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Note

Edge-pancyclicity of coupled graphs

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Abstract

The coupled graph $c(G)$ of a plane graph G is the graph defined on the vertex set $V(G) \cup F(G)$ so that two vertices in $c(G)$ are joined by an edge if and only if they are adjacent or incident in G. We prove that the coupled graph of a 2-connected plane graph is edge-pancyclic. However, there exists a 2-edge-connected plane graph G such that $c(G)$ is not Hamiltonian. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges unless stated otherwise. A *plane graph G* is a particular drawing in the Euclidean plane of a certain planar graph. For a plane graph G, we denote its vertex set, edge set, face set, and order by $V(G), E(G)$, $F(G)$, and $|G|$, respectively. The *total* graph of a graph G is defined on the vertex set $V(G) \cup E(G)$ such that two vertices are joined by an edge if and only if they are adjacent or incident in G. For a plane graph G, its *coupled graph* $c(G)$ (or *entire graph* $e(G)$) is defined on the vertex set $V(G) \cup F(G)$ (or $V(G) \cup E(G) \cup F(G)$) such that two vertices in $c(G)$ (or $e(G)$) are joined by an edge if and only if they are adjacent or incident in G. A graph G is *pancyclic* if it possesses cycles of all lengths ranging from 3 to the order of G. We call G *vertex-pancyclic* (or *edge-pancyclic*) if, for every vertex v (or every edge e), there

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exist cycles of all lengths ranging from 3 to the order of G each of which contains v (or e). We call G *panconnected* if, for every pair of distinct vertices, there exist paths joining them of all possible lengths greater than or equal to the distance between the vertices.

Fleischner [5] proved that the total graph of every 2-edge-connected graph with at least three vertices is Hamiltonian. Fleischner and Hobbs [6] further showed that the total graph of a graph G of order at least two is Hamiltonian if and only if G contains an EPS-subgraph. An *EPS-subgraph* of a graph G is a connected spanning subgraph S of G such that S is the edge-disjoint union of an Euler graph (not necessarily connected) and a (possibly empty) forest F such that each of the components of F is a path.

Mitchem [8] first investigated Hamiltonian and Eulerian properties of entire graphs. Hobbs and Mitchem [7] proved that the entire graph of a 2-edge-connected plane graph is Hamiltonian and the entire graph of a 2-connected plane graph is Hamiltonian connected and pancyclic. Faudree and Schelp [4] strengthened this result to show that the entire graph of a 2-edge-connected plane graph is panconnected.

A *k-coupled coloring* of a plane graph G is a k-coloring of the vertices and the faces of G so that any two distinct adjacent or incident elements in $V(G) \cup F(G)$ receive different colors. Obviously, G is k-coupled colorable if and only if $c(G)$ is k-colorable. Ringel [9] conjectured that every plane graph is 6-coupled colorable. Finally, Borodin [3] established the truth of Ringel's conjecture. The reader is referred to [1,2,10] for further properties of coupled graphs. The purpose of this paper is to study Hamiltonian properties of coupled graphs.

2. Results

Let G be a plane graph. The unique unbounded face of G is called the *outer* face and is denoted by $f_{\text{out}}(G)$ (or simply f_{out}). The other faces of G are called *inner* faces. Let $\tau(G)$ (or simply τ) denote the number of inner faces of G. Thus $\tau(G) = |F(G)| - 1$. Given a cycle C in G, let $\text{IN}(C)$ denote the subgraph of G induced by the vertices on and inside C and let $\text{in}(C)$ denote the number of edges in $E(\text{IN}(C)) \setminus E(C)$. For $f \in F(G)$, we use $b(f)$ to denote the boundary of f.

The following is a refined version of the well-known ear decomposition of Whitney [11] applied to 2-connected plane graphs.

Lemma 1. *Let G be a 2-connected plane graph and P_0 be an edge of G. Then G can be decomposed into an edge-disjoint union of paths $G = P_0 \cup P_1 \cup \cdots \cup P_{\tau(G)}$ such that the following properties hold.*

(1) *Let $H_i = P_0 \cup P_1 \cup \cdots \cup P_i$ for $0 \leqslant i \leqslant \tau = \tau(G)$. Then the path P_{i+1}, $0 \leqslant i < \tau = \tau(G)$, with end vertices x_{i+1} and y_{i+1} passes through the outer face of the subgraph H_i and it moves from x_{i+1} to y_{i+1} in the clockwise direction along the outer face of H_{i+1}.*

(2) $P_{i+1} \cap H_i = \{x_{i+1}, y_{i+1}\}$.

(3) *There is a path Q_{i+1} moving from x_{i+1} to y_{i+1} in the clockwise direction along the boundary of the outer face of H_i such that $P_{i+1} \cup Q_{i+1}$ forms an inner face of G.*

Proof. Since G is 2-connected, P_0 forms an inner face with a certain path P_1. Suppose that $P_0, P_1, \ldots, P_i$, $1 \leqslant i < \tau$, have been determined. If $G = H_i$, then we are done. If $E(G) \setminus E(H_i) \neq \emptyset$, we choose an H_i-bridge B, i.e., B is a component of the subgraph induced by $E(G) \setminus E(H_i)$. Since G is 2-connected, $|V(B) \cap V(H_i)| \geqslant 2$. Note that every inner face of H_i is an inner face of G. It follows that all vertices of $V(B) \cap V(H_i)$ belong to the boundary of the outer face of H_i. There exists a path $P = u_1 u_2 \cdots u_s$ in B such that $s \geqslant 2$, $u_1, u_s \in V(B) \cap V(H_i)$, and $u_2, \ldots, u_{s-1} \in V(B) \setminus V(H_i)$. We may also assume that moving clockwise from u_1 to u_s along the boundary of the outer face of H_i forms a path Q. Thus $C_0 = P \cup Q$ becomes a cycle of G. Now consider the set $\Gamma = \{C \mid C$ is a cycle in $\mathrm{IN}(C_0)$ and C contains at least one edge of the boundary of the outer face of H_i.$\}$ The set Γ is non-empty since it contains C_0. Note that $\mathrm{IN}(C)$ is 2-connected for every $C \in \Gamma$. Among the elements $C \in \Gamma$, we choose a certain C' having the smallest value of $\mathrm{in}(C)$. If some $e \in E(\mathrm{IN}(C')) \setminus E(C')$, then the 2-connectedness of $\mathrm{IN}(C')$ implies that there is a cycle C^* in $\mathrm{IN}(C')$ through both e and an edge of Q. Since $E(\mathrm{IN}(C^*)) \setminus E(C^*) \subseteq E(\mathrm{IN}(C')) \setminus E(C')$ and $e \notin E(\mathrm{IN}(C^*)) \setminus E(C^*)$, we have $\mathrm{in}(C^*) < \mathrm{in}(C')$. This contradicts the choice of C'. It follows that $\mathrm{in}(C') = 0$ and C' forms the boundary of an inner face of G. It is straightforward to define P_{i+1}, Q_{i+1}, x_{i+1}, and y_{i+1} from C'. Since we add one more inner face in each stage, the construction is finished in τ stages. □

We note that every H_i, $1 \leqslant i \leqslant \tau$, is a 2-connected plane graph in the proof of Lemma 1.

Theorem 2. *Let G be a 2-connected plane graph. Then $c(G)$ is edge-pancyclic.*

Proof. Let $e = uv$ be an arbitrary edge of $c(G)$.

Case 1: Both vertices $u, v \in V(G)$.

Without loss of generality, we may assume that uv lies on the common boundary of f_1 and $f_{\mathrm{out}}(G)$. We first decompose G into the form $P_0 \cup P_1 \cup \cdots \cup P_\tau$ guaranteed by Lemma 1 so that P_0 is the edge uv. As we add the P_i's, the inner faces of G can be simultaneously numbered as $f_1, f_2, \ldots, f_\tau$. Again let $H_i = P_0 \cup P_1 \cup \cdots \cup P_i$ for $0 \leqslant i \leqslant \tau$.

We proceed by induction on τ. Actually, in each induction stage we construct the cycles in a systematic way such that, when the next path P_i is added, a certain property (★) is preserved.

When $\tau = 1$, G is the cycle $u_1 u_2 \cdots u_s u_1$, where $u = u_1$ and $v = u_s$, moving along the clockwise direction. Since G is the common boundary of f_1 and f_{out}, we have $V(c(G)) = V(G) \cup \{f_1, f_{\mathrm{out}}\}$ and $E(c(G)) = E(C) \cup \{f_1 f_{\mathrm{out}}\} \cup \{u_i f_1, u_i f_{\mathrm{out}} \mid i = 1, 2, \ldots, s\}$. In $c(G)$, we construct a particular sequence of cycles C_n of length n, $3 \leqslant n \leqslant s+2$, each of which contains the edge uv. Let $C_3 = u f_{\mathrm{out}} v u$, $C_4 = u f_1 f_{\mathrm{out}} v u$, $C_5 = u u_2 f_1 f_{\mathrm{out}} v u, \ldots, C_{s+2} = u u_2 \cdots u_{s-1} f_1 f_{\mathrm{out}} v u$. Note that each C_{i+1}, $i \geqslant 4$, is obtained from C_i by inserting a new vertex of $c(G)$ prior to a fixed vertex of C_i. We call this type of construction a monotone expansion of cycles.

Now rename C_{s+2} temporarily as $z_1 z_2 \cdots z_{s+2} z_1$, where $z_1 = u$ and $z_{s+2} = v$. When the path $P_2 = p_1 p_2 \cdots p_q$ is added along the clockwise direction to H_1, we may assume that $p_1 = z_i$ and $p_q = z_j$ for some $i < j$. Then the following property (★) holds: z_{i+1} is either a vertex of H_1 that is incident to f_2 in H_2 or a face of H_1 that is adjacent to f_2 in H_2.

Assume that the theorem holds for $\tau = k \geqslant 1$. Let G be a 2-connected plane graph with $k+1$ inner faces. We decompose G into the form $P_0 \cup \cdots \cup P_k \cup P_{k+1}$ as in Lemma 1. By the induction hypothesis, $c(H_k) = c(P_0 \cup P_1 \cup \cdots \cup P_k)$ is edge-pancyclic through uv. We further assume that all the cycles in $c(H_k)$ through uv are constructed by inductive stages and, within each stage, by a monotone expansion of cycles. For $m = |H_k| + |F(H_k)| = |H_k| + k + 1$, let $C_m = z_1 z_2 \cdots z_m z_1$, where $z_1 = u$ and $z_m = v$, be the Hamiltonian cycle so constructed. Suppose that $P_{k+1} = v_1 v_2 \cdots v_t$, where $v_1 = z_i$ and $v_t = z_j$ for some $i < j$. Assume that P_{k+1} moves from v_1 to v_t in the clockwise direction along the boundary of the outer face of H_{k+1}. By our assumption, f_{k+1} is the inner face of G formed by H_k and P_{k+1}. Now the property (★) holds by the induction hypothesis, i.e., z_{i+1} is either a vertex of H_k that is incident to f_{k+1} in H_{k+1} or a face of H_k that is adjacent to f_{k+1} in H_{k+1}.

In $c(G)$, a monotone expansion of cycles C_l of length l, $m+1 \leqslant l \leqslant |c(G)|$, each of which contains uv can be constructed as follows:

$$
\begin{aligned}
&C_{m+1} = z_1 \cdots z_i f_{k+1} z_{i+1} \cdots z_m z_1,\\
&C_{m+2} = z_1 \cdots z_i v_2 f_{k+1} z_{i+1} \cdots z_m z_1,\\
&\cdots\\
&C_{|c(G)|} = z_1 \cdots z_i v_2 \cdots v_{t-1} f_{k+1} z_{i+1} \cdots z_m z_1.
\end{aligned}
$$

Note that the path P_{k+2} will be added in the clockwise direction along the boundary of the outer face of H_{k+1}, and $C_{|c(G)|}$ is obtained from C_m by inserting a consecutive segment $v_2 \cdots v_{t-1} f_{k+1}$. If the initial end of P_{k+2} does not belong to $\{z_i, v_2, \ldots, v_{t-1}\}$, then the property (★) holds by induction. However, it is easy to see that the property (★) is preserved if the initial end of P_{k+2} belongs to $\{z_i, v_2, \ldots, v_{t-1}\}$.

Case 2: At least one vertex $u \in F(G)$.

If $v \in F(G)$, we suppose that $u = f_1$ and $v = f_{\text{out}}$. If $v \in V(G)$, we let $u = f_{\text{out}}$ and $v = u_s$, where $u_s \in b(f_1) \cap b(f_{\text{out}}(G))$, as defined in Case 1. We let $C_3 = u_s f_1 f_{\text{out}} u_s$. For $4 \leqslant n \leqslant |c(G)|$, we may take the same cycles C_n as in Case 1 since each C_n always contains both the edge $f_1 f_{\text{out}}$ and the edge $f_{\text{out}} u_s$. □

Once the edge-disjoint decomposition into paths is given, the inductive proof of Theorem 2 actually supplies a polynomial-time algorithm for finding a Hamiltonian cycle in the coupled graph of a 2-connected plane graph. The next theorem provides examples to show that Theorem 2 is best possible in the sense that there exists a 2-edge-connected plane graph G such that $c(G)$ is not Hamiltonian.

The *block graph* $B(G)$ of a graph G is the graph whose vertices are the blocks of G and two vertices in $B(G)$ are adjacent if and only if the corresponding blocks of G share a common vertex. Note that two blocks of G can share at most one vertex. Suppose that x is a cut vertex of G. Let the components of $G-x$ have vertex sets $V_1, V_2, \ldots, V_n$. Then the induced subgraphs $G[V_i \cup \{x\}]$, $i=1,2,\ldots,n$, are called the *x-components* of G. For $S \subseteq V(G)$, let $\omega(G-S)$ denote the number of components of the graph $G-S$.

Theorem 3. *Let G be a plane graph. If $B(G)$ contains a vertex of degree at least 3, then $c(G)$ is not Hamiltonian.*

Proof. Let B_0 be a block of G having degree $m \geqslant 3$ in $B(G)$. Let $B_1, B_2, \ldots, B_m$ be the blocks of G that are neighbors of B_0 in $B(G)$. There are vertices x_i in G for all $i \in [m] = \{1,2,\ldots,m\}$ such that $V(B_0) \cap V(B_i) = \{x_i\}$. Each x_i is a cut vertex of G as well as a cut vertex of $c(G) - f_{\text{out}}$. Moreover, $(V(B_i) \cap V(B_j)) \setminus V(B_0) = \emptyset$ for all $i,j \in [m]$ and $i \neq j$. We have the following two cases.

Case 1: There exist $i,j \in [m]$ such that $i \neq j$ and $x_i = x_j$.

Let $S = \{x_i, f_{\text{out}}\}$. Clearly S is a cut set of $c(G)$. Since $m \geqslant 3$, the number of x_i-components of G is at least 3. It follows that $\omega(c(G)-S) \geqslant 3 > 2 = |S|$. Hence $c(G)$ violates the necessary condition to be Hamiltonian.

Case 2: The vertices $x_1, x_2, \ldots, x_m$ are all distinct.

Note that G has exactly two x_i-components for each $i \in [m]$. One of the x_i-components, called G_i, is a supergraph of B_i. The plane drawing of G induces a natural plane embedding of G_i and we may assume $f_{\text{out}}(G_i) = f_{\text{out}}(G) = f_{\text{out}}$. Since $|G_i| \geqslant 2$, the vertex set of $c(G_i) - \{x_i, f_{\text{out}}\}$ is nonempty.

Suppose that $c(G)$ has a Hamiltonian cycle. Then $c(G) - f_{\text{out}}$ has a Hamiltonian path $P = z_1 z_2 \cdots z_t$, where $t = |c(G)| - 1$. Since $m \geqslant 3$, we may pick three vertices z_i, z_j, and z_k such that $1 \leqslant i < j < k \leqslant t$ and $z_i \in c(G_p) - \{x_p, f_{\text{out}}\}$, $z_j \in c(G_q) - \{x_q, f_{\text{out}}\}$, and $z_k \in c(G_r) - \{x_r, f_{\text{out}}\}$ for distinct p, q, and r in $[m]$. Since x_q is a cut vertex of $c(G) - f_{\text{out}}$, the path P has to traverse x_q twice to include z_j. This contradicts the definition of P. It follows that $c(G)$ is not Hamiltonian. □

It is easy to construct infinitely many 2-edge-connected plane graphs that satisfy the assumption of Theorem 3. We conclude this paper by posing the following problem.

Problem 4. *Let G be a 2-edge-connected plane graph. Is its coupled graph $c(G)$ edge-pancyclic when its block graph $B(G)$ is a path?*

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