Note

Edge-pancyclicity of coupled graphs

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Abstract

The coupled graph $c(G)$ of a plane graph $G$ is the graph defined on the vertex set $V(G) \cup F(G)$ so that two vertices in $c(G)$ are joined by an edge if and only if they are adjacent or incident in $G$. We prove that the coupled graph of a 2-connected plane graph is edge-pancyclic. However, there exists a 2-edge-connected plane graph $G$ such that $c(G)$ is not Hamiltonian. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges unless stated otherwise. A plane graph $G$ is a particular drawing in the Euclidean plane of a certain planar graph. For a plane graph $G$, we denote its vertex set, edge set, face set, and order by $V(G), E(G), F(G)$, and $|G|$, respectively. The total graph of a graph $G$ is defined on the vertex set $V(G) \cup E(G)$ such that two vertices are joined by an edge if and only if they are adjacent or incident in $G$. For a plane graph $G$, its coupled graph $c(G)$ (or entire graph $e(G)$) is defined on the vertex set $V(G) \cup F(G)$ (or $V(G) \cup E(G) \cup F(G)$) such that two vertices in $c(G)$ (or $e(G)$) are joined by an edge if and only if they are adjacent or incident in $G$. A graph $G$ is pancyclic if it possesses cycles of all lengths ranging from 3 to the order of $G$. We call $G$ vertex-pancyclic (or edge-pancyclic) if, for every vertex $v$ (or every edge $e$), there

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exist cycles of all lengths ranging from 3 to the order of $G$ each of which contains $v$ (or $e$). We call $G$ panconnected if, for every pair of distinct vertices, there exist paths joining them of all possible lengths greater than or equal to the distance between the vertices.

Fleischner [5] proved that the total graph of every 2-edge-connected graph with at least three vertices is Hamiltonian. Fleischner and Hobbs [6] further showed that the total graph of a graph $G$ of order at least two is Hamiltonian if and only if $G$ contains an EPS-subgraph. An EPS-subgraph of a graph $G$ is a connected spanning subgraph $S$ of $G$ such that $S$ is the edge-disjoint union of an Euler graph (not necessarily connected) and a (possibly empty) forest $F$ such that each of the components of $F$ is a path.

Mitchem [8] first investigated Hamiltonian and Eulerian properties of entire graphs. Hobbs and Mitchem [7] proved that the entire graph of a 2-edge-connected plane graph is Hamiltonian and the entire graph of a 2-connected plane graph is Hamiltonian connected and pancyclic. Faudree and Schelp [4] strengthened this result to show that the entire graph of a 2-edge-connected plane graph is panconnected.

A $k$-coupled coloring of a plane graph $G$ is a $k$-coloring of the vertices and the faces of $G$ so that any two distinct adjacent or incident elements in $V(G) \cup F(G)$ receive different colors. Obviously, $G$ is $k$-coupled colorable if and only if $c(G)$ is $k$-colorable. Ringel [9] conjectured that every plane graph is 6-coupled colorable. Finally, Borodin [3] established the truth of Ringel’s conjecture. The reader is referred to [1,2,10] for further properties of coupled graphs. The purpose of this paper is to study Hamiltonian properties of coupled graphs.

2. Results

Let $G$ be a plane graph. The unique unbounded face of $G$ is called the outer face and is denoted by $f_{out}(G)$ (or simply $f_{out}$). The other faces of $G$ are called inner faces. Let $\tau(G)$ (or simply $\tau$) denote the number of inner faces of $G$. Thus $\tau(G) = |F(G)| - 1$. Given a cycle $C$ in $G$, let $IN(C)$ denote the subgraph of $G$ induced by the vertices on and inside $C$ and let $in(C)$ denote the number of edges in $E(IN(C)) \setminus E(C)$. For $f \in F(G)$, we use $b(f)$ to denote the boundary of $f$.

The following is a refined version of the well-known ear decomposition of Whitney [11] applied to 2-connected plane graphs.

**Lemma 1.** Let $G$ be a 2-connected plane graph and $P_0$ be an edge of $G$. Then $G$ can be decomposed into an edge-disjoint union of paths $G = P_0 \cup P_1 \cup \cdots \cup P_{\tau(G)}$ such that the following properties hold.

1. Let $H_i = P_0 \cup P_1 \cup \cdots \cup P_i$ for $0 \leq i \leq \tau = \tau(G)$. Then the path $P_{i+1}$, $0 \leq i < \tau = \tau(G)$, with end vertices $x_{i+1}$ and $y_{i+1}$ passes through the outer face of the subgraph $H_i$ and it moves from $x_{i+1}$ to $y_{i+1}$ in the clockwise direction along the outer face of $H_{i+1}$.

2. $P_{i+1} \cap H_i = \{x_{i+1}, y_{i+1}\}$. 
(3) There is a path $Q_{i+1}$ moving from $x_{i+1}$ to $y_{i+1}$ in the clockwise direction along the boundary of the outer face of $H_i$ such that $P_{i+1} \cup Q_{i+1}$ forms an inner face of $G$.

**Proof.** Since $G$ is 2-connected, $P_0$ forms an inner face with a certain path $P_i$. Suppose that $P_0, P_1, \ldots, P_i, 1 \leq i < \tau$, have been determined. If $G = H_i$, then we are done. If $E(G) \setminus E(H_i) \neq \emptyset$, we choose an $H_i$-bridge $B$, i.e., $B$ is a component of the subgraph induced by $E(G) \setminus E(H_i)$. Since $G$ is 2-connected, $|V(B) \cap V(H_i)| \geq 2$. Note that every inner face of $H_i$ is an inner face of $G$. It follows that all vertices of $V(B) \cap V(H_i)$ belong to the boundary of the outer face of $H_i$. There exists a path $P = u_1u_2 \cdots u_s$ in $B$ such that $s \geq 2$, $u_1, u_s \in V(B) \cap V(H_i)$, and $u_2, \ldots, u_{s-1} \in V(B) \setminus V(H_i)$. We may also assume that moving clockwise from $u_1$ to $u_s$ along the boundary of the outer face of $H_i$ forms a path $Q_i$. Thus $C_0 = P \cup Q_i$ becomes a cycle of $G$. Now consider the set $\Gamma = \{C | C$ is a cycle in $IN(C_0)$ and $C$ contains at least one edge of the boundary of the outer face of $H_i\}$. The set $\Gamma$ is non-empty since it contains $C_0$. Note that $IN(C)$ is 2-connected for every $C \in \Gamma$. Among the elements $C \in \Gamma$, we choose a certain $C'$ having the smallest value of $in(C)$. If some $e \in E(IN(C')) \setminus E(C')$, then the 2-connectedness of $IN(C')$ implies that there is a cycle $C''$ in $IN(C')$ through both $e$ and an edge of $Q$. Since $E(IN(C')) \setminus E(C') \subset E(IN(C')) \setminus E(C')$ and $e \notin E(IN(C')) \setminus E(C')$, we have $in(C') < in(C'')$. This contradicts the choice of $C'$. It follows that $in(C') = 0$ and $C'$ forms the boundary of an inner face of $G$. It is straightforward to define $P_{i+1}$, $Q_{i+1}$, $x_{i+1}$, and $y_{i+1}$ from $C'$. Since we add one more inner face in each stage, the construction is finished in $\tau$ stages. \hfill $\square$

We note that every $H_i, 1 \leq i \leq \tau$, is a 2-connected plane graph in the proof of Lemma 1.

**Theorem 2.** Let $G$ be a 2-connected plane graph. Then $c(G)$ is edge-pancyclic.

**Proof.** Let $e = uv$ be an arbitrary edge of $c(G)$.

**Case 1:** Both vertices $u, v \in V(G)$.

Without loss of generality, we may assume that $uv$ lies on the common boundary of $f_1$ and $f_{out}(G)$. We first decompose $G$ into the form $P_0 \cup P_1 \cup \cdots \cup P_\tau$ guaranteed by Lemma 1 so that $P_0$ is the edge $uv$. As we add the $P_i$'s, the inner faces of $G$ can be simultaneously numbered as $f_1, f_2, \ldots, f_\tau$. Again let $H_i = P_0 \cup P_1 \cup \cdots \cup P_i$ for $0 \leq i \leq \tau$.

We proceed by induction on $\tau$. Actually, in each induction stage we construct the cycles in a systematic way such that, when the next path $P_i$ is added, a certain property (★) is preserved.

When $\tau = 1$, $G$ is the cycle $u_1u_2 \cdots u_s$, where $u = u_1$ and $v = u_s$, moving along the clockwise direction. Since $G$ is the common boundary of $f_1$ and $f_{out}$, we have $V(c(G)) = V(G) \cup \{f_1, f_{out}\}$ and $E(c(G)) = E(G) \cup \{f_1, f_{out}\} \cup \{u_if_1, u_if_{out} \mid i = 1, 2, \ldots, s\}$. In $c(G)$, we construct a particular sequence of cycles $C_n$ of length $n$, $3 \leq n \leq s+2$, each of which contains the edge $uv$. Let $C_3 = u_1f_{out}vu$, $C_4 = u_1f_{out}vu$, $C_5 = u_2f_1f_{out}vu, \ldots, C_{s+2} = u_s-1f_1f_{out}vu$. Note that each $C_{i+1}, i \geq 4$, is
obtained from $C_l$ by inserting a new vertex of $c(G)$ prior to a fixed vertex of $C_l$. We call this type of construction a monotone expansion of cycles.

Now rename $C_{l+2}$ temporarily as $z_1z_2\cdots z_{l+2}$, where $z_1 = u$ and $z_{l+2} = v$. When the path $P_2 = p_1p_2\cdots p_q$ is added along the clockwise direction to $H_1$, we may assume that $p_1 = z_i$ and $p_q = z_j$ for some $i < j$. Then the following property ($\star$) holds: $z_{i+1}$ is either a vertex of $H_1$ that is incident to $f_2$ in $H_2$ or a face of $H_1$ that is adjacent to $f_2$ in $H_2$.

Assume that the theorem holds for $\tau = k \geq 1$. Let $G$ be a 2-connected plane graph with $k + 1$ inner faces. We decompose $G$ into the form $P_0 \cup \cdots \cup P_k \cup P_{k+1}$ as in Lemma 1. By the induction hypothesis, $c(H_k) = c(P_0 \cup P_1 \cup \cdots \cup P_k)$ is edge-pancyclic through $uv$. We further assume that all the cycles in $c(H_k)$ through $uv$ are constructed by inductive stages and, within each stage, by a monotone expansion of cycles. For $m = |H_k| + |F(H_k)| = |H_k| + k + 1$, let $C_m = z_1z_2\cdots z_{m-1}$, where $z_1 = u$ and $z_m = v$, be the Hamiltonian cycle so constructed. Suppose that $P_{k+1} = v_1v_2\cdots v_t$, where $v_1 = z_i$ and $v_j = z_j$ for some $i < j$. Assume that $P_{k+1}$ moves from $v_1$ to $v_t$ in the clockwise direction along the boundary of the outer face of $H_{k+1}$. By our assumption, $f_{k+1}$ is the inner face of $G$ formed by $H_k$ and $P_{k+1}$. Now the property ($\star$) holds by the induction hypothesis, i.e., $z_{i+1}$ is either a vertex of $H_k$ that is incident to $f_{k+1}$ in $H_{k+1}$ or a face of $H_k$ that is adjacent to $f_{k+1}$ in $H_{k+1}$.

In $c(G)$, a monotone expansion of cycles $C_l$ of length $l$, $m + 1 \leq l \leq |c(G)|$, each of which contains $uv$ can be constructed as follows:

\begin{align*}
C_{m+1} &= z_1 \cdots z_if_{k+1}z_{i+1} \cdots z_{m}z_1, \\
C_{m+2} &= z_1 \cdots z_iv_2f_{k+1}z_{i+1} \cdots z_{m}z_1, \\
&\vdots \\
C_{|c(G)|} &= z_1 \cdots z_iv_2 \cdots v_{t-1}f_{k+1}z_{i+1} \cdots z_{m}z_1.
\end{align*}

Note that the path $P_{k+2}$ will be added in the clockwise direction along the boundary of the outer face of $H_{k+1}$, and $C_{|c(G)|}$ is obtained from $C_m$ by inserting a consecutive segment $v_2 \cdots v_{t-1}f_{k+1}$ if the initial end of $P_{k+2}$ does not belong to $\{z_i, v_2, \ldots, v_{t-1}\}$, then the property ($\star$) holds by induction. However, it is easy to see that the property ($\star$) is preserved if the initial end of $P_{k+2}$ belongs to $\{z_i, v_2, \ldots, v_{t-1}\}$.

**Case 2:** At least one vertex $u \in F(G)$.

If $v \in F(G)$, we suppose that $u = f_1$ and $v = f_{out}$. If $v \in V(G)$, we let $u = f_{out}$ and $v = u_k$, where $u_k \in b(f_1) \cap b(f_{out}(G))$, as defined in Case 1. We let $C_3 = u_kf_1f_{out}u_k$. For $4 \leq n \leq |c(G)|$, we may take the same cycles $C_n$ as in Case 1 since each $C_n$ always contains both the edge $f_1f_{out}$ and the edge $f_{out}u_k$. "

Once the edge-disjoint decomposition into paths is given, the inductive proof of Theorem 2 actually supplies a polynomial-time algorithm for finding a Hamiltonian cycle in the coupled graph of a 2-connected plane graph. The next theorem provides examples to show that Theorem 2 is best possible in the sense that there exists a 2-edge-connected plane graph $G$ such that $c(G)$ is not Hamiltonian.
The block graph $B(G)$ of a graph $G$ is the graph whose vertices are the blocks of $G$ and two vertices in $B(G)$ are adjacent if and only if the corresponding blocks of $G$ share a common vertex. Note that two blocks of $G$ can share at most one vertex. Suppose that $x$ is a cut vertex of $G$. Let the components of $G - x$ have vertex sets $V_1, V_2, \ldots, V_n$. Then the induced subgraphs $G[V_i \cup \{x\}]$, $i = 1, 2, \ldots, n$, are called the $x$-components of $G$. For $S \subseteq V(G)$, let $\omega(G - S)$ denote the number of components of the graph $G - S$.

**Theorem 3.** Let $G$ be a plane graph. If $B(G)$ contains a vertex of degree at least 3, then $c(G)$ is not Hamiltonian.

**Proof.** Let $B_0$ be a block of $G$ having degree $m \geq 3$ in $B(G)$. Let $B_1, B_2, \ldots, B_m$ be the blocks of $G$ that are neighbors of $B_0$ in $B(G)$. There are vertices $x_i$ in $G$ for all $i \in [m] = \{1, 2, \ldots, m\}$ such that $V(B_0) \cap V(B_i) = \{x_i\}$. Each $x_i$ is a cut vertex of $G$ as well as a cut vertex of $c(G) - f_{out}$. Moreover, $(V(B_i) \cap V(B_j)) \setminus V(B_0) = \emptyset$ for all $i, j \in [m]$ and $i \neq j$. We have the following two cases.

**Case 1:** There exist $i, j \in [m]$ such that $i \neq j$ and $x_i = x_j$.

Let $S = \{x_i, f_{out}\}$. Clearly $S$ is a cut set of $c(G)$. Since $m \geq 3$, the number of $x_i$-components of $G$ is at least 3. It follows that $\omega(c(G) - S) \geq 3 > 2 = |S|$. Hence $c(G)$ violates the necessary condition to be Hamiltonian.

**Case 2:** The vertices $x_1, x_2, \ldots, x_m$ are all distinct.

Note that $G$ has exactly two $x_i$-components for each $i \in [m]$. One of the $x_i$-components, called $G_i$, is a supergraph of $B_i$. The plane drawing of $G$ induces a natural plane embedding of $G_i$ and we may assume $f_{out}(G_i) = f_{out}(G) = f_{out}$. Since $|G_i| \geq 2$, the vertex set of $c(G_i) - \{x_i, f_{out}\}$ is nonempty.

Suppose that $c(G)$ has a Hamiltonian cycle. Then $c(G) - f_{out}$ has a Hamiltonian path $P = z_1z_2 \cdots z_t$, where $t = |c(G)| - 1$. Since $m \geq 3$, we may pick three vertices $z_i, z_j, \text{ and } z_k$ such that $1 \leq i < j < k \leq t$ and $z_i \in c(G_p) - \{x_p, f_{out}\}$, $z_j \in c(G_q) - \{x_q, f_{out}\}$, and $z_k \in c(G_r) - \{x_r, f_{out}\}$ for distinct $p, q, \text{ and } r \in [m]$. Since $x_q$ is a cut vertex of $c(G) - f_{out}$, the path $P$ has to traverse $x_q$ twice to include $z_j$. This contradicts the definition of $P$. It follows that $c(G)$ is not Hamiltonian.

It is easy to construct infinitely many 2-edge-connected plane graphs that satisfy the assumption of Theorem 3. We conclude this paper by posing the following problem.

**Problem 4.** Let $G$ be a 2-edge-connected plane graph. Is its coupled graph $c(G)$ edge-pancyclic when its block graph $B(G)$ is a path?

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