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(d,m)-DOMINATING NUMBERS OF HYPERCUBE

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Abstract. This paper shows that the (d,m)-dominating number of the m-dimensional hypercube $Q_m(m\geqslant4)$ is 2 for any integer d. $\left(\left\lfloor\frac{m}{2}\right\rfloor+2\leqslant d\leqslant m\right)$.

§ 1 Introduction

In this paper we use graphs to represent networks. We quote from [1] the terminology and notations not defined here. In addition, the length of a path $P:=v_1\rightarrow v_2\rightarrow v_3\rightarrow\dots\rightarrow v_p\rightarrow v_{p+1}$ is the number p of edges of P and will be denoted by $|P|$, where v_1 and v_{p+1} are called end-vertices of P and $v_2,v_3,\dots,v_p$ internal vertices. For a nonempty and proper subset S of the vertex set $V(G)$ and $x\in V(G-S)$, an (x,S)-path is a path in G connecting x to some vertex in S.

The m-dimensional hypercube Q_m has 2^m vertices which are labeled with the binary strings of length m. There is an edge between $x_1x_2\dots x_m$ and $y_1y_2\dots y_m$ if and only if $\sum_{i=1}^{m}|x_i-y_i|=1$. For any vertex $x=x_1x_2\dots x_m$, we say that the ith coordinate of x is x_i, being equal to 0 or 1, and $\bar{x}_i=1-x_i$. It is well known that Q_m is m-connected and its diameter is equal to m. Hypercube Q_m is widely used in network theory.

In order to characterize the reliability of transmission delay in a network, Flandrin and Li[2], Hsu and Lyuu[3] independently introduced m-diameter (i. e. wide-diameter) as follows. For any pair (x,y) of vertices in a graph G, the minimum integer d such that there are at least m internally vertex-disjoint paths of length at most d between x and y is called the m-distance of x and y and is denoted by $D_m(x,y)_G$. The m-diameter of G, denoted by $D_m(G)$, is the maximum of $D_m(x,y)_G$ over all pairs (x,y) of vertices of G. General results on the m-diameter of m-connected graphs can be found in [2~4] and results for some particular classes of graphs in [5~7]. In particular, for Q_m, its m-diameter is $m+1$. (see [3]).

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Recently, Li and Xu in [8] defined a new parameter *(d,m)-dominating number* in m-connected graphs, in some sense, which can more accurately characterize the reliability of networks than the wide-diameter does.

Definition. Let G be a m-connected graph, S a nonempty and proper subset of $V(G)$, y a vertex in $G-S$. For a given positive integer d, y is (d,m)-dominated by S in the graph if there are at least m internally vertex-disjoint (y,S)-paths in G such that each of which is of length at most d. S is said to be a (d,m)-dominating set of G, denoted by $S_{d,m}(G)$ if either $S=V(G)$ or S can (d,m)-dominate every vertex in $G-S$. The parameter

$$s_{d,m}(G) = \min\{|S_{d,m}(G)| : S_{d,m}(G) \text{ is a } (d,m)\text{-dominating set of } G\}$$

will be called the (d,m)-dominating number of G.

[8] discovered some general properties of the (d,m)-dominating set and the (d,m)-dominating numbers of m-connected graphs. In particular, [8] proved that for any $m\geqslant 2$, the (d,m)-dominating number $(m-1\leqslant d\leqslant m)$ of the m-dimensional hypercube Q_m is 2.

In this paper, we will prove that for $m\geqslant 4$, the (d,m)-dominating number of Q_m is also 2 for any integer d, $\left(\left\lfloor \frac{m}{2}\right\rfloor+2\leqslant d\leqslant m\right)$. So, the result shown above in [8] follows as a corollary when $m\geqslant 5$.

§ 2 Main Results

Theorem. The (d,m)-dominating number of $Q_m (m\geqslant 4)$ is 2 for any integer d with $\left\lfloor \frac{m}{2}\right\rfloor+2\leqslant d\leqslant m$.

In order to prove the theorem we first give two lemmas.

Lemma 1. Let G be an m-connected $(m\geqslant 2)$ graph of order n and d a positive integer,

(a) if $d=D_m(G)$, then $s_{d,m}(G)=1$;

(b) if $d'>d''$, then $s_{d',m}(G)\leqslant s_{d'',m}(G)$.

Lemma 1 can be obtained directly by the definitions.

Lemma 2. For m-dimensional hypercube Q_m, $(m\geqslant 2)$, $s_{m+1,m}(Q_m)=1$ and $s_{d,m}(Q_m)\geqslant 2$ for any positive integer $d<m+1$.

Proof. Since m-diameter of Q_m is $m+1$ and Q_m is vertex transitive, it is easy to prove $s_{m+1,m}(Q_m)=1$ and $s_{d,m}(Q_m)\geqslant 2$ for $d<m+1$ by Lemma 1.

Proof of Theorem. We say $z=x+y$ if $z_i=x_i+y_i$ for $i=1,2,\ldots,m$ (here Boolean addition is used). Let $S=\{u,v\}$ with $d(u,v)=m$. Note that for any $w\in V(G-S)$ with $d(u,w)=k$, then $d(v,w)=m-k$ $(1\leqslant k\leqslant m-1)$.

Without loss of generality, we suppose that

$$u = x_1x_2\ldots x_m, v = \bar{x}_1\bar{x}_2\ldots\bar{x}_m$$

and

$$w = w' + w'' = 0\ldots 0\bar{x}_{t_1}0\ldots 0\bar{x}_{t_2}0\ldots 0\bar{x}_{t_k}0\ldots 0 + 0\ldots 0x_{t_{k+1}}0\ldots 0x_{t_{k+2}}0\ldots 0x_{t_m}0\ldots 0$$

such that

$$\{t_1,t_2,\ldots,t_k,t_{k+1},\ldots,t_m\}=\{1,2,\ldots,m\}.$$

And let $w^{t_1t_2\cdots t_g}$ denote

$$w_1\ldots w_{i_1-1}\overline{w}_{i_1}w_{i_1+1}\ldots w_{i_2-1}\overline{w}_{i_2}w_{i_2+1}\ldots w_{i_g-1}\overline{w}_{i_g}w_{i_g+1}\ldots w_m.$$

Case 1. $k\leqslant\left\lfloor\frac{m}{2}\right\rfloor$.

We claim that there exist m-internally vertex-disjoint paths between u and w, each of which is of length at most $k+2$.

$P_1: w\rightarrow w^{t_1}\rightarrow w^{t_1t_2}\rightarrow\ldots\rightarrow w^{t_1t_2\cdots t_k}=u$

$P_2: w\rightarrow w^{t_2}\rightarrow w^{t_2t_3}\rightarrow\ldots\rightarrow w^{t_2t_3\cdots t_k}\rightarrow w^{t_2t_3\cdots t_kt_1}=u$

$P_3: w\rightarrow w^{t_3}\rightarrow w^{t_3t_4}\rightarrow\ldots\rightarrow w^{t_3t_4\cdots t_kt_1}\rightarrow w^{t_3t_4\cdots t_1t_2}=u$

$\vdots$ $\qquad\qquad\qquad$ $\vdots$

$P_k: w\rightarrow w^{t_k}\rightarrow w^{t_kt_1}\rightarrow\ldots\rightarrow w^{t_kt_1\cdots t_{k-1}}=u$

$P_{k+1}: w\rightarrow w^{t_{k+1}}\rightarrow w^{t_{k+1}t_1}\rightarrow w^{t_{k+1}t_1t_2}\rightarrow\ldots\rightarrow w^{t_{k+1}t_1\cdots t_k}=$

$\quad x_1x_2\ldots x_{t_{k+1}-1}\overline{x}_{t_{k+1}}x_{t_{k+1}+1}\ldots x_m\rightarrow u$

$P_{k+2}: w\rightarrow w^{t_{k+2}}\rightarrow w^{t_{k+2}t_1}\rightarrow w^{t_{k+2}t_1t_2}\rightarrow\ldots\rightarrow w^{t_{k+2}t_1\cdots t_k}=$

$\quad x_1x_2\ldots x_{t_{k+2}-1}\overline{x}_{t_{k+2}}x_{t_{k+2}+1}\ldots x_m\rightarrow u$

$\vdots$ $\qquad\qquad\qquad$ $\vdots$

$P_m: w\rightarrow w^{t_m}\rightarrow w^{t_mt_1}\rightarrow w^{t_mt_1t_2}\rightarrow\ldots\rightarrow w^{t_mt_1\cdots t_k}=$

$\quad x_1x_2\ldots x_{t_m-1}\overline{x}_{t_m}x_{t_m+1}\ldots x_m\rightarrow u.$

We easily know the lengths of $P_1,P_2,\ldots,P_k$ are k and the lengths of P_{k+1},P_{k+2},P_m are $k+2$. It is obvious that $P_1,P_2,\ldots,P_m$ are internally vertex-disjoint.

Case 2. $k\geqslant\left\lfloor\frac{m}{2}\right\rfloor+1$, i.e. $m-k\leqslant\left\lfloor\frac{m}{2}\right\rfloor$.

We consider w and v as above.

Thus $s_{\left\lfloor\frac{m}{2}\right\rfloor+2,m}(Q_m)\leqslant 2$.

Since $\left\lfloor\frac{m}{2}\right\rfloor+2<m+1$ if $m\geqslant 4$, $s_{\left\lfloor\frac{m}{2}\right\rfloor+2,m}(Q_m)=2$ by Lemma 2. And then, by Lemma 1, $s_{d,m}(Q_m)=2$ when $\left\lfloor\frac{m}{2}\right\rfloor+2\leqslant d\leqslant m$.

The proof of Theorem is complete.

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