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A NECESSARY AND SUFFICIENT CONDITION FOR ARC-PANCYCLICITY OF TOURNAMENTS

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ABSTRACT

In this paper, it is proved that a tournament T with p vertices has arc-pancyclicity, if and only if T has both 3-arc-cyclicity and p-arc-cyclicity.

Let $T=(V, A)$ be a tournament with p vertices. A tournament T is called *k-arc-cyclic* if for any arc $(v_0, v_1) \in A$, there is a $(k-1)$-path $\mu_{k-1}(v_1, v_0)$ from v_1 to v_0 in T, $k (3 \leqslant k \leqslant p)$ being an integer. A tournament T is called *arc-pancyclic if for any integer k $(3 \leqslant k \leqslant p)$, T has k-arc-cyclicity.*

In 1967, B. Alspach first studied the arc-pancyclicity of tournaments. A sufficient condition on arc-pancyclicity of tournaments was presented in [1]. In 1979, Zhu Yongjin and Tian Feng studied the problem again.[2] And Zhu Yongjin put forth the following problem in [3]. "If the squared graph of a tournament T is completely symmetrical, has T arc-pancyclicity?" By Lemma 2 in [4], we know that this problem is equivalent to one asking whether or not the 3-arc-cyclicity of a tournament T is the necessary and sufficient condition for its arc-pancyclicity. In another paper[1)], we have given a negative answer to this problem in a more general case. We have proved: "For $p \geqslant 6$ and $p \neq 7.9$, there exists a tournament with p vertices which is k-arc-cyclic $(k=3, 4, \cdots, p-1)$ but not arc-pancyclic". On the other hand, in [2] it has been shown that "For $p=2q>4$, there exists an almost regular tournament which is k-arc-cyclic $(k=4, 5, \cdots, p)$ but not arc-pancyclic". So, it is very likely to suggest a conjecture as stated in the following Theorem 1. The main purpose of this paper is to prove this conjecture.

Theorem 1. *A tournament $T=(V, A)$ with p vertices has arc-pancyclicity if and only if T has both 3-arc-cyclicity and p-arc-cyclicity.*

The necessity is very clear. The sufficiency will be also clear after the following Theorem 2 is proved.

Theorem 2. *Suppose a tournament $T=(V, A)$ with p vertices has 3-arc-cyclic-*

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1\) See: 吴正声,张克民,邹园,"关于竞赛图弧泛回路性的一类反例"。

ity. Let $(v_0, v_1) \in A$. *If there is a* $(p-1)$*-path* $\mu_{p-1}(v_1, v_0)$ *from* v_1 *to* v_0 *in* T *then there exists an* $(h-1)$*-path* $\mu_{h-1}(v_1, v_0)$ $(h = 4, 5, \cdots, p-1)$ *from* v_1 *to* v_0 *in* T.

Proof. By the above assumptions, it is evident that the Theorem is equivalent to the following proposition. If for any integer $k(3 \leqslant k < p-1)$, there is a $(k-1)$-path:

$$\mu_{k-1}(v_1, v_0) = \{v_1, v_2, \cdots, v_k = v_0\},$$

then there exists a k-path $\mu_k(v_1, v_k)$ from v_1 to v_k.

Now let us prove this proposition.

For convenience, $v_i (i = 1, 2, \cdots, k)$ will be simply denoted by i, and let

$$W = V - \mu_{k-1} = V - \{1, 2, \cdots, k\}.$$

Since $k < p-1$, we have $|W| > 1$, $|W|$ being the number of the vertices of W.

Now, for exhausting all possible cases, we shall prove that there always exists a path $\mu_k(1, k)$.

(I) If $(l, w_0), (w_0, m) \in A$, where $w_0 \in W$ and $1 \leqslant l < m \leqslant k$, then there exist (t, w_0) and $(w_0, t+1) \in A$, where $l \leqslant t < m$. Thus we obtain:

$$\mu_k(1, k) = \{1, \cdots, t, w_0, t+1, \cdots, k\}.$$

So, in the following, we always assume that for any vertex $w \in W$, there exists an integral index $s(w)(1 \leqslant s(w) \leqslant k+1)$ such that

$$(w, 1), (w, 2), \cdots, (w, s(w)-1). (s(w), w), (s(w) + 1, w), \cdots, (k, w) \in A^{1)}.$$

Let $s_1 = \min\limits_{w \in W} \{s(w)\}$, $s_2 = \max\limits_{w \in W} \{s(w)\}$.

(II) Suppose $s_1 = 1$. Then there exists $w_1 \in W$ such that $s(w_1) = s_1 = 1$. Note that $k \geqslant 3$ and $(3, w_1) \in A$. By the assumptions of the theorem, there exists a 2-path $\{w_1, u, 3\}$. Clearly, $u \bar{\in} \mu_{k-1}$. Hence $u \in W$, and thus we have

$$\mu_k(1, k) = \{1, w_1, u, 3, \cdots, k\}.$$

Similarly, we can prove that when $s_2 = k+1$, there also exists $\mu_k(1, k)$.

Thus in the following, we always assume that $2 \leqslant s_1 \leqslant s_2 \leqslant k$. In other words, for any vertex $w \in W$, we have $(w, 1)$, $(k, w) \in A$.

(III) Now, we shall prove $s_1 \neq 2$. Otherwise, there exists $w_1 \in W$ such that $s(w_1) = s_1 = 2$. We consider the case of $(k, w_1) \in A$. By the assumptions of the theorem, there exists a 2-path $\{w_1, u, k\}$. By the final assumption of (II), for any vertex $w \in W$, we have $(k, w) \in A$. Thus by $(u, k) \in A$ we have $u \bar{\in} W$, i. e. $u \in \mu_{k-1}$. And we have $u = 1$ by $(w_1, u) \in A$. Hence, $(1, k) \in A$. It contradicts the assumption $(k, 1) = (0, 1) \in A$. Thus we have $s_1 \neq 2$.

Similarly, we have $s_2 \neq k$.

Therefore, in the following, we always assume $3 \leqslant s_1 \leqslant s_2 \leqslant k-1$ (thus $k \geqslant 4$).

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1) By $s(w) = 1$, we mean that $(1, w), (2, w), \cdots, (k, w) \in A$, and by $s(w) = k+1$, we mean that $(w, 1), (w, 2), \cdots, (w, k) \in A$.

Then for any vertex $w \in W$, we have $(w, 1), (w, 2), (k-1, w), (k, w) \in A$.

(IV) Suppose $s_1 \neq s_2$ satisfying $3 \leqslant s_1 < s_2 \leqslant k-1$. Thus $k \geqslant 5$. Let $s(w_1) = s_1, s(w_2) = s_2$, where $w_1, w_2 \in W$. We consider the case of $(k, w_1) \in A$. By the assumptions of the theorem, there exists a 2-path $\{w_1, l, k\}$. Clearly, $l \bar{\in} W$ and $2 \leqslant l \leqslant s_1 - 1$. Thus we have proved that there exists an integer $l(2 \leqslant l \leqslant s_1 - 1)$ such that $(l, k) \in A$. Consider the case of $(w_2, 1) \in A$. Similarly, there exists an integer $m(s_2 \leqslant m \leqslant k-1)$, satisfying $(1, m) \in A$. Since $s_1 < s_2$, we have $l + 1 \leqslant m - 1$. Now, two cases may be considered separately.

1. When $l + 1 = m - 1$, from $l + 1 \leqslant s_1 \leqslant s_2 - 1 \leqslant m - 1$, we have $l + 1 = s_1 = s_2 - 1 = m - 1$. Since $|W| > 1$, there are $w', w'' \in W$. Without loss of generality, assume that $(w', w'') \in A$. By the final assumption of (III) there are $(k-1, w')$ and $(w'', 2) \in A$. Thus we have

$$\mu_k(1, k) = \{1, m = s_2, \cdots, k-1, w', w'', 2, \cdots, s_2 - 2 = l, k\}.$$

2. When $l + 1 < m - 1$, we consider two possible cases.

(1) For $l + 2 < s_2$, we have

$$\mu_k(1, k) = \{1, m, \cdots, k-1, w_2, l+2, \cdots, m-1, w_1, 2, \cdots, l, k\}.$$

(2) For $l + 2 \geqslant s_2$, noting $s_1 + 1 \geqslant l + 2 \geqslant s_2$, we have $s_1 + 1 = s_2$. Thus $m - 2 > l \geqslant s_2 - 2 = s_1 - 1$ and we have

$$\mu_k(1, k) = \{1, m, \cdots, k-1, w_2, l+1, \cdots, m-2, w_1, 2, \cdots, l, k\}.$$

Therefore, in the following, we always assume that $s_1 = s_2 = s$ and $3 \leqslant s \leqslant k-1$, i. e. for any vertex $w \in W$, we have $(w, 1), (w, 2), \cdots, (w, s-1), (s, w), \cdots, (k-1, w), (k, w) \in A$.

At this point, the following lemmas are valid.

Lemma 1. *$T[W] = (W, A_W)$, the subgraph of T induced by W is a 3-arc-cyclic tournament. Hence $|W| \geqslant 3$, and $T[W]$ is strongly connected.*

Proof. Clearly, $T[W]$ is a tournament. Note that T has 3-arc-cyclicity. Thus for any $(w_1, w_2) \in A_W (\subset A)$, there must be a 2-path $\{w_2, u, w_1\}$. Clearly $u \in \mu_{k-1}$, so that $u \in W$. Hence $\{w_2, u, w_1\}$ is a 2-path in $T[W]$. Q. E. D.

Lemma 2. If *$(\alpha, \gamma), (\gamma - 1, \delta) \in A$, where $1 \leqslant \alpha \leqslant s-2$, $\alpha + 1 < \gamma < \delta \leqslant k$ and $s + 1 \leqslant \delta$, then there is a k-path $\mu_k(1, k)$ from 1 to k.*

Proof. Taking $w_0 \in W$, we have

$$\mu_k(1, k) = \{1, \cdots, \alpha, \gamma, \cdots, \delta - 1, w_0, \alpha + 1, \cdots, \gamma - 1, \delta, \cdots, k\}. \text{ Q. E. D.}$$

Lemma 3. *If $(\alpha, \gamma), (\beta, \delta) \in A$, where $1 \leqslant \alpha < \beta < s - 1$ and $s < \gamma < \delta \leqslant k$, then there is a μ_k-path $\mu_k(1, k)$ from 1 to k.*

Proof. Two cases may be considered.

(1) When $\gamma > s + 1$, taking $w_1, w_2 \in W$, we have

$$\mu_k(1, k) = \{1, \cdots, \alpha, \gamma, \cdots, \delta - 1, w_1, \beta + 1, \cdots, \gamma - 2, w_2, \alpha + 1, \cdots, \beta, \delta, \cdots, k\}.$$

(2) When $\gamma = s+1$, noting $\beta \leqslant s-2$, we examine two possible cases

(a) For $\beta < s-2$, taking $w_1, w_2 \in W$, we have

$$\mu_k(1,k) = \{1, \cdots, \alpha, \gamma, \cdots, \delta - 1, w_1, \beta + 2, \cdots, s = \gamma - 1, w_2, \alpha + 1, \cdots, \beta, \delta, \cdots, k\}.$$

(b) For $\beta = s-2$, by Lemma 1, there is a 2-path $\{w_1, w_2, w_3\}$ in $T[W]$. Then we have

$$\mu_k(1,k) = \{1, \cdots, \alpha, \gamma = s+1, \cdots, \delta - 1, w_1, w_2, w_3, \alpha + 1, \cdots, s - 2 = \beta, \delta, \cdots, k\}.$$

The proof of the lemma is completed.

Lemma 4. *Let*

$$R(l) = \{i \mid (l,i) \in A, s \leqslant i \leqslant k\} \qquad (1 \leqslant l \leqslant s-1),$$

and

$$L(r) = \{i \mid (i,r) \in A, 1 \leqslant i \leqslant s-1\} \qquad (s \leqslant r \leqslant k).$$

Then for any integer $l (1 \leqslant l \leqslant s-1)$, *we have* $R(l) \neq \phi$, *and* $k \bar{\in} R(1)$. *And for any integer* $r (s \leqslant r \leqslant k)$, *we have* $L(r) \neq \phi$, *and* $1 \bar{\in} L(k)$.

Proof. Taking $w_0 \in W$, we have $(w_0, l) \in A$. By the assumptions of the theorem, there is a 2-path $\{l, u, w_0\}$ in T, where $u \in V$. Clearly $u \bar{\in} W \cup \{1, 2, \cdots, s-1\}$. Hence $u \in \{s, s+1, \cdots, k\}$ and $(l, u) \in A$, so that $u \in R(l)$, $R(l) \neq \phi$. Since $(1, k) \bar{\in} A$, we have $k \bar{\in} R(1)$. Similarly, we can prove the rest of the lemma. Q. E. D.

Let

$$\phi(l) = \max R(l), \quad \phi_1(l) = \min R(l), \qquad (1 \leqslant l \leqslant s-1);$$

and

$$\varphi(r) = \min L(r), \quad \varphi_1(r) = \max L(r), \qquad (s \leqslant r \leqslant k).$$

Thus $(l, \phi(l)), (l, \phi_1(l)), (\varphi(r), r), (\varphi_1(r), r) \in A$, and

$$s \leqslant \phi_1(l) \leqslant \phi(l) \leqslant k, \qquad (2 \leqslant l \leqslant s-1),$$
$$s \leqslant \phi_1(1) \leqslant \phi(1) \leqslant k-1;$$
$$1 \leqslant \varphi(r) \leqslant \varphi_1(r) \leqslant s-1, \qquad (s \leqslant r \leqslant k-1),$$
$$2 \leqslant \varphi(k) \leqslant \varphi_1(k) \leqslant s-1.$$

(V) Let $4 \leqslant s \leqslant k-2$. Thus $k \geqslant 6$. Two cases may be considered.

1. When $2 \leqslant \varphi(k) < s-1$, we examine two possible cases.

(1) For $s < \phi(1) \leqslant k-1$, letting $\alpha = 1$, $\beta = \varphi(k)$, $\gamma = \phi(1)$ and $\delta = k$, we have $\mu_k(1,k)$ by Lemma 3.

(2) For $\phi(1) = s$, by the following 8 steps, we shall prove that there always exists $\mu_k(1,k)$.

1) If $\varphi_1(s+1) = 1$, letting $\alpha = \varphi_1(s+1) = 1$, $\beta = \varphi_1(k)$, $\gamma = s+1$ and $\delta = k$, we have $\mu_k(1,k)$ by Lemma 3.

If $\varphi_1(s+1) = s-1$, letting $\alpha = 1$, $\gamma = \phi(1) = s$, and $\delta = s+1$, we have $\mu_k(1,k)$ by Lemma 2.

If $\varphi_1(s+1) = s-2$, taking $w_1, w_2 \in W$, and assuming $(w_1, w_2) \in A$, we have

$$\mu_k(1, k) = \{1, \phi(1) = s, w_1, w_2, 2, \cdots, s-2 = \varphi_1(s+1), s+1, \cdots, k\}.$$

Thus, in the following, we shall assume that $2 \leqslant \varphi_1(s+1) \leqslant s-3$.

2) If $s < \phi(s-1) \leqslant k$, letting $\alpha = 1, \gamma = \phi(1) = s$, and $\delta = \phi(s-1)$, we have $\mu_k(1, k)$ by Lemma 2.

Thus, in the following, we shall assume that $\phi(s-1) = s$, i. e. $(s-1, s) \in A$ and $(i, s-1) \in A$, where $s < i \leqslant k$.

3) If $(s, k) \in A$, letting $\alpha = \varphi_1(s+1), \gamma = s+1$, and $\delta = k$, we have $\mu_k(1, k)$ by Lemma 2.

Thus, in the following, we shall assume that $(k, s) \in A$.

4) By the final assumption of 2) we have $(k, s-1) \in A$. Thus by the assumptions of the theorem, there is a 2-path $\{s-1, q, k\}$, where $q \in V$. Clearly, $q \bar{\in} W$. Thus by the final assumptions of 2) and 3) once again, and $(k, 1) \in A$, we have $2 \leqslant q \leqslant s-2$.

If $\varphi_1(s+1) < q \leqslant s-2$, letting $\alpha = \varphi_1(s+1), \beta = q, \gamma = s+1$ and $\delta = k$, we have $\mu_k(1, k)$ by Lemma 3.

Thus, in the following, we shall assume that there are $(s-1, q), (q, k) \in A$, where $2 \leqslant q \leqslant \varphi_1(s+1)$.

5) If there is $(q-1, r_0) \in A$, where $s < r_0 < k$, letting $\alpha = q-1, \beta = q, \gamma = r_0$ and $\delta = k$, we have $\mu_k(1, k)$ by Lamma 3.

Thus, in the following, we shall assume that there are $(r, q-1) \in A, (s < r < k)$, i. e. $\phi(q-1) = s$ or k.

6) If $\phi(q-1) = s$, taking $w_0 \in W$, by the final assumptions of 1) and 4), we have

$$\mu_k(1, k) = \{1, \cdots q-1, \phi(q-1) = s, w_0, \varphi_1(s+1)+1, \cdots, s-1, q, \cdots, \phi_1(s+1), s+1, \cdots, k\}.$$

Thus in the following, let $\phi(q-1) = k$ so that $q \neq 2$. Hence, we have $3 \leqslant q \leqslant \varphi_1(s+1)$.

7) If $(\varphi_1(s+1)+1, k-1) \in A$, by the final assumption of 1), and the definition of $\varphi_1(s+1)$, we have $k-1 \neq s+1$. Letting $\alpha = \varphi_1(s+1), \beta = \varphi_1(s+1)+1, \gamma = s+1$ and $\delta = k-1$, we have $\mu_k(1, k)$ by Lemma 3.

If $(2, \varphi_1(s+1)+1) \in A$, letting $\alpha = 2, \gamma = \varphi_1(s+1)+1$ and $\delta = s+1$, and noting $s \geqslant 4$, by the final assumptions of 6) and 1), we have $\mu_k(1, k)$ by Lemma 2

Thus, in the following, we shall assume $(k-1, \varphi_1(s+1)+1)$, $(\varphi_1(s+1)+1, 2) \in A$.

8) Noting all the assumptions mentioned above, and taking a vertex $w_0 \in W$, we have

$$\mu_k(1, k) = \{1, \phi(1) = s, w_0, \varphi_1(s+1)+2, \cdots, s-1, q, \cdots, \varphi_1(s+1), s+1, \cdots, k-1, \varphi_1(s+1)+1, 2, \cdots, q-1, \phi(q-1) = k\}.$$

Therefore, we have proved that there always exists $\mu_k(1, k)$.

2. When $\varphi(k) = s - 1$, we examine two possible cases.

1) When $\psi(1) = s$, letting $\alpha = 1, \gamma = \psi(1) = s$ and $\delta = k$, we have $\mu_k(1, k)$, by Lemma 2.

2) When $s < \psi(1) \leqslant k - 1$, we consider the converse of T, denoted by T^*. Clearly T^* is of Case 1(2). Hence, there exists $\mu_k^*(k, 1)$ in T^*. Then we invert the direction of the arcs of $\mu_k^*(k, 1)$, and obtain $\mu_k(1, k)$ in T.

Thus, in the following, we always let $s = 3$ or $k - 1$.

(VI) When $s = 3$, for any vertex $w \in W$, there are $(w, 1), (w, 2), (3, w), \cdots, (k, w) \in A$. By the assumptions of the theorem, there exists $\mu_{p-1}(1, k)$. At this point, $\mu_{p-1}(1, k)$ must be in the form

$$\{1, i_3, \cdots, i_l, \mu, 2, i_{l+1}, \cdots, i_{k-1}, k\},$$

where μ is a Hamilton path in $T[W]$, and $i_3, i_4, \cdots, i_{k-1}$ is a permutation of $3, 4, \cdots, k - 1$, and $l \geqslant 3$. Thus taking a vertex $w_0 \in W$, we have

$$\mu_k(1, k) = \{1, i_3, \cdots, i_l, w_0, 2, i_{l+1}, \cdots, i_{k-1}, k\}.$$

Thus, in the following, we always suppose $s = k - 1$.

(VII) When $s = k - 1$, we consider the converse T^*. Clearly, T^* is of Case (VI). Hence there is a path $\mu_k^*(k, 1)$ in T^*. Then we invert the direction of the arcs of $\mu_k^*(k, 1)$, and obtain $\mu_k(1, k)$ in T.

Thus, the proof of Theorem 2 is completed.

Remark. Recently, Hong Yuan of Hua-dong Normal University, who suggests a general conjecture as follows: Suppose a tournament T with p vertices has h-arc-cyclicity ($h = 3, 4, \cdots, l; l \leqslant p$) if and only if T has both 3-arc-cyclicity and l-arc-cyclicity. This conjecture can be proved by the same method used in proving Theorems 1 and 2.

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