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An Ore-type Condition for Cyclability†

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A graph G is said to be cyclable if for each orientation D of G, there exists a set $S(D) \subseteq V(G)$ such that reversing all the arcs with one end in S results in a Hamiltonian digraph. Let G be a simple graph of even order $n \geq 8$. In this paper, we show that if the degree sum of any two nonadjacent vertices is not less than $n + 1$, then G is cyclable and the lower bound is sharp.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a finite simple graph without loops. The neighbourhood $N(v)$ of a vertex v is the set of vertices adjacent to v. The degree $d(v)$ of v is $|N(v)|$. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a vertex $v \in V(G)$ and a subset $S \subseteq V(G)$, $N_S(v)$ is the set of neighbours of v contained in S, i.e., $N_S(v) = N(v) \cap S$. We let $d_S(v) = |N_S(v)|$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S. Let H be a subgraph of G. If $h_1h_2 \in E(G)$ for any $h_1, h_2 \in V(H)$, then we say H is a *clique*. A path with one end u is called a *u-path*. Let $u, v \in V(G)$. A spanning subgraph H of G is called a (u, v)*-path-factor* if H contains two components, one of them is a u-path and the other is a v-path. Let P be a path. We denote by $\overrightarrow{P}$ the path P with a given direction, and by $\overleftarrow{P}$ the path P with the reverse direction. If $u, v \in V(P)$, then $u\overrightarrow{P}v$ denotes the consecutive vertices of P from u to v in the direction specified by $\overrightarrow{P}$. The same vertices, in reverse order, are given by $v\overleftarrow{P}u$. If a path or cycle includes every vertex of $V(G)$, then it is called a *Hamilton path* or *cycle*. If G contains a Hamilton cycle, then we say G is *Hamiltonian*. Furthermore, we define

$$\sigma_2(G) = \min\{d(u) + d(v) \mid u, v \in V(G) \text{ and } uv \notin E(G)\}.$$

Let D be orientation of G and $C = v_1 \cdots v_m$ be an even cycle of G. We define

$$f_C(v_iv_{i+1}) = \begin{cases} 1, & \text{if } v_iv_{i+1} \in A(D), \\ 0, & \text{if } v_{i+1}v_i \in A(D), \end{cases}$$

and

$$f(C) = \sum_{e \in E(C)} f_C(e),$$

where $v_{m+1} = v_1$ and $A(D)$ is the arc set of D.

If $f(C)$ is even, then we say C is *good* under the orientation. Otherwise, we say C is *bad*.

Switch at a vertex v of a graph G removes from G all the edges incident with v and adds the new edges between v and all the vertices originally nonadjacent to v. This operation has been studied by Colbourn and Corneil [1], Mallows and Sloane [5], Rubinson and Goldman [12, 13], Stanley [14], Taylar [15], and others. *Pushing* a vertex v in a digraph reverses all the orientations of all arcs incident with v. We say that a digraph D can be pushed to a digraph H if a digraph isomorphic to H can be obtained by applying a sequence of pushes to D. The push operation has been studied by Pretzel [9–11]. In [4], Klostermeyer *et al.* introduced a Hamiltonian-like property of graphs, that is, cyclability. A graph is said to be *cyclable* if each

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of its orientations can be pushed to one that contains a directed Hamilton cycle. The following is the first result on cyclability due to Klostermeyer.

THEOREM 1 (KLOSTERMEYER [3]). *Let G be a graph with order n. If n is odd, then G is cyclable if and only if G is Hamiltonian. If n is even, then an orientation D of G can be pushed to one that contains a directed Hamilton cycle if and only if D contains a good Hamilton cycle.*

Clearly, if a graph is cyclable, then it is Hamiltonian. However, the reverse is not true. Furthermore, as pointed out in [4], neither Hamilton connectivity nor cycle extendibility is stronger than cyclability and vice versa. Hence, for any theorem on hamiltonicity, it is of interest to give an analogous result for cyclable graphs. The following is a fundamental result on hamiltonicity due to Dirac.

THEOREM 2 (DIRAC [2]). *Let G be a simple graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian.*

Dirac's Theorem is important since it has many generalizations and the following well known one of them is due to Ore.

THEOREM 3 (ORE [8]). *Let G be a simple graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is Hamiltonian.*

The following is a generalization of Dirac's Theorem to digraphs.

THEOREM 4 (NASH-WILLIAMS [7]). *Let D be a strict digraph on $n \geq 3$ vertices with minimum in-degree δ^- and minimum out-degree δ^+. If $\min\{\delta^-, \delta^+\} \geq n/2$, then D contains a directed Hamilton cycle.*

A far-reaching generalization of Theorems 2, 3 and 4, which was given by Meyniel, is the following.

THEOREM 5 (MEYNIEL [6]). *Let D be a strict strong digraph on n vertices, where $n \geq 2$. If $\sigma_2(D) \geq 2n - 1$, then D contains a directed Hamilton cycle.*

In this paper, we give an Ore-type condition for cyclability. The main result of this paper is the following theorem.

THEOREM 6. *Let G be a graph with even order $n \geq 8$. If $\sigma_2(G) \geq n+1$, then G is cyclable.*

REMARK. The lower bound of the condition is best possible in the following sense.
Let $G = K_{2t+1,2t+1} = (A, B)$ be a complete bipartite graph on $4t + 2$ vertices with bipartition (A, B), where $t \geq 1$. Suppose D is an orientation of G such that each edge is oriented from A to B. It is not difficult to see that $\sigma_2(G) = 4t + 2$ and G is not cyclable since each Hamilton cycle of D is bad.

As a direct consequence of Theorem 6, we have the following Dirac-type condition for cyclability.

COROLLARY 1. *Let G be a graph with even order $n \geq 8$. If $\delta(G) \geq n/2 + 1$, then G is cyclable.*

Let $\delta(n)$ be the smallest positive integer δ such that each n-vertex graph with minimum degree at least δ is cyclable ($n \geq 5$). Klostermeyer showed that $\delta(6) = 5$ and asked in [4] the precise values of n for all positive even integers n. By Corollary 1 and the remark, we have $\delta(n) = n/2 + 1$ for $n \equiv 2 \pmod 4$ and $n \geq 10$. However, we do not know whether it is true for $n \equiv 0 \pmod 4$ and $n \geq 8$. It is of interest to determine the precise values for all $n \equiv 0 \pmod 4$ and $n \geq 8$.

2. Some Lemmas

In order to prove Theorem 6, we need the following lemmas. The first three lemmas can be extracted from [4].

LEMMA 1 (KLOSTERMEYER *et al.* [4]). *Let G be a simple graph of even order. If for each orientation D of G, D contains a good 4-cycle with a diagonal, say $a_1a_2a_3a_4$ with $a_1a_3 \in E(G)$, such that there exists a Hamilton path in $G - \{a_1, a_3\}$ connecting a_2 and a_4, then G is cyclable.*

LEMMA 2 (KLOSTERMEYER *et al.* [4]). *Let $K_{2,3} = (A, B)$ be a complete bipartite graph with bipartition $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$. Then for any orientation of $K_{2,3}$, at least one of the cycles $a_1b_1a_2b_2$, $a_1b_1a_2b_3$ and $a_1b_2a_2b_3$ is good.*

LEMMA 3 (KLOSTERMEYER *et al.* [4]). *Let G be a graph, $xy \in E(G)$ and $\{v_1, v_2, v_3\} \subseteq N(x) \cap N(y)$. If for any two vertices $v_i, v_j \in \{v_1, v_2, v_3\}$, there exists a Hamilton path in $G - \{x, y\}$ connecting v_i and v_j, then G is cyclable.*

The following lemma is a consequence of Theorem 3, so we omit its proof.

LEMMA 4. *Let G be a graph of order n. If $\sigma_2(G) \geq n - 1$, then G has a Hamilton path.*

LEMMA 5. *Let G be a graph of order n and $P = v_1v_2 \cdots v_n$ a Hamilton path of G. If G is not Hamiltonian, then $d(v_1) + d(v_n) \leq n - 1$.*

PROOF. Since G is not Hamiltonian, we have $v_1v_n \notin E(G)$ and for any $v_i \in N(v_n)$, $v_{i+1} \notin N(v_1)$. Otherwise, $v_1 \overrightarrow{P} v_i v_n \overleftarrow{P} v_{i+1} v_1$ is a Hamilton cycle. This implies that there are at least $d(v_n)$ vertices among $v_2, \ldots, v_n$ that are not adjacent to v_1 and hence $d(v_1) + d(v_n) \leq n - 1$. □

LEMMA 6. *Let n be an even integer and G a graph of order n. If $\sigma_2(G) \geq n - 1$, then for any two vertices $u, v \in V(G)$, G contains a (u, v)-path-factor.*

PROOF. By Lemma 4, G contains a Hamilton path, say $P = v_1 \cdots v_n$. Suppose $u = v_i$, $v = v_j$ with $i < j$. If $i = 1$ or $j = n$ or $j = i + 1$, then it is easy to see the conclusion holds. Hence we may assume $1 < i < j - 1 < j < n$. If G is Hamiltonian, then the conclusion holds. Hence we may assume G is not Hamiltonian.

Suppose to the contrary that G contains no (u, v)-path-factor. Then $\{v_1, v_{i+1}, v_n\}$ is an independent set. Let $P_1 = v_1 \overrightarrow{P} v_{i-1}$, $P_2 = v_{i+1} \overrightarrow{P} v_{j-1}$ and $P_3 = v_{j+1} \overrightarrow{P} v_n$. Since $\sigma_2(G) \geq n - 1$, by Lemma 5, we have

$$d(v_1) + d(v_n) = n - 1. \tag{1}$$

We now show that $\{u, v\} \subseteq N(v_1) \cap N(v_n)$. Since G is not Hamiltonian, by the proof of Lemma 5, we can see that for any $v_k \in N(v_n)$, $v_{k+1} \notin N(v_1)$. Clearly, $v_{i-1}v_n \notin E(G)$. Otherwise, G has a (u, v)-path-factor. This implies that there are at least $d_{P_1}(v_n)$ vertices among $v_2, \ldots, v_{i-1}$ that are not adjacent to v_1 and hence $d_{P_1}(v_1) + d_{P_1}(v_n) \leq |P_1| - 1$. By symmetry, we have $d_{P_3}(v_1) + d_{P_3}(v_n) \leq |P_3| - 1$. Noting that $v_{i+1}, v_{j-1} \notin N(v_1) \cap N(v_n)$, by a similar argument, we find that $d_{P_2}(v_1) + d_{P_2}(v_n) \leq |P_2| - 1$. Thus, $d(v_1) + d(v_n) \leq |P_1| + |P_2| + |P_3| - 3 + 4 = n - 1$. Since $\sigma_2(G) \geq n - 1$, we have $d(v_1) + d(v_n) = n - 1$, which implies that $\{u, v\} \subseteq N(v_1) \cap N(v_n)$.

Since $v_1v \in E(G)$, $v_{j-1}\overleftarrow{P}v_1v\overrightarrow{P}v_n$ is a Hamilton path of G. By a similar argument as above, we have $uv_{j-1} \in E(G)$. Thus, noting that $uv_n \in E(G)$, we can see that both $v_1\overrightarrow{P}uv_n\overleftarrow{P}v_{i+1}$ and $v_{i+1}\overrightarrow{P}v_{j-1}u\overleftarrow{P}v_1v\overrightarrow{P}v_n$ are Hamilton paths of G. Since $\sigma_2(G) \geq n-1$, by Lemma 5, we have

$$d(v_1) + d(v_{i+1}) = n - 1 \tag{2}$$

and

$$d(v_{i+1}) + d(v_n) = n - 1. \tag{3}$$

By (1), (2) and (3), we obtain

$$2(d(v_1) + d(v_{i+1}) + d(v_n)) = 3(n - 1).$$

Noting that n is even, this is a contradiction. □

LEMMA 7. *Let $G = K_4$ with $V(G) = \{1, 2, 3, 4\}$ and D be an orientation of G such that the cycle $C_1 = 1234$ is good. Then either $C_2 = 1324$ or $C_3 = 1243$ is good.*

PROOF. Suppose to the contrary that both C_2 and C_3 are bad. It is not difficult to show that $f_{C_2}(14) = f_{C_2}(23)$ if and only if $f_{C_3}(12) \neq f_{C_3}(34)$. On the other hand, it is easy to check that $f_{C_2}(14) = f_{C_2}(23)$ if and only if $f_{C_1}(14) \neq f_{C_1}(23)$ and $f_{C_3}(12) \neq f_{C_3}(34)$ if and only if $f_{C_1}(12) = f_{C_1}(34)$. Thus, we find that $f_{C_1}(14) \neq f_{C_1}(23)$ if and only if $f_{C_1}(12) = f_{C_1}(34)$. Hence we can see that $f(C_1)$ is odd and then C_1 is bad, a contradiction. □

LEMMA 8. *Let G be a graph of order $n \geq 8$. If $\sigma_2(G) \geq n + 1$, then there exists an edge $xy \in E(G)$ such that $|N(x) \cap N(y)| \geq 3$.*

PROOF. Let $V_1 = \{v|v \in V(G)$ and $d(v) \geq n/2+1\}$ and $V_2 = V(G)-V_1$. Clearly, $V_1 \neq \emptyset$. We first show that for any $u \in V_1$, $N(u) \cap V_1 \neq \emptyset$. If $N(u) \cap V_1 = \emptyset$, then $N(u) \subseteq V_2$, which implies $|V_2| \geq n/2 + 1$. Since $\sigma_2(G) \geq n + 1$, $G[V_2]$ is a clique. Thus, for any $v \in V_2$, we have $d(v) \geq n/2 + 1$, a contradiction.

Choose $uv \in E(G)$ with $u, v \in V_1$ such that $d(u) + d(v)$ is as large as possible. If $d(u) + d(v) \geq n+3$, then uv is the edge as required. Thus we may assume $d(u)+d(v) \leq n+2$ and hence $d(u)+d(v) = n+2$. By the choice of uv, we have $d(u) = n/2+1$ for any $u \in V_1$. This implies $\Delta(G) = n/2 + 1$. Since $\sigma_2(G) \geq n + 1$ and $n \geq 8$, we have $\delta(G) \geq n/2 \geq 4$. Since $d(u)+d(v) = n+2$, we have $|N(u)\cap N(v)| \geq 2$. If $|N(u)\cap N(v)| \geq 3$, then the result holds. Hence we may assume $|N(u) \cap N(v)| = 2$. Let $N(u) \cap N(v) = \{a, b\}$, $N(u) - \{a, b, v\} = X$ and $N(v) - \{a, b, u\} = Y$. Since $d(u) + d(v) = n + 2$ and $|N(u) \cap N(v)| = 2$, we have $V(G)-\{u, v, a, b\} = X \cup Y$. If $ab \in E(G)$, then since $\delta(G) \geq 4$, $N(a)\cap(X \cup Y) \neq \emptyset$. Thus, au is an edge as required if $N(a)\cap X \neq \emptyset$ and av is an edge as required if $N(a)\cap Y \neq \emptyset$. Now let $ab \notin E(G)$. Then $d(a) + d(b) \geq n + 1$. Assume $d(a) \geq d(b)$, then $d(a) \geq n/2 + 1 \geq 5$. This implies $|N(a)\cap X| \geq 2$ or $|N(a)\cap Y| \geq 2$. Thus, au is an edge as required in the former case and av is an edge as required in the latter case. □

3. PROOF OF THEOREM 6

PROOF OF THEOREM 6. By Lemmas 2, 3 and 8, for any orientation D of G, D contains a good 4-cycle with a diagonal. Assume $a_1a_2a_3a_4$ is a good 4-cycle with $a_1a_3 \in E(G)$. Let

$G^* = G - \{a_1, a_3\}$. If G^* contains a Hamilton path connecting a_2 and a_4, then by Lemma 3, G is cyclable. Hence we may assume G contains no Hamilton path connecting a_2 and a_4. Clearly, $|G^*| = n - 2$ and $\sigma_2(G^*) \geq n - 3$. Thus by Lemma 6, G^* contains an (a_2, a_4)-path-factor. Choose an (a_2, a_4)-path-factor $P_1 = u_0 u_1 \cdots u_s$, $P_2 = v_0 v_1 \cdots v_t$ such that

$$|s - t| \text{ is as large as possible,} \tag{$*$}$$

where $a_2 = u_s$ and $a_4 = v_t$. Without loss of generality, we assume $s \leq t$. Write $U = \{u_0, u_1, \ldots, u_{s-1}\}$, $V = \{v_0, v_1, \ldots, v_{t-1}\}$, $S = \{a_1, a_2, a_3, a_4\}$ and let $U_0 = U \cup S$, $V_0 = V \cup S$.

CLAIM 1. *For any $v_i \in V$, if $v_i \in N(u_0)$, then $v_{i+1} \notin N(v_0)$.*

PROOF. Otherwise, $a_2 \overleftarrow{P_1} u_0 v_i \overleftarrow{P_2} v_0 v_{i+1} \overrightarrow{P_2} a_4$ is a Hamilton path connecting a_2 and a_4 in G^*, a contradiction. □

CLAIM 2. *If $N(u_0) \cap V \neq \emptyset$, then $d_V(u_0) + d_V(v_0) \leq t - s$ and if $d_V(u_0) + d_V(v_0) = t - s$, then $v_{t-s-1} \in N(u_0)$.*

PROOF. By the choice of s and t, we have $v_{t-1}, \ldots, v_{t-s} \notin N(u_0)$. Since $N(u_0) \cap V \neq \emptyset$, we have $N(u_0) \cap v_1 \overrightarrow{P_2} v_{t-s-1} \neq \emptyset$, which implies $v_{t-1}, \ldots, v_{t-s} \notin N(v_0)$ by the choice of s and t. Suppose u_0 has k neighbours among $v_1, \ldots, v_{t-s-1}$. If $v_{t-s-1} \notin N(u_0)$, then by Claim 1, there are at least k vertices among $v_1, \ldots, v_{t-s-1}$ that are not adjacent to v_0. Thus,

$$d_V(u_0) + d_V(v_0) \leq |v_1 \overrightarrow{P_2} v_{t-s-1}| = t - s - 1.$$

That is to say, $d_V(u_0) + d_V(v_0) \leq t - s$ and if $d_V(u_0) + d_V(v_0) = t - s$, then we must have $v_{t-s-1} \in N(u_0)$. □

If $s = 0$, then we have $d_V(a_2) + d_V(v_0) \leq t$ by Claim 2. If $v_{t-1} \notin N(a_2)$, then $d_V(a_2) + d_V(v_0) \leq t - 1$, which implies $d(a_2) + d(v_0) \leq (t-1) + d_S(a_2) + d_S(v_0) \leq (t-1) + 3 + 3 = t + 5 = n + 1$. Since $\sigma_2(G) \geq n + 1$, we have $d(a_2) + d(v_0) = n + 1$, which implies $d_S(a_2) = d_S(v_0) = 3$ and hence $a_2 a_4, v_0 a_1, v_0 a_3 \in E(G)$. If $v_{t-1} \in N(a_2)$, then $a_4 = v_t \notin N(v_0)$ by Claim 1. This implies $d_S(v_0) \leq 2$. Thus we have $d(a_2) + d(v_0) \leq t + d_S(a_2) + d_S(v_0) \leq t + 3 + 2 = t + 5 = n + 1$. By a similar argument, we find that $a_2 a_4, v_0 a_1, v_0 a_3 \in E(G)$. On the other hand, by Lemma 7 we know either $a_1 a_3 a_2 a_4$ or $a_1 a_2 a_4 a_3$ is good. Thus, if in the former case, $a_1 a_3 a_2 a_4$ is a good 4-cycle with a diagonal $a_1 a_2$ and $a_3 v_0 \overrightarrow{P_2} a_4$ is a Hamilton path in $G - \{a_1, a_2\}$ and if in the latter case, $a_1 a_2 a_4 a_3$ is a good 4-cycle with a diagonal $a_2 a_3$ and $a_1 v_0 \overrightarrow{P_2} a_4$ is a Hamilton path in $G - \{a_1, a_4\}$, then, by Lemma 1, G is cyclable. Hence in the following we may assume $s \geq 1$.

CLAIM 3. $d_S(u_0) + d_S(v_0) \geq 7$.

PROOF. By the choice of s and t, we have $d_U(v_0) = 0$. If $d_V(u_0) = 0$, then it is easy to see that Claim 3 holds. If $d_V(u_0) \neq 0$, then, by Claim 2, we have $d_V(u_0) + d_V(v_0) \leq t - 1$. Clearly, $d_U(u_0) \leq s - 1$. Thus we have

$$\begin{aligned} n + 1 &\leq d(u_0) + d(v_0) \\ &= d_U(u_0) + d_S(u_0) + d_V(u_0) + d_V(v_0) + d_S(v_0) + d_U(v_0) \\ &\leq (s - 1) + (t - 1) + (d_S(u_0) + d_S(v_0)). \end{aligned}$$

Since $s + t = n - 4$, we find that $d_S(u_0) + d_S(v_0) \geq 7$. □

CLAIM 4.

(1) *Suppose* $s = 1$ *and* $d_S(u_0) = 4$. *If* $G[V(P_2)]$ *contains a Hamilton path* $P = a_4 w_{t-1} \cdots w_1 w_0$ *such that* $\{a_2, a_3\} \subseteq N(w_0)$, *then* G *is cyclable.*

(2) *If* $s \geq 2$ *and* $U_0 - \{u_i\} \subseteq N(u_i)$ *for any* $u_i \in U$, *then* G *is cyclable.*
Similarly, if $V_0 - \{v_i\} \subseteq N(v_i)$ *for any* $v_i \in V$, *then* G *is cyclable.*

PROOF. (1) In this case, it is easy to see that $a_2 w_0 \overrightarrow{P} a_4 a_3$, $a_3 a_2 w_0 \overrightarrow{P} a_4$ and $a_2 a_3 w_0 \overrightarrow{P} a_4$ are Hamilton paths in $G - \{a_1, u_0\}$. By Lemma 3, G is cyclable.

(2) By Claim 3, $\{a_1, a_3\} \cap N(v_0) \neq \emptyset$. By the symmetry of a_1 and a_3, we may assume $a_1 \in N(v_0)$. Now, consider the edge $u_0 u_1$ and the vertices $a_2, a_3, a_4 \in N(u_0) \cap N(u_1)$. It is not difficult to see that $a_2 \overleftarrow{P_1} u_2 a_1 v_0 \overrightarrow{P_2} a_4 a_3$, $a_3 a_2 \overleftarrow{P_1} u_2 a_1 v_0 \overrightarrow{P_2} a_4$ and $a_2 \overleftarrow{P_1} u_2 a_3 a_1 v_0 \overrightarrow{P_2} a_4$ are Hamilton paths in $G - \{u_0, u_1\}$. By Lemma 3, G is cyclable. For the remainder part, noting that $n \geq 8$ and $s \leq t$ implies $t \geq 2$, we can obtain the conclusion by a similar argument as above. □

We now consider the following two cases.

Case 1. $N(u_0) \cap V = \emptyset$.

In this case, we have $d(u_0) \leq (s-1) + 4 = s + 3$ and $d(v_0) \leq (t-1) + 4 = t + 3$. Subject to $(*)$, we choose u_0 and v_0 such that

$$d(u_0) + d(v_0) \text{ is as small as possible.} \qquad (**)$$

CLAIM 5.

(1) *If* $V_0 - \{v_0\} \subseteq N(v_0)$, *then* $V_0 - \{v_i\} \subseteq N(v_i)$ *for any* $v_i \in V$.

(2) *If* $U_0 - \{u_0\} \subseteq N(u_0)$, *then* $U_0 - \{u_i\} \subseteq N(u_i)$ *for any* $u_i \in U$.

PROOF. (1) If $V_0 - \{v_0\} \subseteq N(v_0)$, then for any $v_i \in V$, $a_4 \overleftarrow{P_2} v_{i+1} v_0 \overrightarrow{P_2} v_i$ and P_1 is an (a_2, a_4)-path-factor of G^* satisfying $(*)$. By $(**)$, we have $d(v_i) \geq d(v_0)$, which implies that $V_0 - \{v_i\} \subseteq N(v_i)$.

(2) If $U_0 - \{u_0\} \subseteq N(u_0)$, then for any $u_i \in U$, $a_2 \overleftarrow{P_1} u_{i+1} u_0 \overrightarrow{P_1} u_i$ and P_2 is an (a_2, a_4)-path-factor of G^* satisfying $(*)$. If $s = 1$, then there is nothing to prove. Hence we may assume $s \geq 2$. If $N(u_i) \cap V \neq \emptyset$, then we have $d_V(v_0) \leq t - 3$ by Claim 2. This implies that $d(u_0) + d(v_0) \leq (s+3) + 4 + (t-3) = n$, which contradicts $\sigma_2(G) \geq n + 1$. Thus we have $N(u_i) \cap V = \emptyset$. By $(**)$, we have $d(u_i) \geq d(u_0)$, which implies that $U_0 - \{u_i\} \subseteq N(u_i)$. □

If $d(v_0) = t + 3$, then $V_0 - \{v_0\} \subseteq N(v_0)$. By Claim 5(1), we have $V_0 - \{v_i\} \subseteq N(v_i)$ for any $v_i \in V$. Thus G is cyclable by Claim 4(2). Hence we may assume $d(v_0) \leq t + 2$. Noting that $\sigma_2(G) \geq n + 1$ and $s + t = n - 4$, we have $d(v_0) = t + 2$ and $d(u_0) = s + 3$.

Since $d(u_0) = s + 3$, we have $U_0 - \{u_0\} \subseteq N(u_0)$. By Claim 5(2), we have $U_0 - \{u_i\} \subseteq N(u_i)$ for any $u_i \in U$. If $s \geq 2$, then by Claim 4(2), G is cyclable. Thus, we may assume $s = 1$. If $a_2, a_3 \in N(v_0)$, then by Claim 4(1), G is cyclable. Hence we may assume $\{a_2, a_3\} \not\subseteq N(v_1)$. This implies $\{a_1, a_4\} \cup V - \{v_0\} \subseteq N(v_0)$. Thus, for any $v_i \in V$, P_1 and $a_4 \overleftarrow{P_2} v_{i+1} v_0 \overrightarrow{P_2} v_i$ is an (a_2, a_4)-path-factor of G^* satisfying $(*)$. By $(**)$, we have $d(v_i) = t + 2$. If $a_2, a_3 \in N(v_i)$, then by Claim 4(1), G is cyclable. Hence we may assume $\{a_2, a_3\} \not\subseteq N(v_i)$ for any $v_i \in V$. This implies $\{a_1, a_4\} \cup V - \{v_i\} \subseteq N(v_i)$. Since $n \geq 8$, $s = 1$ and $s + t = n - 4$, we have $t \geq 3$. By Claim 3, we have $\{a_2, a_3\} \cap N(v_{t-1}) \neq \emptyset$. Now, consider the edge $v_0 v_1$ and $v_2, a_1, a_4 \in N(v_0) \cap N(v_1)$. We can see that $a_1 u_0 a_2 a_3 a_4 \overleftarrow{P_2} v_2$,

$v_2\overrightarrow{P_2}v_{t-1}a_1u_0a_2a_3a_4$ and $a_1v_2\overrightarrow{P_2}v_{t-1}a_2a_3u_0a_4$ (if $a_2v_{t-1} \in E(G)$) or $a_1v_3\overrightarrow{P_2}v_{t-1}a_3a_2u_1a_4$ (if $a_3v_{t-1} \in E(G)$) are Hamilton paths in $G - \{v_0, v_1\}$. By Lemma 3, G is cyclable.

Case 2. $N(u_0) \cap V \neq \emptyset$.

By the choice of s and t, we have $d_U(v_0) = 0$. By Claim 2, we have $d_V(u_0)+d_V(v_0) \leq t-s$. Thus, we have $d(u_0) + d(v_0) = d_U(u_0) + d_S(u_0) + d_V(u_0) + d_V(v_0) + d_S(v_0) + d_U(v_0) \leq (s-1)+4+(t-s)+4+0 = t+7$. Noting that $s+t = n-4$, we have $d(u_0)+d(v_0) \leq n-s+3$. If $s \geq 3$, then we have $d(u_0) + d(v_0) \leq n$, which contradicts $\sigma_2(G) \geq n + 1$. Therefore we have $s \leq 2$.

If $s = 2$, then $d_V(u_0)+d_V(v_0) \leq t-2$ by Claim 2. Thus we have $d(u_0)+d(v_0) \leq (s-1)+4+(t-2)+4 = n+1$ and hence $d(u_0)+d(v_0) = n+1$. This implies $d_S(u_0) = d_S(v_0) = 4$. Clearly, $a_2u_0u_1$ and P_2 is an (a_2, a_4)-path-factor of G^* satisfying $(*)$. If $N(u_1) \cap V = \emptyset$, then we have $d(u_1) + d(v_0) \leq (s + 3) + (t - 3) + 4 = n$ since $d_V(u_0) + d_V(v_0) \leq t - 2$ and $N(u_0) \cap V \neq \emptyset$. This is a contradiction. Hence we have $N(u_1) \cap V \neq \emptyset$. By a similar argument, we have $d_S(u_1) = 4$. Thus, we have $U_0 - \{u_i\} \subseteq N(u_i)$ for any $u_i \in U$. By Claim 4(2), G is cyclable.

If $s = 1$, then subject to $(*)$, we choose u_0 such that $d_S(u_0)$ is as large as possible. By Claim 3, we have $d_S(u_0) \geq 3$. If $d_S(u_0) = 3$, then we have $d_S(v_0) = 4$ by Claim 3. Thus, if we replace P_2 with $a_4v_0\overrightarrow{P_2}v_{t-1}$, we can obtain $d_S(v_{t-1}) = 4$. By the choice of u_0, we have $v_{t-2} \notin N(u_0)$, otherwise we can choose $v_{t-1}a_4$ and $a_2u_0v_{t-2}\overleftarrow{P_2}v_0$ instead of P_1 and P_2. Thus, by Claim 2, we have $d_V(u_0) + d_V(v_0) \leq t - 2$. This implies $d(u_0) + d(v_0) \leq (s - 1) + 3 + (t - 2) + 4 = n$, which contradicts $\sigma_2(G) \geq n + 1$. Hence we have $d_S(u_0) = 4$. On the other hand, since $N(u_0) \cap V \neq \emptyset$ and $s = 1$, by $(*)$, $G[V]$ is not Hamiltonian and hence $d_V(v_0) + d_V(v_{t-1}) \leq |V| - 1 = t - 1$ by Lemma 5. This implies $n + 1 \leq d(v_0) + d(v_{t-1}) \leq (t - 1) + d_S(v_0) + d_S(v_{t-1})$. Noting that $s + t = n - 4$ and $s = 1$, we have $d_S(v_0) + d_S(v_{t-1}) \geq 7$. If $\{a_2, a_3\} \subseteq N(v_0)$, then by Claim 4(1), G is cyclable. Hence we may assume $\{a_2, a_3\} \not\subseteq N(v_0)$ and hence $a_4 \in N(v_0)$ by Claim 3. Now, replacing P_2 with $a_4v_0\overleftarrow{P_2}v_{t-1}$, we have $d_S(v_{t-1}) = 4$ since $d_S(v_0) + d_S(v_{t-1}) \geq 7$. Thus, G is cyclable by Claim 4(1).

The proof of Theorem 6 is completed. □

REFERENCES

1. C. Colbourn and D. Corneil, On deciding switching equivalence of graphs, *Discrete Appl. Math.*, **2** (1980), 181–184.
2. G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.*, **2** (1952), 69–81.
3. W. Klostermeyer, Pushing vertices and orienting edges, *Ars Combinatorial*, **51** (1999), 65–75.
4. W. Klostermeyer and L. Soltes, Hamiltonicity and reversing arcs in digraphs, *J. Graph Theory*, **28** (1998), 13–30.
5. C. Mallows and N. Sloane, Two-graphs, switching classes and Euler graphs are equal in number, *SIAM J. Appl. Math.*, **28** (1975), 876–880.
6. H. Meyniel, Une condition suffisante d'existence d'un circuit hamiltonien dans un graphe orienté, *J. Comb. Theory, Ser. B*, **14** (1973), 137–147.
7. C. St. J. A. Nash-Williams, Hamiltonian circuits in graphs and digraphs, in: *The Many Facets of Graph Theory, Lecture Notes in Mathematics*, **110**, G. Chartrand and S. F. Kapoor (eds), Springer, Berlin, 1969, pp. 237–243. MR40:5484.
8. O. Ore, Note on Hamilton circuits, *Am. Math. Mon.*, **67** (1960), 55.
9. O. Pretzel, On graphs that can be oriented as diagrams of ordered sets, *Order*, **2** (1985), 25–40.
10. O. Pretzel, On reorienting graphs by pushing down maximal vertices, *Order*, **3** (1986), 135–153.

11. O. Pretzel, Orientations and edges functions on graphs, in: *Surveys in Combinatorics*, *London Mathematical Society Lecture Notes Series*, **166**, A. D. Keedwell (ed.), 1991, pp. 161–185.
12. A. Robinson and A. Goldman, On Ringeisen's isolation game, *Discrete Math.*, **80** (1990), 261–283.
13. A. Robinson and A. Goldman, The isolation game for regular graphs, *Discrete Math.*, **112** (1993), 173–184.
14. R. Stanley, Reconstruction from switching, *J. Comb. Theory, Ser. B*, **38** (1965), 132–138.
15. R. Taylor, Switchings constrained to 2-connectivity in simple graphs, *SIAM J. Algebr. Discrete Math.*, **3** (1983), 114–121.

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