THE RAMSEY NUMBERS $R(T_n, W_6)$ FOR T_n WITHOUT CERTAIN DELETABLE SETS*

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Abstract. Let T_n denote a tree of order n and W_m a wheel of order $m+1$. In this paper, we determine the Ramsey numbers $R(T_n, W_6)$ for T_n without certain deletable sets.

Key words. Ramsey number, tree, wheel.

1 Introduction

All graphs considered in this paper are finite simple graph without loops. For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest positive integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. Let G be a graph. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The independence number of G is denoted by $\alpha(G)$. The neighborhood $N(v)$ of a vertex v is the set of vertices adjacent to v in G. For a vertex $v \in V(G)$ and a subgraph H of G, $N_H(v)$ is the set of neighbors of v contained in H, i.e., $N_H(v) = N(v) \cap V(H)$; and if $U \subseteq V(G)$, then $N_H(U) = \bigcup_{u \in U} N_H(u)$. We let $d_H(v) = |N_H(v)|$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. Let m be a positive integer. We use mG to denote m vertex disjoint copies of G. A path and a cycle of order n are denoted by P_n and C_n respectively. A Star S_n $(n \geq 3)$ is a bipartite graph $K_{1,n-1}$. A complete graph of order n is denoted by K_n. A Wheel $W_n = K_1 + C_n$ is a graph of $n+1$ vertices, where K_1 called the hub of the wheel. $S_n(l, m)$ is a tree of order n obtained from $S_{n-l\times m}$ by subdividing each of l chosen edges m times. $S_n(l)$ is a tree of order n obtained from an S_l and an S_{n-l} by adding an edge joining the centers of them. $S_n[l]$ is a tree of order n obtained from an S_l and an S_{n-l} by adding an edge joining a vertex of degree one of S_l to the center of S_{n-l}. Define

$$\mathcal{T} = \{S_n \mid n \geq 5\} \cup \{S_n(1,1) \mid n \geq 5\} \cup \{S_n(1,2) \mid n \geq 6 \text{ and } n \equiv 0 \pmod 3\}.$$

For a tree T, we define $L(T) = \{v \mid v \in V(T) \text{ and } d(v) = 1\}$. Let $V \subseteq L(T)$ and $|V| = k$. Write $T_V = T - V$. If $T_V \notin \mathcal{T}$, we call V a k-deletable set. If $k = 2$ and $|N(V)| = 2$, we call V a II-set. If $k = 3$ and $|N(V)| = 3$, we call V a III-set. If $k = 3$ and $|N(V)| = 2$, we call V a IV-set. If V is a II-set and $T_V \notin \mathcal{T}$, we call V a II-deletable set. Similarly, we can define III-deletable set and IV-deletable set. A graph on n vertices is pancyclic if it contains cycles of every length l, $3 \leq l \leq n$.

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In [1], Baskoro et al. obtain the following

Theorem 1[1] *Let T_n be a tree of order n other than S_n. Then $R(T_n, W_4) = 2n - 1$ for $n \geq 3$; $R(T_n, W_5) = 3n - 2$ for $n \geq 4$.*

Motivated by Theorem 1, Baskoro et al.[1] pose the following

Conjecture 1 Let T_n be a tree other than S_n and $n \geq m - 1$. Then $R(T_n, W_m) = 2n - 1$ for $m \geq 6$ even; $R(T_n, W_m) = 3n - 2$ for $m \geq 7$ and odd.

In [2] we show Conjecture 1 holds for $T_n = P_n$.

Theorem 2 *$R(P_n, W_m) = 3n - 2$ for m odd and $n \geq m - 1 \geq 2$; $R(P_n, W_m) = 2n - 1$ for m even and $n \geq m - 1 \geq 3$.*

In [3], we consider $R(T_n, W_6)$ for $\Delta(T_n) \geq n - 3$ and establish the following

Theorem 3[3] *$R(S_n(1, 1), W_6) = 2n$ for $n \geq 4$.*

Theorem 4[3] *$R(S_n(1, 2), W_6) = 2n$ for $n \geq 6$ and $n \equiv 0 \ (mod\ 3)$.*

Theorem 5[3] *$R(S_n(3), W_6) = R(S_n(2, 1), W_6) = 2n-1$ for $n \geq 6$; $R(S_n(1, 2), W_6) = 2n-1$ for $n \geq 6$ and $n \not\equiv 0 \ (mod\ 3)$.*

By Theorems 3 and 4, we can see that Conjecture 1 is not true when m is even. However, we believe that $R(T_n, W_6) = 2n - 1$ for $T_n \notin \mathcal{T}$ and $n \geq 5$. In [4] we show this is true for $n \leq 8$. In order to determine $R(T_n, W_6)$ for a general tree T_n, we need to use induction on n. However, if you delete some vertices of degree one from T_n, the resulting tree maybe belongs to $\mathcal{T}$ and induction does not work in this case. So it is necessary to consider trees with this property. Before giving the main result of this paper, we first define several classes of special trees. Let

$$\mathcal{T}_1 = \{T_{9b}, T_{9c}, T_{9d}, T_{9e}, T_{9f}, T_{9g}, T_{9h}, T_{9i}\},$$

$$\mathcal{T}_2 = \{T_a, T_b, T_c, T_d, T_e, T_f, T_g\},$$

$$\mathcal{S} = \{S_n(3), S_n(2, 1), S_n(1, 2) | n \geq 6\} \cup \{S_n[4], S_n(1, 3) | n \geq 8\} \cup \{S_9[5], S_9(4, 1), T_{9a}\},$$

$$\mathcal{S}' = \{S_9(3, 1), S_{10}[5], S_{10}(4, 1), S_{11}(5, 1)\} \cup \mathcal{T}_1 \cup \mathcal{T}_2,$$

where, $T_{9a}, T_{9b}, T_{9c}, T_{9d}, T_{9e}, T_{9f}, T_{9g}, T_{9h}, T_{9i}$ denote the nine trees of order 9, and T_a, T_b, T_c, T_d, T_e, T_f, T_g the seven trees of order 10, respectively, as shown in Figure 1.

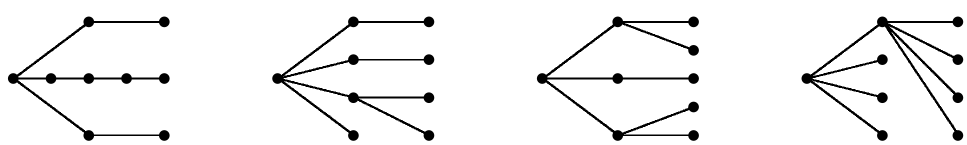
In this paper, we first give a characterization of trees without II-deletable set or III-deletable set and IV-deletable set or 4-deletable set, and then we determine $R(T_n, W_6)$ for T_n with this property. The main result of this paper is the following

Theorem 6 *Let $T \notin \mathcal{T}$ be a tree of order $n \geq 9$. If*

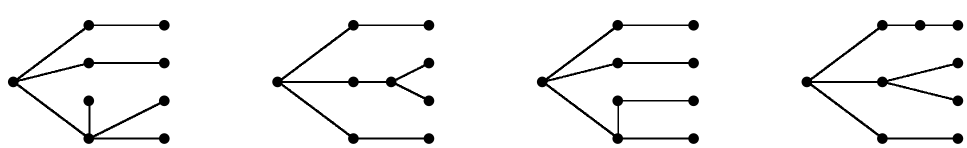
(1) T contains no II-deletable set, or

(2) $|L(T)| \geq 3$ and T contains neither III-deletable set nor IV-deletable set, or

(3) $|L(T)| \geq 4$ and T contains no 4-deletable set, then $R(T, W_6) = 2n - 1$.



T_{9a} T_{9b} T_{9c} T_{9d}



T_{9e} T_{9f} T_{9g} T_{9h}

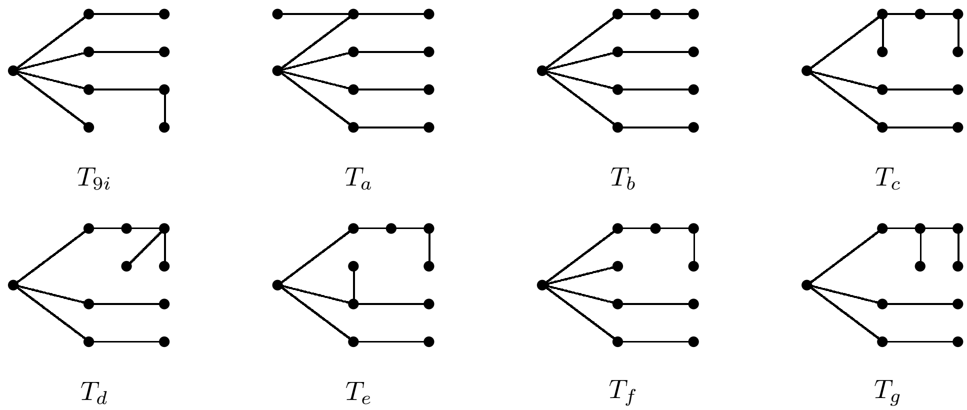


Figure 1

2 Deletable Sets in Trees

In this section, we will characterize trees without II-deletable set or III-deletable set and IV-deletable set or 4-deletable set.

Proposition 1 *Let $T \notin \mathcal{T}$ be a tree of order $n \geq 9$. If T contains no II-deletable set, then $T \in \{S_n(2,1), S_n(3), S_n(1,2), S_n[4], S_n(1,3)\}$.*

Proof Since $T \notin \mathcal{T}$, T contains a II-set $U = \{u_1, u_2\}$. Suppose $T_U \in \mathcal{T}$. If $T_U = S_{n-2}$, then it is easy to see $T = S_n(2,1)$. If $T_U = S_{n-2}(1,1)$, we let $V(T_U) = \{v_0, v_1, \cdots, v_{n-4}, w_1\}$ and $E(T_U) = \{v_0v_i \mid 1 \leq i \leq n-4\} \cup \{v_1w_1\}$. If $v_0 \in N(U)$, we let $v_0 \in N(u_1)$. By symmetry, we may assume $N(u_2) \subseteq \{v_2, v_1, w_1\}$. If $v_2 \in N(u_2)$, then $T = S_n(2,1)$. If $v_1 \in N(u_2)$, then $T = S_n(3)$. If $w_1 \in N(u_2)$, then $T = S_n(1,2)$. If $v_0 \notin N(U)$, then by symmetry, we need to consider the following four cases: (1) $v_2 \in N(u_1)$ and $v_3 \in N(u_2)$, (2) $v_2 \in N(u_1)$ and $v_1 \in N(u_2)$, (3) $v_2 \in N(u_1)$ and $w_1 \in N(u_2)$, (4) $v_1 \in N(u_1)$ and $w_1 \in N(u_2)$. Thus, taking $V = \{u_2, v_4\}$ in all cases, we have $T_V \notin \mathcal{T}$. If $T_U = S_{n-2}(1,2)$, we let $T_U = \{v_0, v_1, \cdots, v_{n-5}, w_1, w_2\}$ and $E(T_U) = \{v_0v_i \mid 1 \leq i \leq n-5\} \cup \{v_1w_1, w_1w_2\}$. If $v_0 \in N(U)$, we let $v_0 \in N(u_1)$. By symmetry, we assume $N(u_2) \subseteq \{v_2, v_1, w_1, w_2\}$. If $N(u_2) \subseteq \{v_2, v_1\}$, then taking $V = \{u_1, w_2\}$, we have $T_V \notin \mathcal{T}$. If $w_1 \in N(u_2)$, then $T = S_n[4]$. If $w_2 \in N(u_2)$, then $T = S_n(1,3)$. If $v_0 \notin N(U)$, then by symmetry, we need to consider the following seven cases: (1) $v_2u_1, v_3u_2 \in E(T)$, (2) $v_2u_1, v_1u_2 \in E(T)$, (3) $v_2u_1, w_1u_2 \in E(T)$, (4) $v_2u_1, w_2u_2 \in E(T)$, (5) $v_1u_2, w_1u_1 \in E(T)$, (6) $v_1u_2, w_2u_1 \in E(T)$, (7) $w_1u_2, w_2u_1 \in E(T)$. Thus, taking $V = \{u_2, v_4\}$ in each case, we have $T_V \notin \mathcal{T}$. ▮

Using the same method, we can prove the following two propositions. Since the proofs are easy but tedious, we leave them to the readers.

Proposition 2 *Let $T \notin \mathcal{T}$ be a tree of order $n \geq 9$ and $|L(T)| \geq 3$. If T contains neither III-deletable set nor IV-deletable set, then $T \in \mathcal{S}$.*

Proposition 3 *Let $T \notin \mathcal{S} \cup \mathcal{T}$ be a tree of order $n \geq 9$ and $|L(T)| \geq 4$. If T contains no 4-deletable set, then $T \in \mathcal{S}'$.*

3 Some Lemmas

In order to prove Theorem 6, we need the following lemmas.

Lemma 1[5] *Let G be a graph of order n. If $\delta(G) \geq n/2$, then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.*

Lemma 2[3] *Let G be a graph of order $2n-1 \geq 7$ and (U, V) a partition of $V(G)$ with $|U| \geq 3$ and $|V| \geq 4$. Suppose $u_i \in U$ and $N_V(u_i) = \emptyset$, $1 \leq i \leq 3$. If $\overline{G}$ contains no W_6, then $\delta(G[V]) \geq |V| - 3$.*

Lemma 3[4] *$R(T, W_6) = 2n-1$ for $T = S_n[4], S_n(1,3), S_n(3,1)$ and $n \geq 8$.*

Lemma 4[4] *Let G be a graph of order 7 and $\delta(G) \geq 4$. Then for any $v \in V(G)$, G contains a tree $T = S_7(3,1)$ such that $d_T(v) = 3$.*

Lemma 5[4] *Let $T_n \notin \mathcal{T}$ be a tree of order n and $5 \leq n \leq 8$. Then $R(T_n, W_6) = 2n-1$.*

Lemma 6[6] *$R(S_n, W_6) = 2n+1$ for $n \geq 3$.*

Lemma 7 *$R(S_n[5], W_6) = 2n-1$ for $n \geq 9$.*

Proof Let G be a graph of order $2n-1$. If $\overline{G}$ contains no W_6, then G contains an $S_n[4]$ by Lemma 3. Let T be an $S_n[4]$ with $V(T) = V = \{v_0, v_1, \cdots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq n-4\} \cup \{v_1w_1, w_1w_2, w_1w_3\}$. Set $U = V(G) - V$. Obviously, $|U| = n-1 \geq 8$.

If G contains no $S_n[5]$, then we have $w_1v_i \notin E(G)$ for $2 \leq i \leq n-4$ and $N_U(w_1) = \emptyset$. For any $u \in U$, if $d_U(u) \geq 3$, then $N(u) \cap N_T(v_0) = \emptyset$. Thus if U contains three vertices u_1, u_2, u_3 such that $d_U(u_i) \geq 3$ for $1 \leq i \leq 3$, then $\overline{G}[w_1, v_2, v_3, v_4, u_1, u_2, u_3]$ contains a W_6 with the hub w_1. Hence we may assume U contains at most two vertices, say u_1, u_2 such that $d_U(u_i) \geq 3$, $i = 1, 2$. Thus, noting that $|U| \geq 8$, U contains a subset U' with $|U'| = 6$ such that $d_{U'}(u) \leq 2$ for each $u \in U'$ which implies $\overline{G}[U']$ contains a C_6 by Lemma 1, and hence $\overline{G}$ contains a W_6 with the hub w_1, a contradiction. Thus we have $R(S_n[5], W_6) \leq 2n-1$. On the other hand, the graph $G = 2K_{n-1}$ shows $R(S_n[5], W_6) \geq 2n-1$ and hence we have $R(S_n[5], W_6) = 2n-1$. ∎

Lemma 8 *$R(T, W_6) = 17$ for $T = S_9(4,1)$, T_{9a}.*

Proof Let G be a graph of order 17. Suppose $\overline{G}$ contains no W_6.

We first show G contains an $S_9(4,1)$. By Lemma 3, G contains an $S_9(3,1)$. Let $T = S_9(3,1)$, $V(T) = V = \{v_0, \cdots, v_5, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 5\} \cup \{v_1w_1, v_2w_2, v_3w_3\}$. Set $U = V(G) - V$. Obviously, $|U| = 8$. If G contains no $S_9(4,1)$, then $v_4v_5 \notin E(G)$, $N_U(v_i) = \emptyset$ for $i = 4, 5$ and if $u \in N_U(v_0)$, then $d_U(u) = 0$. Thus if $d_U(v_0) \geq 2$, say $u_1, u_2 \in N_U(v_0)$, then it is not difficult to see that $\overline{G}[v_4, v_5, u_1, u_2, u_3, u_4, u_5]$ contains a W_6 with the hub v_4 for any three vertices $u_3, u_4, u_5 \in U - \{u_1, u_2\}$, a contradiction. Hence we have $d_U(v_0) \leq 1$. Let $U' \subseteq U - N_U(v_0)$ and $|U'| = 7$. By Lemma 2, we have $\delta(G[U']) \geq 4$ and then $N_T(u) = \emptyset$ for any $u \in U'$ by Lemma 4. Thus we have $\delta(G[V]) \geq 6$ by Lemma 2. Noting that $v_4v_5 \notin E(G)$, after an easy check, we can see $G[V]$ contains an $S_9(4,1)$, and hence we have $R(S_9(4,1), W_6) \leq 17$.

Next, we show G contains a T_{9a}. Let T be an $S_9(4,1)$ with $V(T) = V \cup W$, where $V = \{v_i \mid 0 \leq i \leq 4\}$ and $W = \{w_i \mid 1 \leq i \leq 4\}$, and $E(T) = \{v_0v_i \mid 1 \leq i \leq 4\} \cup \{v_iw_i \mid 1 \leq i \leq 4\}$. Set $U = V(G) - V$. Obviously, $|U| = 8$. If G contains no T_{9a}, then W is an independent set. Since $|U| = 8$, by Lemma 6, $G[U]$ contains a star S_3. Assume $u_1, u_2, u_3 \in U$ and $u_1u_2, u_2u_3 \in E(G)$. Since G contains no T_{9a}, we have $N(w_i) \cap \{u_1, u_2, u_3\} = \emptyset$ for $1 \leq i \leq 4$. Thus $\overline{G}[w_1, w_2, w_3, w_4, u_1, u_2, u_3]$ contains a W_6 with the hub w_1, a contradiction. Thus we have $R(T_{9a}, W_6) \leq 17$.

Since $2K_8$ contains no trees of order 9 and its complement contains no W_6, we have $R(T, W_6) \geq 17$, and hence $R(T, W_6) = 17$, for $T = S_9(4,1)$, T_{9a}. ∎

Lemma 9 *Let G be a graph of order n. If $\alpha(G) = 2$ and $\delta(G) \geq n-3$, then for any maximum independent set $I = \{u_1, u_2\}$ and any two vertices $v_1, v_2 \in V(G) - I$, either $u_1v_1, u_2v_2 \in E(G)$ or $u_1v_2, u_2v_1 \in E(G)$.*

Proof Since $\delta(G) \geq n-3$, we have $N(u_1) \cap \{v_1, v_2\} \neq \emptyset$ and $N(u_2) \cap \{v_1, v_2\} \neq \emptyset$. Assume $u_1v_1 \in E(G)$. If $u_2v_2 \in E(G)$, then we are done. Hence we have $u_2v_2 \notin E(G)$ which implies $u_2v_1 \in E(G)$. If $u_1v_2 \in E(G)$, then we are done, and hence $u_1v_2 \notin E(G)$ which implies $\alpha(G) \geq 3$, a contradiction. ∎

Lemma 10 *Let G be a graph of order $2n-1$. If $\alpha(G) \leq 2$, then G contains all trees of order n.*

Proof If $\alpha(G) = 1$, then it is trivial, and hence we may assume $\alpha(G) = 2$. We use induction on n. If $n = 3$, then it holds. Assume it holds for small values of n. Let $I = \{u, v\}$ be a maximum independent set of G and T any given tree of order n. Let $v_1v_0 \in E(T)$ with $d_T(v_0) = 1$ and $T' = T - v_0$. By induction hypothesis, $G - I$ contains T'. Since any vertex in $G - I$, especially v_1, must be adjacent to at least one of $\{u, v\}$ as v_0, G contains T. Thus G contains all trees of order n. ▮

4 Proof of Theorem 6

Proof of Theorem 6 Let G be a graph of order $2n-1$. Suppose $\overline{G}$ contains no W_6. Before starting to prove Theorem 6, we first show the following claims under the assumption $\alpha(G) = 3$.

- **Claim 1** *G contains an $S_n(4,1)$ for $n \geq 10$.*
  Proof By Lemma 3, we may assume that $T = S_n(3,1)$ is a tree in G. Let $V(T) = \{v_0, \cdots, v_{n-4}, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq n-4\} \cup \{v_iw_i \mid 1 \leq i \leq 3\}$. Set $U = V(G) - V(T)$. If G contains no $S_n(4,1)$, then $\{v_4, v_5, v_6\}$ is an independent set and $N_U(v_i) = \emptyset$ for $i = 4, 5, 6$ which implies $\alpha(G) \geq 4$, a contradiction.
- **Claim 2** *If $n = 11$, then G contains an $S_{11}(5,1)$.*
  Proof By Claim 1, G contains a tree $S_{11}(4,1)$. Let $T = S_{11}(4,1)$ with $V(T) = \{v_0, \cdots, v_6, w_1, \cdots, w_4\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 6\} \cup \{v_iw_i \mid 1 \leq i \leq 4\}$. Set $U = V(G) - V(T)$. If G contains no $S_{11}(5,1)$, then $v_5v_6 \notin E(G)$ and $N_U(v_i) = \emptyset$ for $i = 5, 6$. Thus, since $\alpha(G) = 3$, we have $G[U] = K_{10}$, and hence $d_U(v) = 0$ for any $v \in V(T)$, since otherwise G contains an $S_{11}(5,1)$. By Lemma 2, we have $\delta(G[V(T)]) \geq 8$ which implies there is some i with $1 \leq i \leq 4$ such that $v_i, w_i \in N(v_5) \cap N(v_6)$, and hence G contains an $S_{11}(5,1)$.
- **Claim 3** *If $n = 9$, then G contains all trees $T \in \mathcal{T}_1$.*
  Proof By Theorem 5, G contains an $S_9(3)$. Let $T = S_9(3)$, $V(T) = \{v_0, \cdots, v_6, w_1, w_2\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 6\} \cup \{v_1w_1, v_1w_2\}$. Set $V_0 = \{v_2, \cdots, v_6\}$ and $U = V(G) - V(T)$. Since $\alpha(G) = 3$, $G[V_0]$ contains at least two edges.
  If G contains no T_{9b} and we assume $v_2v_3 \in E(G)$, then $\{v_4, v_5, v_6\}$ is an independent set and $N_U(v_i) = \emptyset$ for $i = 4, 5, 6$ which implies $\alpha(G) \geq 4$, a contradiction. Hence G contains a T_{9b}.
  If $\Delta(G[V_0]) = 1$, then $G[V_0] = 2K_2 \cup K_1$. Assume $E(G[V_0]) = \{v_2v_3, v_4v_5\}$. If G contains no T_{9c}, then $N_U(v_i) = \emptyset$ for $2 \leq i \leq 5$. Since $\alpha(G) = 3$, we have $G[U \cup \{v_6\}] = K_9$ which implies G contains a T_{9c}, a contradiction. Hence we have $\Delta(G[V_0]) \geq 2$. Assume $v_2v_3, v_2v_4 \in E(G)$. If G contains no T_{9c}, then $v_5v_6 \notin E(G)$ and $N_U(v_i) = \emptyset$ for $i = 5, 6$. Since $\alpha(G) = 3$, we have $G[U] = K_8$, and hence $d_U(v) = 0$ for any $v \in V(T)$ which implies $\delta(G[V(T)]) \geq 6$ by Lemma 2 and $\{v_5, v_6\}$ is a maximum independent set of $G[V(T)]$. Thus by Lemma 9 we can assume $v_1 \in N(v_5)$ and $w_1, w_2 \in N(v_6)$ which implies G contains a T_{9c}.
  If G contains no T_{9d}, then $v_1v_i \notin E(G)$ for $2 \leq i \leq 6$, $N_U(v_1) = \emptyset$ and $d_U(v_i) \leq 2$ for $2 \leq i \leq 6$. Thus, since $|U| = 8$, we can choose three vertices $u_1, u_2, u_3 \in U$ such that there is at most one edge between $\{v_2, v_3, v_4\}$ and $\{u_1, u_2, u_3\}$, and hence $\overline{G}[v_1, v_2, v_3, v_4, u_1, u_2, u_3]$ contains a W_6 with the hub v_1, a contradiction. Thus G contains a T_{9d}.
  Let $T = T_{9d}$, $V(T) = \{v_0, \cdots, v_5, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 5\} \cup \{v_1w_1, v_1w_2, v_1w_3\}$. Set $U = V(G) - V(T)$. Since $\alpha(G) = 3$, $G[v_2, \cdots, v_5]$ contains at least one edge. Assume $v_2v_3 \in E(G)$. If G contains no T_{9e}, then $v_4v_5 \notin E(G)$ and $N_U(v_i) = \emptyset$ for $i = 4, 5$. Thus, since $\alpha(G) = 3$, we have $G[U] = K_8$. And then $d_U(v) = 0$ for any $v \in V(T)$ which implies $\alpha(G[V(T)]) = 2$. By Lemma 2, we have $\delta(G[V(T)]) \geq 6$. By Lemma 9, we can assume $v_4v_2, v_5v_3 \in E(G)$ and hence G contains a T_{9e}.
  By Lemma 3, G contains an $S_9[4]$. Let $T = S_9[4]$, $V(T) = \{v_0, \cdots, v_5, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 5\} \cup \{v_1w_1, w_1w_2, w_1w_3\}$. Set $V_1 = \{v_2, \cdots, v_5\}$ and $U =$

$V(G)-V(T)$. Since $\alpha(G)=3$, $G[V_1]$ contains at least one edge. Assume $v_2v_3 \in E(G)$. By an argument similar to that for the case when G contains a T_{9e}, we can see G contains a T_{9f}.

Let $T=T_{9e}$, $V(T)=\{v_0,\cdots,v_3,w_1,\cdots,w_5\}$ and $E(T)=\{v_0v_i \mid 1\le i\le 3\}\cup\{v_1w_1,v_1w_2,v_1w_3,v_2w_4,v_3w_5\}$. Set $U=V(G)-V(T)$. If G contains no T_{9g}, then $\{w_1,w_2,w_3\}$ is an independent set and $N_U(w_i)=\emptyset$ for $i=1,2,3$ which implies $\alpha(G)\ge 4$, a contradiction. Hence G contains a T_{9g}.

Let $T=T_{9c}$, $V(T)=\{v_0,\cdots,v_3,w_1,\cdots,w_5\}$ and $E(T)=\{v_0v_i \mid 1\le i\le 3\}\cup\{v_1w_1,v_1w_2,v_2w_3,v_3w_4,v_3w_5\}$. Set $U=V(G)-V(T)$. If G contains no T_{9h}, then $\{w_1,w_2,w_3\}$ is an independent set and $N_U(w_i)=\emptyset$ for $i=1,2,3$ which implies $\alpha(G)\ge 4$, a contradiction. Hence G contains a T_{9h}.

By Lemma 8, G contains an $S_9(4,1)$. Let $T=S_9(4,1)$. If G contains no T_{9i}, then $L(T)$ is an independent set which implies $\alpha(G)\ge 4$ and hence G contains a T_{9i}.

- **Claim 4** *If $n=10$, then G contains all trees $T\in\mathcal{T}_2$.*

Proof We first show G contains T_a. By Theorem 5, G contains a tree $T=S_{10}(3)$. Let $V(T)=\{v_0,\cdots,v_7,w_1,w_2\}$ and $E(T)=\{v_0v_i \mid 1\le i\le 7\}\cup\{v_1w_1,v_1w_2\}$. Set $V_0=\{v_2,\cdots,v_7\}$ and $U=V(G)-V(T)$. Since $\alpha(G)=3$, $G[V_0]$ contains at least two independent edges. Assume $v_2v_3,v_4v_5\in E(G)$. If G contains no T_a, then $v_6v_7\notin E(G)$ and $N_U(v_i)=\emptyset$ for $i=6,7$. Thus we have $G[U]=K_9$, and hence $d_U(v)=0$ for any $v\in V(T)$ which implies $\alpha(G[V(T)])=2$ and $\delta(G[V(T)])\ge 7$ by Lemma 2. By Lemma 9, we can assume $v_4v_6,v_5v_7\in E(G)$ which implies G contains a T_a.

Next, we show G contains T_b,T_c,T_d,T_e. Let $T=T_a$, $V(T)=\{v_0,\cdots,v_4,w_1,\cdots,w_5\}$ and $E(T)=\{v_0v_i \mid 1\le i\le 4\}\cup\{v_iw_i \mid 1\le i\le 4\}\cup\{v_4w_5\}$. Set $U=V(G)-V(T)$.

If G contains no T_b, then $\{w_3,w_4,w_5\}$ is an independent set and $N_U(w_i)=\emptyset$ for $i=3,4,5$ which implies $\alpha(G)\ge 4$, a contradiction. Thus, G contains a T_b.

If G contains no T_c, then $w_4v_i\notin E(G)$ for $i=1,2,3$. Since $|U|=9$, $G[U]$ contains S_4 by Lemma 6. Assume $u_0,u_1,u_2,u_3\in U$ and $u_0u_1,u_0u_2,u_0u_3\in E(G)$. Since G contains no T_c, we have $u_iv_j\notin E(G)$ for $i,j=1,2,3$ and $w_4u_i\notin E(G)$ for $i=1,2,3$. Thus $\overline{G}[w_4,v_1,v_2,v_3,u_1,u_2,u_3]$ contains a W_6 with the hub w_4, a contradiction. Hence G contains a T_c.

If G contains no T_d, then $v_4w_i\notin E(G)$ for $i=1,2,3$. Since $|U|=9$, $G[U]$ contains an S_4 by Lemma 6. Assume $u_0,u_1,u_2,u_3\in U$ and $u_0u_1,u_0u_2,u_0u_3\in E(G)$. Since G contains no T_d, we have $v_4u_i\notin E(G)$ for $i=1,2,3$. If there is some $w_iu_j\in E(G)$ with $i,j\in\{1,2,3\}$, say $w_1u_1\in E(G)$, then $N(u_1)\cap\{w_2,w_3,u_2,u_3\}=\emptyset$; otherwise G contains a T_d. Since $\alpha(G)=3$, $\{w_2,w_3,u_2,u_3\}$ is a clique which implies G contains a T_d. Hence $w_iu_j\notin E(G)$ for $i,j=1,2,3$ which implies $\overline{G}[w_4,w_1,w_2,w_3,u_1,u_2,u_3]$ contains a W_6 with the hub w_4, a contradiction. Thus G contains a T_d.

If G contains no T_e, then $\{w_1,w_2,w_3\}$ is an independent set. Since $\alpha(G)=3$ and $|U|=9$, $G[U]$ contains an edge u_1u_2. Noting that $\{w_1,w_2,w_3\}$ is an maximum independent set, we have $N(u_1)\cap\{w_1,w_2,w_3\}\ne\emptyset$ which implies G contains a T_e.

And then we show G contains T_f. By Lemma 3, G contains a tree $T=S_{10}(1,3)$. Let $V(T)=\{v_0,\cdots,v_6,w_1,w_2,w_3\}$ and $E(T)=\{v_0v_i \mid 1\le i\le 6\}\cup\{v_1w_1,w_1w_2,w_2w_3\}$. Set $V_0=\{v_2,\cdots,v_6\}$ and $U=V(G)-V(T)$. Since $\alpha(G)=3$, $G[V_0]$ contains at least one edge. Assume $v_2v_3\in E(G)$. If G contains no T_f, then $\{v_4,v_5,v_6\}$ is an independent set and $N_U(v_i)=\emptyset$ for $i=4,5,6$ which implies $\alpha(G)\ge 4$, a contradiction. Hence G contains a T_f.

Finally, we show G contains T_g. Let $T=T_b$, $V(T)=\{v_0,\cdots,v_4,w_1,\cdots,w_5\}$ and $E(T)=\{v_0v_i \mid 1\le i\le 4\}\cup\{v_iw_i \mid 1\le i\le 4\}\cup\{w_1w_5\}$. Set $U=V(G)-V(T)$. Since $|U|=9$, $G[U]$ contains P_5 by Theorem 2. Let $P=u_1\cdots u_5$ be a P_5 in $G[U]$. If G contains no T_g, then we have $w_1v_i\notin E(G)$, $w_1u_i\notin E(G)$ for $2\le i\le 4$ and $v_iu_j\notin E(G)$ for $2\le i,j\le 4$.

Thus, $\overline{G}[w_1, v_2, v_3, v_4, u_2, u_3, u_4]$ contains a W_6 with the hub w_1, a contradiction. Thus G contains a T_g.

We now begin to prove Theorem 6.

(1) By Proposition 1 we have $T \in \{S_n(2,1), S_n(3), S_n(1,2), S_n[4], S_n(1,3)\}$, and hence $R(T, W_6) = 2n - 1$ by Theorem 5 and Lemma 3.

(2) If T contains neither III-deletable set nor IV-deletable set, then $T \in \mathcal{S}$ by Proposition 2, and hence $R(T, W_6) = 2n - 1$ by Theorem 5 and Lemmas 3, 7, 8.

(3) Suppose T contains no 4-deletable set. If $T \in \mathcal{S}$, then $R(T, W_6) = 2n - 1$ by Theorem 5 and Lemmas 3, 7, 8. If $T \notin \mathcal{S}$, then $T \in \mathcal{S}'$ by Proposition 3. By Lemmas 3 and 7 we may assume $T \in \mathcal{S}' - \{S_9(3,1), S_{10}[5]\} = \{S_{10}(4,1), S_{11}(5,1)\} \cup \mathcal{T}_1 \cup \mathcal{T}_2$.

We now show $R(T, W_6) = 2|T| - 1$ for $T \in \{S_{10}(4,1), S_{11}(5,1)\} \cup \mathcal{T}_1 \cup \mathcal{T}_2$. Obviously, $2K_{|T|-1}$ shows $R(T, W_6) \geq 2|T| - 1$ for any tree T. In the following, we will prove G contains T.

Since $\overline{G}$ contains no W_6, we have $\alpha(G) \leq 6$. If $\alpha(G) \leq 2$, then G contains T by Lemma 10. If $\alpha(G) = 3$, then G contains T by Claims 1–4. Now, we assume $4 \leq \alpha(G) \leq 6$.

Let I be a maximum independent set of G. If $\alpha(G) = 4$, then since $T \notin \mathcal{S}$, by Proposition 1, T contains a II-deletable set U_0. If $n = 9, 10$, then $G - I$ contains T_{U_0} by Lemma 5. If $n = 11$, then $T = S_{11}(5,1)$ and for any II-deletable set U_0, $T_{U_0} = S_9(3,1)$, and hence $G - I$ contains T_{U_0} by Lemma 3. Let $N_T(U_0) = U$. If $|N_I(U)| \geq 2$, then G contains T. Thus we have $|N_I(U)| = 1$ which implies G contains an induced subgraph $3K_1 \cup K_3$ since otherwise $\alpha(G) \geq 5$. Let $G' = 3K_1 \cup K_3$ with $V(G') = W = \{w_i \mid 1 \leq i \leq 6\}$ and $E(G') = \{w_4w_5, w_4w_6, w_5w_6\}$. Since $T \notin \mathcal{S}$, by Proposition 2, T contains a 3-deletable set U_0. Let $N_T(U_0) = U$. By Lemma 5, $G - W$ contains T_{U_0}. If $d_W(u) \geq 3$ for each $u \in U$, then G contains T. Hence there is some vertex $u_0 \in U$ such that $d_W(u_0) \leq 2$. Since $\alpha(G) = 4$, we have $|N(u_0) \cap \{w_4, w_5, w_6\}| \leq 1$. Since $d_W(u_0) \leq 2$, we may assume $w_1 \notin N(u_0)$. Thus $\overline{G}[w_1, w_2, w_3, w_4, w_5, w_6, u_0]$ contains a W_6 with the hub w_1, a contradiction.

For $\alpha(G) = 5, 6$, since $T \notin \mathcal{S}$, by Proposition 2, T contains a 3-deletable set U_0. Let $N_T(U_0) = U$. By Lemma 5, $G - I$ contains T_{U_0}. If $d_I(u) \geq 3$ for each $u \in U$, then G contains T. Hence there is some vertex $u \in U$ such that $d_I(u) \leq 2$. Thus, if $\alpha(G) = 5$, then G contains an induced subgraph $3K_1 \cup P_3$ or $4K_1 \cup K_2$. By an analogous argument of $\alpha(G) = 4$, we can get a contradiction. If $\alpha(G) = 6$, then $\overline{G}[I \cup \{u\}]$ contains a W_6, a contradiction.

The proof of Theorem 6 is completed. ▮

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