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Completely Strong Path-Connected Tournaments

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Let $T=(V,A)$ be a tournament with p vertices. T is called completely strong path-connected if for each arc $(a,b)\in A$ and k $(k=2,3,\ldots,p)$, there is a path from b to a of length k (denoted by $P_k(a,b)$) and a path from a to b of length k (denoted by $P'_k(a,b)$). In this paper, we prove that T is completely strong path-connected if and only if for each arc $(a,b)\in A$, there exist $P_2(a,b)$, $P'_2(a,b)$ in T, and T satisfies one of the following conditions: (a) $T\not\cong T_0$-type graph, (b) T is 2-connected, (c) for each arc $(a,b)\in A$, there exists a $P'_{p-1}(a,b)$ in T.

1. INTRODUCTION

Let $D=(V,A)$ be a digraph with p vertices. D is called *arc-pancyclic* (resp. *arc-antipancyclic*) if for each arc $(a,b)\in A$, there is a path from b to a (resp. from a to b) of length k $(k=2,3,\ldots,p-1)$ in D, denoted by $P_k(a,b)$, or briefly P_k (resp. $P'_k(a,b)$, P'_k). D is called *strong path-connected* if for each two vertices $a,b\in V$, there is a path from a to b of length k $(k=d, d+1,\ldots,p-1$, where $d=d_D(a,b)$ is a distance from a to b) in D.

Clearly, a strong path-connected digraph is arc-antipancyclic.

A tournament T is called *completely strong path-connected* if T is arc-pancyclic and arc-antipancyclic.

Faudree and Schelp [3] defined the concept of strong path-connectedness in undirected graphs. The concept of strong path-connectedness in digraphs is a natural generalization of that concept. Thomassen [5] defined a concept of *strongly panconnected*. Although a completely strong path-connected tournament is strongly panconnected, both the probabilities of the existence of these two classes of tournaments approach one as $p\to\infty$ in the case of random tournaments with p vertices. (See [4, sects. 5 and 9].) In [1, 5, 8], the authors studied strong panconnectedness and obtained several sufficient conditions for that. But they do not consider the existence of the P_2 and P'_2. In this paper, we are going to study the action of the P_2, P'_2 in the completely

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strong path-connected tournaments, and obtain three necessary and sufficient conditions which are stated in Theorems 1–3. Obviously, all of these conditions are rather easy to verify.

2. The Main Results

THEOREM 1. *A tournament $T=(V,A)$ with p vertices is completely strong path-connected if and only if for each arc $e \in A$, there exist $P_2(e)$, $P_2'(e)$ in T, and $T \not\cong T_0$-type graph (see Fig. 1).*

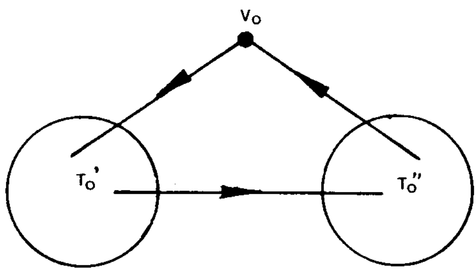

FIG. 1. T_0-type graph. (Here T_0', T_0'' are tournaments and $(T_0', T_0''), (T_0'', v_0), (v_0, T_0') \subset A(T_0)$.)

By Theorem 1, it is easy to obtain Theorems 2 and 3 as follows:

THEOREM 2. *A tournament $T=(V,A)$ with p vertices is completely strong path-connected if and only if T is 2-connected and for each arc $e \in A$, there exist $P_2(e)$, $P_2'(e)$ in T.*

THEOREM 3. *A tournament $T=(V,A)$ with p vertices is completely strong path-connected if and only if for each arc $e \in A$, there exist $P_2(e)$, $P_2'(e)$, and $P_r'(e)$ (where $r=r(e) \geqslant p/2$) in T.*

We have immediately the following:

COROLLARY (Zhang and Wu [7]). *A tournament $T=(V,A)$ with p vertices is completely strong path-connected if and only if for each $e \in A$, there exist $P_2(e)$, $P_2'(e)$, and $P_{p-1}'(e)$ in T.*

The corollary is a conjecture in [7], its general form is still an open problem as follows:

Conjecture. A tournament $T=(V,A)$ with p vertices is strong path-connected if and only if for each arc $e \in A$, there exist $P_2'(e)$ and $P_{p-1}'(e)$ in T.

3. PROOF OF THEOREM 1

Necessity. Obvious.

Sufficiency. For T_6 or T_8-type graph (see Figs. 2, 3), it is easy to prove

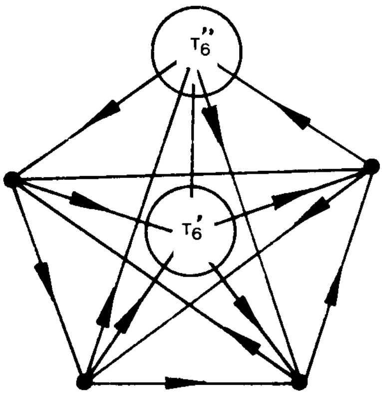

FIG. 2. T_6-type graph. (Where T_6', T_6'' are tournaments, the directions of the edges without arrow heads can be chosen arbitrary.)

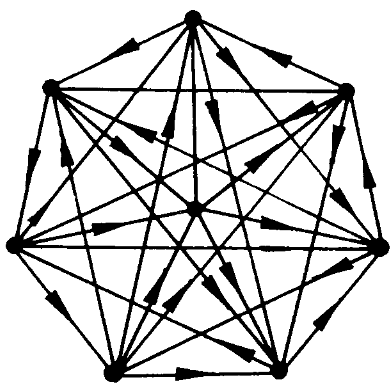

FIG. 3. T_8-type graph. (The directions of the edges without arrow heads can be chosen arbitrary.)

directly that there exists some arc such that there is no P_2' with respect to that arc. So, T is not a T_6- or T_8-type graph. By [6, Theorem 1], T is an arc-pancyclic tournament. And by [4, Sect. 9], there always exists a $P_k'(a, b)$ in T for $k \leqslant 6$. Then it is only necessary to prove the following:

PROPOSITION. *For any k ($7 \leqslant k \leqslant p-1$), if there exists a $P_{k-1}'(a, b)$ in T, then there exists a $P_k'(a, b)$ in T.*

Proof. From now on, we shall assume that there is a $P'_{k-1}(a, b)$ in T, and denote it by $[1, 2,..., k]$, where a and 1 represent the same vertex in T, so do b and k. The set of vertices $\{1, 2,..., k\}$ of $P'_{k-1}(a, b)$ is also denoted by P'_{k-1}. Let $W = V \backslash P'_{k-1}$. Hence $|W| \geqslant 1$. If the conclusion of the proposition were false, we should assume:

$$\text{There does not exist any } P'_k(a, b) \text{ in } T. \qquad (*)$$

We could immediately obtain:

(I) There are no (i, w), $(w, j) \in A$, where $w \in W$ and $i < j$, $i.\ j \in P'_{k-1}$.

(II) There is no $w \in W$ such that $(i, w) \in A$ (resp. $(w, i) \in A$) for each $i \in P'_{k-1}$.

Before discussing (III) and (IV), it is convenient to introduce some notation. Let $D = (V, A)$ be a digraph, $v \in V$, set $I_D(v) = \{u \mid u \in V, (u, v) \in A\}$ and $O_D(v) = \{u \mid u \in V, (v, u) \in A\}$ (without ambiguity, they may be denoted as $I(v)$ and $O(v)$, respectively). An *index function* $s(w)$ on W is defined as follows: For each $w \in W$, there is an index $s(w)$ satisfying $1 < s(w) \leqslant k$, such that $O'(w) \equiv O(w) \cap P'_{k-1} = \{1, 2,..., s(w) - 1\}$ and $I'(w) \equiv I(w) \cap P'_{k-1} = \{s(w), s(w) + 1,..., k\}$. From (I), (II), it is obvious that $s(w)$ exists for each $w \in W$.

LEMMA 1. *For any $v_0 \in V$ in T, there exists a cycle in the induced subgraph $T[O(v_0)]$ (resp. $T[I(v_0)]$). Furthermore, $|O(v_0)| \geqslant 3$, (resp. $|I(v_0)| \geqslant 3$).*

Proof. Since T is strongly connected and anti-symmetrical, the conclusion of Lemma 1 is obvious. ∎

Set $s_1 = s(w_1) = \min\{s(w) \mid w \in W\}$ and $s_2 = s(w_2) = \max\{s(w) \mid w \in W\}$.

LEMMA 2. *If $s_1 < s_2$, then there are not n, m, u, and v in T such that $u < n \leqslant s_1 - 1 < s_2 \leqslant v < m$ and (n, m), $(u, v) \in A$.*

Proof. Otherwise, it will contradict $(*)$. ∎

Now, (n, m), $(u, v) \in A$ are called cis-*crosswise arcs with respect to the* $P'_k(a, b)$ (briefly cis-*crosswise arcs*) if n, m, u, and v are on $P'_k(a, b)$ such that $u < n < v < m$.

LEMMA 3. *If $s_1 < s_2$, $(s_1 - 1, s_2) \in A$ and $(s_1 - 1, s_2) \neq (a, b)$, then there exists an arc (u, v) such that (u, v) and $(s_1 - 1, s_2)$ are* cis-*crosswise arcs.*

Proof. First, we have that:

(i) For each $i \in \{3, 4, \dots, s_i - 1\}$, we have $(i, 1) \in A$.

Otherwise, there exists i_0, $(1, i_0) \in A$. By assumption, there is a $P_2(w_2, i_0 - 1)$: $[i_0 - 1, u, w_2]$, according to the definition of s_1, w_2, $u \bar{\in} W$, $u \bar{\in} O'(w_2)$, hence we must have $u \in I'(w_2)$. Thus there is a $P'_k(a, b)$ in T: $[1, i_0, \dots, u - 1, w_1, 2, \dots, i_0 - 1, u, \dots, k]$. This contradicts $(*)$. Similarly, we have:

(ii) For each $j \in \{s_2, s_2 + 1, \dots, k - 2\}$, we have $(k, j) \in A$.

Now, if the Lemma were not true, we would have:

(a) If $s_1 - 1 = 1$, we have $s_2 \neq k$ and $I(k) = \{1, k - 1\}$ by (ii).

(b) If $s_2 = k$, we have $s_1 - 1 \neq 1$ and $O(1) = \{2, k\}$ by (i).

(c) If $1 < s_1 - 1 < s_2 < k$, when $(j, 1) \in A$ for each $j \in \{s_2, s_2 + 1, \dots, k - 1\}$, then, by (i), $O(1) = \{2, k\}$. When there is j_0 such that $(1, j_0) \in A$, then, by Lemma 2, $(k, i) \in A$ for each $i \in \{2, 3, \dots, s_1 - 1\}$. Thus, by (ii), $I(k) = \{1, k - 1\}$.

These conclusions of (a)–(c) contradict Lemma 1. ∎

(III) Suppose $s_1 < s_2$.

There is a $P_2(s_2, w_1)$: $[w_1, u, s_2]$, $u \bar{\in} W$, $u \bar{\in} I'(w_1)$ by assumption, hence we have $u \in O'(w_1)$, that is, there exists u such that $1 \leqslant u \leqslant s_1 - 1$, $(u, s_2) \in A$. Similarly, by means of a $P_2(w_2, s_1 - 1)$, there exists m such that $s_2 \leqslant m \leqslant k$, $(s_1 - 1, m) \in A$. Since $u < s_1 - 1$, $m > s_2$ contradict Lemma 2, $(s_1 - 1, s_2) \in A$.

Case 1: $s_2 - s_1 \geqslant 4$

There exists a $P'_k(a, b)$ in T: $[1, \dots, s_1, w_1, w_2, s_1 + 2, \dots, s_2, \dots, k]$, (resp. $[1, \dots, s_1, w_1, w_3, w_2, s_1 + 3, \dots, s_2, \dots, k]$) for $(w_1, w_2) \in A$ (resp. $(w_2, w_1) \in A$).

Case 2: $s_2 - s_1 \leqslant 3$

Since $k \geqslant 7$, $(s_1 - 1, s_2) \neq (a, b)$. There exists an arc (u, v) such that (u, v) and $(s_1 - 1, s_2)$ are cis-crosswise arcs by Lemma 3.

We may assume, without loss of generality, that $u < s_1 - 1 < v < s_2$ (otherwise, we consider the converse of T). For $v = s_1, s_1 + 1$, and $s_1 + 2$, there exist $P'_k(a, b)$ in T, respectively, e.g., $v = s_1 + 2$, there exists a $P'_k(a, b)$: $[1, \dots, u, s_1 + 2 = v, \dots, s_2 - 1, w_1, w_3, w_2, u + 1, \dots, s_1 - 1, s_2, \dots, k]$.

Summing up Cases 1 and 2, there always exist $P'_k(a, b)$ in T when $s_1 < s_2$. Thus (III) contradicts $(*)$. So, it follows that

(IV) $s_1 = s_2 = s$, i.e., $s(w) \equiv s$ on W.

In order to deduce that (IV) is in contradiction with ($*$), we need a Lemma as follows:

LEMMA 4. *If* $u < n < v < m$, *and* (u, v), (n, m) *are* cis-*crosswise arcs, then* (a) $v \neq n+1$ *when* $n < s < m$, (b) $n = s-1$, $v = s$ *can not hold simultaneously*, (c) $n = s-2$, $v = s+1$ *can not hold simultaneously*.

Proof. Conditions (a) and (b) are obvious.

(c) Let $n = s-2$, $v = s+1$. Case 1: If $(s-1, m-1) \in A$, we have $(s, u+1) \in A$ by (b), hence there is a $P_k'(a, b)$ in T: $[1,..., u, s+1 = v,..., m-1, \ w, \ s-1, \ s, \ u+1,..., \ s-2 = n, m,..., k]$. Case 2: If $(m-1, s-1) \in A$, there is a $P_k'(a, b)$: $[1,..., u, s+1 = v,..., m-1, \ s-1, s, w, u+1,..., s-2 = n, m,..., k]$. They are in contradiction with ($*$). So, (c) is valid. ∎

Now, by (IV), we have:

(1) $3 \leqslant s \leqslant k-1$.

Note that $1 < s \leqslant k$, T is a T_0-type graph when $s = 2$ or k, this contradicts the assumption. Therefore (1) is valid.

(2) There exists an arc $(n', m') \in A$ such that $n' < s-1 < s < m'$ and $(n', m') \neq (a, b)$.

Case 1

If $s = k-1$, we have $(k, s-1) \in A$ by Lemma 4(b) and Lemma 1. Hence from $|I(k)| \geqslant 3$, there always exists $i_0 \in \{2,..., s-2\}$ such that $(i_0, k) \in A$. Set $i_0 = n'$, $k = m'$. Since $k \geqslant 7$, $(n', m') \neq (a, b)$. Similarly, we can also verify conclusion (2) in the case $s = 3$.

Case 2

If $3 < s < k-1$, there are $u' \in O'(w)$ and $v' \in I'(w)$ such that $(2, v')$, $(u', k-1) \in A$ by $P_2(w, 2)$ and $P_2(k-1, w)$, respectively. When $v' > s$, we may set $n' = 2$, $m' = v'$; When $v' = s$, we have $u' < s-1$ by Lemma 4(b). Hence we may set $n' = u'$, $m' = k-1$.

Thus Cases 1 and 2 imply that (2) is valid.

Let A' denote the totality of $(n', m') \in A$ mentioned above, and let $\tilde{n} = \max\{n' \mid (n', m') \in A'\}$, $\tilde{m} = \min\{m' \mid (\tilde{n}, m') \in A'\}$. Obviously, $(\tilde{n}, \tilde{m}) \in A' \subset A$, $(\tilde{n}, \tilde{m}) \neq (a, b)$ and $\tilde{n} < s-1 < s < \tilde{m}$. Furthermore, if $\tilde{m}_1 = \min\{m' \mid (n', m') \in A'\}$ and $\tilde{n}_1 = \max\{n' \mid (n', \tilde{m}) \in A'\}$, we have $\tilde{n} = \tilde{n}_1$ and $\tilde{m} = \tilde{m}_1$. In fact, $\tilde{n} \geqslant \tilde{n}_1$, $\tilde{m} \geqslant \tilde{m}_1$, $(\tilde{n}_1, \tilde{m}_1) \in A' \subset A$ and $\tilde{n}_1 <$

$s-1<s<\tilde{m}_1$. When $\tilde{n}_1<\tilde{n}$, $\tilde{m}_1<\tilde{m}$, we need only consider two subcases by Lemma 4(c): (i) $\tilde{n}<s-2$ and (ii) $\tilde{m}_1>s+1$. There exist the following $P'_k(a,b)$ in T: $[1,\ldots,\tilde{n}_1,\tilde{m}_1,\ldots,\tilde{m}-1,\tilde{n}+1,\ldots,\tilde{m}_1-1,w,\tilde{n}_1+1,\ldots,\tilde{n},\tilde{m},\ldots,k]$ and $[1,\ldots,\tilde{n}_1,\tilde{m}_1,\ldots,\tilde{m}-1,w,\tilde{n}+1,\ldots,\tilde{m}_1-1,\tilde{n}_1+1,\ldots,\tilde{n},\tilde{m},\ldots,k]$, respectively. These contradict $(*)$. Hence we must have either $\tilde{n}=\tilde{n}_1$ or $\tilde{m}=\tilde{m}_1$ which give $\tilde{m}=\tilde{m}_1$ or $\tilde{n}=\tilde{n}_1$, respectively. Therefore $\tilde{n},\tilde{m}$ are independent of the order of selection.

(3) There always exists an arc (u',v') in A such that (u',v') and $(\tilde{n},\tilde{m})$ are cis-crosswise arcs.

First, we assume that there does not exist any (u',v') as mentioned above. Then T has the following three properties:

(3i) For each $i\in\{3,4,\ldots,\tilde{n}\}$, we have $(i,1)\in A$.

In fact, if there is an i_0 such that $(1,i_0)\in A$, there does not exist, by Lemma 4(a), any $P_2(w,i_0-1)$ in T. This contradicts the assumption. Similarly, we can prove:

(3ii) For each $j\in\{\tilde{m},\tilde{m}+1,\ldots,k-2\}$, we have $(k,j)\in A$.

(3iii) If $u_1<n_1\leqslant\tilde{n}<\tilde{m}\leqslant v_1<m_1$, (u_1,v_1) and (n_1,m_1) can not belong to A simultaneously.

In fact, if (u_1,v_1), $(n_1,m_1)\in A$, we shall consider four subcases separately: (i) $u_1\leqslant\tilde{n}-2$, $m_1\geqslant\tilde{m}+2$. Then there is a $P'_k(a,b)$: $[1,\ldots,u_1,v_1,\ldots,m_1-1,s,\ldots,v_1-1,w,n_1+1,\ldots,s-1,u_1+1,\ldots,n_1,m_1,\ldots,k]$. (ii) $u_1\leqslant\tilde{n}-2$, $m_1=\tilde{m}+1$. Then there is a $P'_k(a,b)$: $[1,\ldots,u_1,\tilde{m}=v_1,w,n_1+1,\ldots,\tilde{m}-1,u_1+1,\ldots,n_1,\tilde{m}+1=m_1,\ldots,k]$. (iii) $u_1=\tilde{n}-1$, $m_1\geqslant\tilde{m}+2$. Then there is a $P'_k(a,b)$: $[1,\ldots,\tilde{n}-1=u_1,v_1,\ldots,m_1-1,\tilde{n}+1,\ldots,v_1-1,w,\tilde{n}=n_1,m_1,\ldots,k]$. (iv) $u_1=\tilde{n}-1$, $m_1=\tilde{m}+1$. We have that: $\tilde{n}<s-2$ or $\tilde{m}>s+1$ by Lemma 4(c). Hence there exist $P'_k(a,b)$: $[1,\ldots,\tilde{n}-1=u_1,\tilde{m}=v_1,\tilde{n}+1,\ldots,\tilde{m}-1,w,\tilde{n}=n_1,\tilde{m}+1=m_1,\ldots,k]$ or $[1,\ldots,\tilde{n}-1=u_1,\tilde{m}=v_1,w,\tilde{n}+1,\ldots,\tilde{m}-1,\tilde{n}=n_1,\tilde{m}+1=m_1,\ldots,k]$, respectively. Since (i)–(iv) contradict $(*)$, (3iii) is valid.

Now, we begin proving (3) by contradiction as follows:

Case 1: $s=k-1$

By (3i), we have $O(1)=\{2,k\}$.

Case 2: $s=3$

By (3ii), we have $I(k)=\{1,k-1\}$.

Case 3: $3<s<k-1$

If for each $j\in\{\tilde{m},\ldots,k-1\}$, we have $(j,1)\in A$. Then, by (3i), $O(1)=\{2,k\}$. Otherwise, by (3ii) and (3iii), we have $I(k)=\{1,k-1\}$.

Cases 1–3 contradict Lemma 1. Therefore, (3) is valid.

We may assume, without loss of generality, that $(u', v') \in A$, $u' < \tilde{n} < v' < \tilde{m}$, $\tilde{n} \geqslant 2$ (otherwise, we consider the converse of T). Let A'' denote the totality of $(u', v') \in A$ mentioned above. Set $\tilde{v} = \min\{v' \mid (u', v') \in A''\}$, $\tilde{u} = \max\{u' \mid (u', \tilde{v}) \in A''\}$. Obviously, $(\tilde{u}, \tilde{v}) \in A'' \subset A$ and $\tilde{u} < \tilde{n} < \tilde{v} < \tilde{m}$. Then we have:

(4) $(t, i) \in A$ for any $i \in \{1, 2, ..., \tilde{n} - 1\}$, $t \in \{\tilde{n} + 1, ..., \tilde{v} - 1\}$.

(5) $\tilde{u} = \tilde{n} - 1$. Furthermore, $(\tilde{n} - 1, \tilde{v}) \in A$. Otherwise, it will contradict (*).

(6) $\tilde{m} = s + 1$. Furthermore, $(\tilde{n}, s + 1) \in A$.

If $\tilde{m} > s + 1$, we have $(s - 1, s + 1) \in A$ by $P_2(s + 1, w)$ and the definition of $\tilde{m}_1$ $(=\tilde{m})$.

Case 1: $\tilde{n} < s - 2$

By $P_2(w, s - 2)$ and the definition of $\tilde{n}$, we have $(t, s - 2) \in A$, where $t \in \{s + 1, ..., k\}$. Hence $(s - 2, s) \in A$. This contradicts Lemma 4(b).

Case 2: $\tilde{n} = s - 2$

Note that $(s - 1, s + 1) \in A$ and by Lemma 4, we have $\tilde{v} \neq s - 1, s, s + 1$, i.e., $\tilde{v} > s + 1$. Thus, there is a $P'_k(a, b)$: $[1, ..., \tilde{u}, \tilde{v}, ..., \tilde{m} - 1, w, s - 1, ..., \tilde{v} - 1, \tilde{u} + 1 = \tilde{n} = s - 2, \tilde{m}, ..., k]$. This contradicts (*). Therefore (6) is valid.

(7) For each $i \in \{\tilde{n} + 1, ..., s\}$. $j \in \{s + 1, ..., k\}$ and $(j, i) \neq (s + 1, s)$, we have $(j, i) \in A$.

Otherwise, we assume that there exists $(i, j) \in A$.

Case 1: $j > s + 1$

If we consider the converse T' of T, then $\tilde{n} = s - 2$ by (6), i.e., $(s - 2, s + 1) \in A$. Also for T, T', we have $(s - 3, s)$, $(s - 1, s + 2) \in A$ by (5) and Lemma 4(a). This contradicts Lemma 4(b).

Case 2: $j = s + 1 = \tilde{m}$

By definition of $\tilde{n}_1$ $(=\tilde{n})$, we have $(s + 1, i) \in A$, for all $i \in \{\tilde{n} + 1, ..., s - 2\}$. It remains to prove that $(s + 1, s - 1) \in A$. In fact, if $(s - 1, s + 1) \in A$, we have $(s, i) \in A$ for each $i \in \{1, 2, ..., s - 2\}$. And by Case 1, there does not exist any $P'_2(s - 1, s)$ in T. Which leeds to a contradiction.

Therefore (7) is valid.

(8) $(t, s) \in A$ for any $t \in \{\tilde{n} + 1, ..., s - 1\}$.

By $P_2(w, t)$ and (7), (8) is valid.

(9) $v \bar{\in} \{\tilde{n} + 1, \tilde{n} + 2, ..., s - 2\}$.

Assume that $v \in \{\tilde{n}+1, \tilde{n}+2,..., s-1\}$. Let T_1 denote the induced subgraph $T[\{\tilde{n}+1,..., s-1\}]$. The condensation $\hat{T}_1$ of T_1 is a transitive tournament (See [2, 10.1.9]. Let $\hat{v}$ denote the dicomponent including $\tilde{v}$ in T_1 and denote it in $\hat{T}$, too. Let L (resp. R) be the set of vertices corresponding to $I_{\hat{T}_1}(\hat{v})$ (resp. $O_{\hat{T}_1}(\hat{v})$) in T. Clearly, we have:

(9i) For any $i \in L$, we have $i < \tilde{v}$. Also for any $j \in R$, we have $\tilde{v} < j$. Obviously, L, R, and $\hat{v}$ have Hamilton paths, denoted by μ_1, μ_2, and μ, respectively.

(9ii) $L \neq \varnothing$.

Otherwise, if $L \neq \varnothing$, there is a $P'_k(a, b)$ in T: $[1,..., \tilde{u} = \tilde{n}-1, \mu, \mu_2, s, w, \tilde{n}, s+1 = \tilde{m},..., k]$ by (8). This contradicts (*).

(9iii) $R = \varnothing$.

In fact, if $R \neq \varnothing$, we have $(L, R) \subset A$. By (4) and (7), $P_2(L, R)$ must be $[R, \tilde{n}, L]$. Hence, there is a $P'_k(a, b)$ in T: $[1,..., \tilde{n}-1 = \tilde{u}, \mu, s, w, \mu_1, \mu_2, \tilde{n}, s+1 = \tilde{m},..., k]$ by (8). This contradicts (*).

(9iv) $\hat{v} = \{\tilde{v}\}$.

Otherwise, if $\hat{v} \neq \{\tilde{v}\}$, $P_2(L, \hat{v})$ must be $[\hat{v}, \tilde{n}, L]$ by (4) and (7). Hence, by (9iii) and (8), there is a $P'_k(a, b)$ in T: $[1,..., \tilde{n}-1 = \tilde{u}, \tilde{v}, s, w, \mu_1, \mu', \tilde{n}, s+1 = \tilde{m},..., k]$, where μ' is a Hamilton path in $\hat{v} \backslash \{\tilde{v}\}$. This contradicts (*).

Finally, by (9i), (9iii), and (9iv), we have $\tilde{v} = s-1$. So, (9) is valid.

(10) $\tilde{v} \bar{\in} \{s-1, s\}$.

We prove (10) by contradiction. Assume that $\tilde{v} = s-1$ or s. By Lemma 4(a), $\tilde{v} > \tilde{n}+1$. In this case, T has the following properties:

(10i) For each $i \in \{1, 2,..., \tilde{u}-1 = \tilde{n}-2\}$, we have $(\tilde{v}, i) \in A$. Furthermore, for each $j \in \{1, 2,..., \tilde{u}-2 = \tilde{n}-3\}$, we have $(s, j) \in A$.

In fact, if $(i, \tilde{v}) \in A$, there is a $P'_k(a, b)$: $[1,..., i, \tilde{v},..., s, w, \tilde{n}+1,..., \tilde{v}-1, i+1,..., \tilde{n}, s+1 = \tilde{m},..., k]$ by (4). If $\tilde{v} = s-1$, $(j, s) \in A$, there is a $P'_k(a, b)$: $[1,..., j, s, w, \tilde{n}+1,..., s-1 = \tilde{v}, j+1,..., \tilde{n}, s+1 = \tilde{m},..., k]$ by (4). These contradict (*).

(10ii) For each $i, j \in \{1, 2,..., \tilde{n}-1\}$ and $i > j+1$, we have $(i, j) \in A$, except the case of $\tilde{v} = s-1$, $(\tilde{n}-2, s) \in A$, and $(i, j) = (\tilde{n}-1, \tilde{n}-3)$.

In fact, except for the case of $\tilde{v} = s-1$, $i = \tilde{n}-1$, and $(\tilde{n}-2, s) \in A$, by (10i) and the same reasoning as in the proof of (3i), we have $(i, j) \in A$. As for the case of $\tilde{v} = s-1$, $i = \tilde{n}-1$, $(\tilde{n}-2, s) \in A$, and $j < \tilde{n}-3$, if $(j, \tilde{n}-1) \in A$, we have, by (10i) and $P_2(w, \tilde{n}-3)$, that $r_0 \in \{s+1,..., k\}$ such that $(\tilde{n}-3, r_0) \in A$. Hence there is a $P'_k(a, b)$: $[1,..., j, \tilde{n}-1,..., s-1 = \tilde{v}, \tilde{n}-2, s,..., r_0-1, w, j+1,..., \tilde{n}-3, r_0,..., k]$. This contradicts (*).

(10iii) For each $i, j \in \{s+1,..., k\}$ and $i > j+1$, we have $(i, j) \in A$.

By (7) and the same reasoning as in (3i), (10iii) follows immediately.

(10iv) There always exists $i_0 \in \{2, 3,..., \tilde{n}\}$ such that $(i_0, k) \in A$.

When $k = s + 1$, the conclusion is trivial. When $k > s + 1$, if $(k, i) \in A$ for each $i \in \{2, 3,..., \tilde{n}\}$, we have $I(k) = \{1, k-1\}$ by (10iii) and (7). This contradicts Lemma 1.

(10v) There do not exist (u_1, v_1), $(u_2, v_2) \in A$ such that $u_1 < u_2 < s-1 < s < v_1 < v_2$.

If (u_1, v_1), $(u_2, v_2) \in A$ and $u_1 < u_2 < s-1 < s < v_1 < v_2$, we have $u_2 \leqslant \tilde{n} \leqslant s-2$, $v_1 \geqslant s+1$.

Case 1

If $u_1 = \tilde{n} - 1$, we have $u_2 = \tilde{n}$. There is a $P'_k(a, b)$: $[1,..., \tilde{n}-1 = u_1, v_1,..., v_2 - 1, \tilde{n}+1,..., v_1 - 1, w, \tilde{n} = u_2, v_2,..., k]$ by (7). This contradicts (*).

Case 2

If $u_1 < \tilde{n} - 1$, $\tilde{n} < s-2$ or $u_1 < \tilde{n} - 1$, $v_2 > s+2$, there is a $P'_k(a, b)$: $[1,..., u_1, v_1,..., v_2 - 1, \tilde{n}+2,..., v_1 - 1, w, u_2 + 1,..., \tilde{n}+1, u_1 + 1,..., u_2, v_2,..., k]$ by (4) and (7). This contradicts (*).

Case 3

If $u_1 < \tilde{n} - 1$, $\tilde{n} = s - 2$, and $v_2 = s + 2$, we have $v_1 = s + 1$, $\tilde{v} = s$.

Subcase 3.1. If $u_1 < s - 4 = \tilde{n} - 2$, there is a $P'_k(a, b)$: $[1,..., u_1, s+1 = v_1, w, u_2 + 1,..., u_1 + 1,..., u_2, s+2 = v_2,..., k]$ by (10i). This contradicts (*).

Subcase 3.2. If $u_1 = s - 4 = \tilde{n} - 2$, we have $u_2 = s - 3$ by Lemma 4(c). Thus $(s-3, s+2)$, $(s-4, s+1) \in A$. By (10ii) and Lemma 4(a), we have $(i, 1) \in A$ for each $i \in \{3, 4,..., s-2\}$. When $k = s+2$ and $s - 4 = 1$, thus $k = 7$ and there exists no $P'_2(1, 2)$ nor $P'_2(1, 6)$ in T. This leads to a contradiction. When $k = s + 2$ and $s - 4 > 1$, if $(1, s+1) \in A$, there is a $P'_k(a, b)$: $[1, s+1, s-1, s, w, s-2, 2,..., s-3, k]$ by (7). This contradicts (*). Hence we have $(s+1, 1) \in A$ and $O(1) = \{2, k\}$ by (10i) and (4). When $k > s+2$, by (10iv) and Case 2, we have $(j, 1) \in A$ for each $j \in \{s+1,..., k-1\}$. Hence, in this case, we always have $s - 4 > 1$ and $O(1) = \{2, k\}$ by (10i) and (4). These contradict Lemma 1.

(10vi) If $s < k - 1$, we have $(k, \tilde{n}) \in A$.

In fact, if $(\tilde{n}, k) \in A$, we have $(k-1, i) \in A$ by (10v), where $i \in \{1, 2,..., \tilde{n}-1\}$. Hence, by (10iii) and (7), $I(k-1) \subset \{\tilde{n}, k-2\}$. This contradicts Lemma 1.

(10vii) $\tilde{n} \geqslant 4$.

Case 1

If $\tilde{n}=2$, (10iv) and (10vi) can not be satisfied simultaneously for $s<k-1$. Hence, we only consider the following subcases:

Subcase 1.1. If $s=k-1$ and $\tilde{v}=s-1$, we have $L=\{3,\dots,k-3\}$ and $(L,\tilde{v})\subset A$ by (9iv). Thus $O(\tilde{v})\subset\{2,k-1\}$ by (7). This contradicts Lemma 1.

Subcase 1.2. If $s=k-1$ and $\tilde{v}=s$, there exists no $P_2'(1,2)$ nor $P_2'(1,k-1)$ in T by (4), which is a contradiction.

Case 2

If $\tilde{n}=3$.

Subcase 2.1. If $\tilde{v}=s-1$, we have $L=\{4,\dots,s-2\}=\varnothing$ and $(L,\tilde{v})\subset A$ by (9). And we have $O(\tilde{v})=\{1,3,s\}$ by (7) and Lemma 1, thus $(\tilde{v},3)\in A$. Hence $(s,1)\in A$, for otherwise, there is a $P_k'(a,b)$: $[1,s,w,4,\dots,s-2,2=\tilde{u},s-1=\tilde{v},3=\tilde{n},s+1=\tilde{m},\dots,k]$. This contradicts (*). Furthermore, by (10i), (10iv), (10v), (4), and Lemma 1, we have $O(1)=\{2,3,k\}$ and $(k,2)$, $(3,k)\in A$. Hence, when $s=k-1$, there does not exist any $P_2'(2,s-1)$ in T. This contradicts the assumption. When $s<k-1$, $(3,k)=(\tilde{n},k)\in A$ contradicts (10vi).

Subcase 2.2. If $\tilde{v}=s$, we have, by (10i), (4), and Lemma 1, that $O(1)=\{2,3,k\}$, $(k,2)\in A$ and $O(2)=\{3,k-1\}$ when $s=k-1$; and we have, by (10iv), (10vi), that $(2,k)$, $(3,1)\in A$ when $s<k-1$. Furthermore, by (10v), we have $(j,1)\in A$ for each $j\in\{s+1,\dots,k-1\}$. Thus we have $O(1)=\{2,k\}$ by (10i) and (4). These contradict Lemma 1.

Thus Cases 1 and 2 imply that (10vii) is valid.

(10viii) $(1,\tilde{n})\in A$.

If $(\tilde{n},1)\in A$, we have, by (10i), (10ii), (10iv), (10v), and (4), that $O(1)=\{2,k\}$ when $\tilde{n}>4$; and we have, by (10i), (10iv), (10v), (4), and Lemma 1, that $O(1)=\{2,3,k\}$ and $(1,3),(k,2)$, $(3,k)\in A$ when $\tilde{n}=4$. Thus, $O(2)=\{3,s\}$. These contradict Lemma 1. So, $(1,\tilde{n})\in A$.

(10ix) $(k,2)\in A$. In particular, when $\tilde{n}=4$, we have $(k,3)\in A$, too.

If $(2,k)\in A$, there is a $P_k'(a,b)$ in T: $[1,\tilde{n},\dots,\tilde{v}-1,3,\dots,\tilde{n}-1=\tilde{u},\tilde{v},\dots,k-1,w,2,k]$ by (10vii), (10viii) and (4). This contradicts (*). When $\tilde{n}=4$, we have $(k,3)\in A$, by (10viii). So, (10ix) is valid.

(10x) $s\neq k-1$.

Otherwise, if $s=k-1$, we need only consider the following two cases by (10vii):

Case 1

If $\tilde{v}=s$ or $\tilde{v}=s-1$, $\tilde{n}>4$, we have, by (10ix) and $P_2(w,2)$, that $(2,k-1)=(2,s)\in A$. Thus, there is a $P'_k(a,b)$ in T: $[1,2,k-1=s,\ w,\ \tilde{n}+1,\dots,s-1,\ 3,\dots,\tilde{n},k]$. This contradicts $(*)$.

Case 2

If $\tilde{v}=s-1$, $\tilde{n}=4$, we have $O(2)=\{3,4,s\}$, $(2,4)$, $(s,3)\in A$, and $(3,k)\in A$ by (10i), (10ix), (4), Lemma 1, and $P_2(w,3)$. This contradicts Lemma 4(a).

Thus Cases 1 and 2 imply that (10x) is valid.

(10xi) $k-1\leqslant s$.

In fact, if $k-1>s$, we have, by (10i), (10ii), (10iv), (10v), and (4), that $O(1)\subset\{2,\tilde{n},k\}$ when $\tilde{n}>4$ and we have $O(1)\subset\{2,3,4=\tilde{n},k\}$ when $\tilde{n}=4$. By virtue of (10vi) and (10ix), there exists no $P'_2(1,k)$ in T. This contradiction implies that (10xi) is valid.

Since (10x), (10xi) contradict (1), (10) is established.

Finally, we have $\tilde{v}\bar{\in}\{\tilde{n}+1,\dots,s-1,s\}$ by (9) and (10). But it contradicts the assumption of the existence of an arc $(\tilde{u},\tilde{v})\in A$ such that $\tilde{u}<\tilde{n}<\tilde{v}<\tilde{m}$. Hence, under the condition of (IV) $s_1=s_2=s$, there always exists a $P'_k(a,b)$ in T.

Up to now, under the conditions of Theorem 1, we have exhausted all possible cases of T and deduced that there always exists a $P'_k(a,b)$ in T. Therefore the proof of Theorem 1 is complete. ∎

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