1\), and \(\delta = k\), we have \(\mu_k(1, k)\) by Lemma 3.

Thus, in the following, we shall assume that there are \((s - 1, q), (q, k) \in A\), where \(2 \leq q \leq q_1(s + 1)\).

5) If there is \((q - 1, r_0) \in A\), where \(s < r_0 < k\), letting \(\alpha = q - 1, \beta = q, \gamma = r_0\) and \(\delta = k\), we have \(\mu_k(1, k)\) by Lemma 3.

Thus, in the following, we shall assume that there are \((r, q - 1) \in A, (s < r < k)\), i.e. \(\phi(q - 1) = s\) or \(k\).

6) If \(\phi(q - 1) = s\), taking \(w_0 \in W\), by the final assumptions of 1) and 4), we have

\[ \mu_k(1, k) = \{1, \ldots, q - 1, \phi(q - 1) = s, w_0, q_1(s + 1) + 1, \ldots, s - 1, q, \ldots, \phi_1(s + 1), s + 1, \ldots, k\}. \]

Thus in the following, let \(\phi(q - 1) = k\) so that \(q \approx 2\). Hence, we have \(3 \leq q \leq q_1(s + 1)\).

7) If \((q_1(s + 1) + 1, k - 1) \in A\), by the final assumption of 1), and the definition of \(q_1(s + 1)\), we have \(k - 1 \approx s + 1\). Letting \(\alpha = q_1(s + 1), \beta = q_1(s + 1) + 1, \gamma = s + 1\) and \(\delta = k - 1\), we have \(\mu_k(1, k)\) by Lemma 3.

If \((2, q_1(s + 1) + 1) \in A\), letting \(\alpha = 2, \gamma = q_1(s + 1) + 1\) and \(\delta = s + 1\), and noting \(s \geq 4\), by the final assumptions of 6) and 1), we have \(\mu_k(1, k)\) by Lemma 2.

Thus, in the following, we shall assume \((k - 1, q_1(s + 1) + 1), (q_1(s + 1) + 1, 2) \in A\).

8) Noting all the assumptions mentioned above, and taking a vertex \(w_0 \in W\), we have

\[ \mu_k(1, k) = \{1, \phi(1) = s, w_0, q_1(s + 1) + 2, \ldots, s - 1, q, \ldots, q_1(s + 1), s + 1, \ldots, k - 1, q_1(s + 1) + 1, 2, \ldots, q - 1, \phi(q - 1) = k\}. \]
Therefore, we have proved that there always exists $\mu_k(1,k)$.

2. When $q(k) = s - 1$, we examine two possible cases.

1) When $\phi(1) = s$, letting $a = 1, \gamma = \phi(1) = s$ and $\delta = k$, we have $\mu_k(1,k)$, by Lemma 2.

2) When $s < \phi(1) \leq k - 1$, we consider the converse of $T$, denoted by $T^*$. Clearly $T^*$ is of Case 1(2). Hence, there exists $\mu^*_k(k,1)$ in $T^*$. Then we invert the direction of the arcs of $\mu^*_k(k,1)$, and obtain $\mu_k(1,k)$ in $T$.

Thus, in the following, we always let $s = 3$ or $k - 1$.

(VI) When $s = 3$, for any vertex $w \in W$, there are $(w,1),(w,2),(3,w), \ldots, (k,w) \in A$. By the assumptions of the theorem, there exists $\mu_{p-1}(1,k)$. At this point, $\mu_{p-1}(1,k)$ must be in the form

$$\{1, i_3, \cdots, i_l, \mu, 2, i_{l+1}, \cdots, i_{k-1}, k\},$$

where $\mu$ is a Hamilton path in $T[W]$, and $i_3, i_4, \cdots, i_{k-1}$ is a permutation of $3, 4, \cdots, k - 1$, and $l \geq 3$. Thus taking a vertex $v_0 \in W$, we have

$$\mu_k(1,k) = \{1, i_3, \cdots, i_l, v_0, 2, i_{l+1}, \cdots, i_{k-1}, k\}.$$

Thus, in the following, we always suppose $s = k - 1$.

(VII) When $s = k - 1$, we consider the converse $T^*$. Clearly, $T^*$ is of Case (VI). Hence there is a path $\mu^*_k(k,1)$ in $T^*$. Then we invert the direction of the arcs of $\mu^*_k(k,1)$, and obtain $\mu_k(1,k)$ in $T$.

Thus, the proof of Theorem 2 is completed.

Remark. Recently, Hong Yuan of Hua-dong Normal University, who suggests a general conjecture as follows: Suppose a tournament $T$ with $p$ vertices has $h$-arc-cyclicity ($h = 3, 4, \cdots, l; l \leq p$) if and only if $T$ has both 3-arc-cyclicity and $l$-arc-cyclicity. This conjecture can be proved by the same method used in proving Theorems 1 and 2.

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References