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The Ramsey number $R(C_8, K_8)$☆

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Abstract

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let C_m denote a cycle of length m and K_n a complete graph of order n. We show that $R(C_8, K_8) = 50$.

Keywords: Ramsey number; Cycle; Complete graph

1. Introduction

All graphs considered in this paper are simple graphs without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The *maximum* and *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Let $V_1, V_2 \subseteq V(G)$. We use $E(V_1, V_2)$ to denote the set of the edges between V_1 and V_2. The *independence number* of a graph G is denoted by $\alpha(G)$. For $U \subseteq V(G)$, we write $\alpha(U)$ for $\alpha(G[U])$, where $G[U]$ is the subgraph induced by U in G. A *cycle* and a *path* of order n are denoted by C_n and P_n, respectively. A *clique* or a *complete graph* of order n is denoted by K_n. We use mK_n to denote the union of m vertex disjoint K_n. Let G_1 and G_2 be two given graphs, G_1+G_2 is a graph with vertex set $V = V(G_1) \cup V(G_2)$ and edge set $E = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. A *Wheel* of order $n + 1$ is $W_n = K_1 + C_n$ and W_n^- is a graph obtained from W_n by deleting a spoke from W_n. A *Book* $B_n = K_2 + \overline{K_n}$ is a graph of order $n + 2$. For notations not defined here, we follow [2].

In 1978, Erdös et al. posed the following conjecture.

Conjecture (*Erdös et al. [5]*). $R(C_m, K_n) = (m - 1)(n - 1) + 1$ *for* $m \geq n \geq 3$ *and* $(m, n) \neq (3, 3)$.

The conjecture was confirmed for $n = 3$ in early works on Ramsey theory [6,8]. Yang et al. [10] proved the conjecture for $n = 4$. Bollobás et al. [1] showed that the conjecture is true for $n = 5$. Schiermeyer [9] confirmed the conjecture for $n = 6$. Recently, Cheng et al. [3,4] solved the conjecture for $n = 7$. All the results as above support

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that the conjecture is true. In this paper, we calculate the value of the Ramsey number $R(C_8, K_8)$. The main result is the following.

Theorem 1. $R(C_8, K_8) = 50$.

2. Some lemmas

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 ([3]). *Let G be a graph of order $7n-6$ $(n \geq 7)$ with $\alpha(G) \leq 7$. If G contains no C_n, then $\delta(G) \geq n-1$.*

Lemma 2 ([3]). *Let G be a graph of order $7n-6$ $(n \geq 7)$ with $\alpha(G) \leq 7$. If G contains no C_n, then G contains no W_{n-2}.*

Lemma 3 ([7]). $R(B_2, K_7) \leq 34$.

3. Proof of Theorem 1

Proof of Theorem 1. Let G be a graph of order 50. Suppose to the contrary that neither G contains a C_8 nor $\overline{G}$ contains a K_8. By Lemma 1, we have $\delta(G) \geq 7$. That is

$$G \text{ contains no } C_8. \tag{1}$$

$$1 \leq \alpha(G) \leq 7. \tag{2}$$

$$\delta(G) \geq 7. \tag{3}$$

Let $k \in \mathbf{N}$ and $4 \leq k \leq 6$. If G contains $K_1 + P_k$ as a subgraph, let $P_k = v_1 \cdots v_k$ and $V(P_k) \subseteq N(v_0)$. If G contains W_k or W_k^-, let $C = v_1 \cdots v_k$, $W_k = \{v_0\} + C$ and $W_k^- = \{v_0\} + C - \{v_0v_1\}$. In both cases, let $I = \{0, 1, \ldots, k\}$ and $S = \{v_i \mid i \in I\}$. If G contains K_k as a subgraph, let $\{v_1, \ldots, v_k\}$ be a clique. If G contains a B_{k-2}, let $v_1v_2 \in E(G)$ and $v_3, \ldots, v_k \in N(v_1) \cap N(v_2)$. In both cases, let $I = \{1, \ldots, k\}$ and $S = \{v_i \mid i \in I\}$. In all cases, set $U = V(G) - S$ and $U_i = N_U(v_i)$ for $i \in I$. By (3), $|U_i| \neq \emptyset$ for $i \in I$. If $U_i \cap U_j \neq \emptyset$ for some $i, j \in I$, let $v_{k+1} \in U_i \cap U_j$. Set $I' = I \cup \{k+1\}$, $X = S \cup \{v_{k+1}\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $i \in I'$. If $k \leq 5$, then $Y_i \neq \emptyset$ for $i \in I'$. If $Y_i \neq \emptyset$, then for each $i \in I'$, let z_i be an arbitrary vertex in Y_i and let $Z_i = N_Y(z_i)$.

Let I be an index set, $A_i \subseteq V(G)$ for $i \in I$, and $I_1 = \{i_1, i_2, \ldots, i_k\} \subseteq I$. We say that $A_{i_1}, \ldots, A_{i_k}$ have **Property A** if

$$A_i \cap A_j = \emptyset \quad \text{for } i \in I_1,\ j \in I \text{ and } j \neq i,$$
$$\text{and } E(A_i, A_j) = \emptyset \quad \text{for } i, j \in I_1 \text{ and } i \neq j.$$

We say that $A_{i_1}, \ldots, A_{i_k}$ have **Property B** if

$$A_i \cap A_j = \emptyset \quad \text{and} \quad E(A_i, A_j) = \emptyset \quad \text{for } i, j \in I_1 \text{ and } i \neq j,$$
$$\text{and} \quad \alpha\left(\bigcup_{i \in I_1} A_i\right) = \sum_{i \in I_1} \alpha(A_i) \geq 8.$$

These notations will be used throughout the proof of Theorem 1.

In order to prove Theorem 1, we need the following claims.

Claim 1. *G contains no $K_1 + P_6$.*

Proof. Suppose, to the contrary, that G contains $K_1 + P_6$. By (1), we have $U_2 \cap U_3 = U_4 \cap U_5 = \emptyset$ and $U_i \cap U_j = \emptyset$ for $i = 1, 6$, $j \in I$ and $j \neq i$.

If $U_2 \cap U_5 \neq \emptyset$. By (1), we have $v_1v_6, v_3v_7 \notin E(G)$. By (3), $Y_i \neq \emptyset$ for $i = 1, 3, 6, 7$. By (1), we have Y_1, Y_3, Y_6, and Y_7 have Property A, thus $|Z_i| \geq 6$ and $\alpha(Z_i) \geq 2$ for $i = 1, 3, 6, 7$ by Lemma 2. By (1), Z_1, Z_3, Z_6, and Z_7 have Property B, a contradiction. Hence $U_2 \cap U_5 = \emptyset$.

If $U_0 \cap U_5 \neq \emptyset$. By Lemma 2, $v_1v_6, v_1v_7 \notin E(G)$. By (3), $Y_1, Y_6, Y_7 \neq \emptyset$. If $Y_2 = \emptyset$, then $N[v_2] = X$. By (1), $v_3v_6 \notin E(G)$, which implies that $Y_3 \neq \emptyset$. It is clear that Y_1, Y_3, Y_6, and Y_7 have Property A, thus $|Z_i| \geq 6$ and

$\alpha(Z_i) \geq 2$ for $i = 1, 3, 6, 7$ by Lemma 2. By (1), Z_1, Z_3, Z_6, and Z_7 have Property B, a contradiction. Now we can assume that $Y_2 \neq \emptyset$. Since $U_2 \cap U_5 = \emptyset$, we have $Y_2 \cap Y_5 = \emptyset$. If $Y_2 \cap Y_0 = \emptyset$ and $Y_2 \cap Y_4 = \emptyset$, then Y_1, Y_2, Y_6, and Y_7 have Property A by (1), thus $|Z_i| \geq 6$ and $\alpha(Z_i) \geq 2$ for $i = 1, 2, 6, 7$ by Lemma 2. By (1), Z_1, Z_2, Z_6, and Z_7 have Property B, a contradiction. Now we assume that $Y_2 \cap Y_0 \neq \emptyset$ or $Y_2 \cap Y_4 \neq \emptyset$, say $v_8 \in Y_2 \cap Y_0$ or $Y_2 \cap Y_4$. Let $X' = X \cup \{v_8\}$ and $Y' = V(G) - X'$. Set $Y'_i = N_{Y'}(v_i)$, $z'_i \in Y'_i$ and $N_{Y'}(z'_i) = Z'_i$ for $0 \leq i \leq 8$. By (1), $\{v_1, v_6, v_7, v_8\}$ is an independent set. Thus $Y'_i \neq \emptyset$ for $i = 1, 6, 7, 8$. By (1), Y'_1, Y'_6, Y'_7, and Y'_8 have Property A, thus $|Z'_i| \geq 6$ and $\alpha(Z'_i) \geq 2$ for $i = 1, 6, 7, 8$ by Lemma 2. By (1), Z'_1, Z'_6, Z'_7, and Z'_8 have Property B, a contradiction. Thus $U_0 \cap U_5 = \emptyset$. By symmetry, $U_0 \cap U_2 = \emptyset$.

If $U_3 \cap U_5 \neq \emptyset$. By (1), $v_2v_7, v_1v_6 \notin E(G)$. By (3), $Y_i \neq \emptyset$ for $i = 1, 2, 6, 7$. By (1), Y_1, Y_2, Y_6, and Y_7 have Property A, so $|Z_i| \geq 6$ and $\alpha(Z_i) \geq 2$ for $i = 1, 2, 6, 7$. Thus Z_1, Z_2, Z_6, and Z_7 have Property B, a contradiction. So $U_3 \cap U_5 = \emptyset$. By symmetry, $U_2 \cap U_4 = \emptyset$.

By the argument above, U_1, U_2, U_5, and U_6 have Property A. For each $i = 1, 2, 5, 6$, let u_i be an arbitrary vertex in U_i and let $V_i = N_U(u_i)$. Then $|V_i| \geq 6$ and $\alpha(V_i) \geq 2$ for each $i = 1, 2, 5, 6$. By (1), V_1, V_2, V_5, and V_6 have Property B, a contradiction. ■

Claim 2. *G contains no* W_6^-.

Proof. Suppose, to the contrary, that G contains a W_6^-. By (1), U_0, U_1, U_3, and U_5 have Property A. For each $i = 0, 1, 3, 5$, let u_i be any vertex in U_i and let $V_i = N_U(u_i)$. Then $|V_i| \geq 6$. By Lemma 2, $\alpha(V_i) \geq 2$ for $i = 0, 1, 3, 5$. By (1), V_0, V_1, V_3, and V_5 have Property B, a contradiction. ■

Claim 3. *G contains no* W_5.

Proof. Suppose, to the contrary, that G contains a W_5. By Claim 1, $U_0 \cap U_i = \emptyset$ for $1 \leq i \leq 5$. By Claim 2, $U_1 \cap U_2 = U_2 \cap U_3 = U_3 \cap U_4 = U_4 \cap U_5 = U_5 \cap U_1 = \emptyset$. If $U_1 \cap U_3 \neq \emptyset$, then $U_4 \cap (U_1 \cup U_2) = \emptyset$ and $U_5 \cap (U_2 \cup U_3) = \emptyset$ by (1). So we can assume that $U_3 \cap (U_1 \cup U_5) = \emptyset$ and $U_4 \cap (U_1 \cup U_2) = \emptyset$. By (1), U_0, U_1, U_3, and U_4 have Property A. For each $i = 0, 1, 3, 4$, let u_i be any vertex in U_i and let $V_i = N_U(u_i)$. Then $|V_i| \geq 6$ and so $\alpha(V_i) \geq 2$ for $i = 0, 1, 3, 4$. By (1), V_0, V_1, V_3, and V_4 have Property B, a contradiction. ■

Claim 4. *G contains no* K_5.

Proof. Suppose, to the contrary, that G contains a K_5. If there are U_i and U_j with $i \neq j$ such that $U_i \cap U_j \neq \emptyset$, we assume, without loss of generality, that $U_4 \cap U_5 \neq \emptyset$. By Claim 1, $(Y_4 \cup Y_5) \cap (Y_1 \cup Y_2 \cup Y_3 \cup Y_6) = \emptyset$. By Claim 2, $Y_6 \cap (Y_1 \cup Y_2 \cup Y_3) = \emptyset$. If $Y_1 \cap Y_2 \neq \emptyset$, then $Y_1 \cap Y_3 = \emptyset$ by (1). So we may assume that $Y_1 \cap Y_2 = \emptyset$. By (1), $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 2, 5, 6\}$ and $j \neq i$. By the argument as above, $|Z_i| \geq 5$ for $i = 1, 2, 5, 6$. By Claim 3, $\alpha(Z_i) \geq 2$. By (1), Z_1, Z_2, Z_5, and Z_6 have Property B, a contradiction. Hence $U_i \cap U_j = \emptyset$ for $1 \leq i < j \leq 5$. For each $1 \leq i \leq 5$, let u_i be an arbitrary vertex in U_i. Let $T = \{u_1, u_2, u_3, u_4, u_5\}$, $U' = U - T$ and $N_{U'}(u_i) = V_i$ for $1 \leq i \leq 5$. By (1), $\Delta(G[T]) \leq 1$, thus $|V_i| \geq 5$ and so $\alpha(V_i) \geq 2$ for $1 \leq i \leq 5$. Thus $V_1, \ldots, V_5$ have Property B by (1), a contradiction. ■

Claim 5. *G contains no* $K_1 + P_5$.

Proof. Suppose, to the contrary, that G contains $K_1 + P_5$. By Claim 1, $U_0 \cap (U_1 \cup U_5) = \emptyset$. By Claim 2, $U_1 \cap U_5 = \emptyset$.

If $U_4 \cap U_5 \neq \emptyset$. By (1), $Y_1 \cap Y_i = \emptyset$ for $i \neq 1$, $Y_2 \cap Y_i = \emptyset$ for $i \neq 0, 2, 4$, $Y_3 \cap Y_i = \emptyset$ for $i \neq 0, 3$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 6$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 2, 3, 6\}$ and $i \neq j$. Then $|Z_i| \geq 4$ for $i = 1, 2, 3, 6$. By Claim 4, $\alpha(Z_i) \geq 2$ for $i = 1, 2, 3, 6$. By (1), Z_1, Z_2, Z_3, and Z_6 have Property B, a contradiction. Hence $U_4 \cap U_5 = \emptyset$. By symmetry, $U_1 \cap U_2 = \emptyset$.

If $U_2 \cap U_5 \neq \emptyset$. By (1), $Y_1 \cap Y_i = \emptyset$ for $i \neq 1$, $Y_3 \cap Y_i = \emptyset$ for $i \neq 2, 3$, $Y_4 \cap Y_i = \emptyset$ for $i \neq 0, 2, 4$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 6$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 3, 4, 6\}$ and $i \neq j$. Then $|Z_i| \geq 4$ for $i = 1, 3, 4, 6$. By Claim 4, $\alpha(Z_i) \geq 2$ for $i = 1, 3, 4, 6$. By (1), Z_1, Z_3, Z_4, and Z_6 have Property B, a contradiction. Hence $U_2 \cap U_5 = \emptyset$. By symmetry, $U_1 \cap U_4 = \emptyset$.

If $U_3 \cap U_2 \neq \emptyset$. By (1), $Y_1 \cap Y_i = \emptyset$ for $i \neq 1, 3$, $Y_4 \cap Y_i = \emptyset$ for $i \neq 0, 2, 4$, $Y_5 \cap Y_i = \emptyset$ for $i \neq 2, 5$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 0, 6$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 4, 5, 6\}$ and $i \neq j$. Then $|Z_i| \geq 4$ for $i = 1, 4, 5, 6$. By Claim 4, $\alpha(Z_i) \geq 2$ for $i = 1, 4, 5, 6$. By (1), Z_1, Z_4, Z_5, and Z_6 have Property B, a contradiction. So $U_3 \cap U_2 = \emptyset$. By symmetry, $U_3 \cap U_4 = \emptyset$.

If $U_3 \cap U_5 \neq \emptyset$. By (1), $Y_i \cap Y_j = \emptyset$ for $i = 1, 6$ and $j \neq i$, $Y_2 \cap Y_i = \emptyset$ for $i \neq 0, 2$, $Y_4 \cap Y_i = \emptyset$ for $i \neq 3, 4$ and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 2, 4, 6\}$ and $i \neq j$. Then $|Z_i| \geq 5$ for $i = 1, 2, 4, 6$. By Claim 4, $\alpha(Z_i) \geq 2$ for $i = 1, 2, 4, 6$. By (1), Z_1, Z_2, Z_4, and Z_6 have Property B, a contradiction. So $U_3 \cap U_5 = \emptyset$. By symmetry, $U_1 \cap U_3 = \emptyset$.

By (1), $E(U_i, U_j) = \emptyset$ for $i, j \in \{1, 3, 4, 5\}$ and $i \neq j$. For each $i = 1, 3, 4, 5$, let u_i be an arbitrary vertex in U_i and let $V_i = N_U(u_i)$. By the argument above, $|V_i| \geq 4$ for $i = 1, 3, 4, 5$. By Claim 4, $\alpha(V_i) \geq 2$ for $i = 1, 3, 4, 5$. By (1), V_1, V_3, V_4, and V_5 have Property B, a contradiction. ■

Claim 6. *G contains no* W_5^-.

Proof. Suppose, to the contrary, that G contains a W_5^-.

If $U_0 \cap U_1 \neq \emptyset$. By (1), $Y_2 \cap Y_i = \emptyset$ for $i \neq 2$, $Y_3 \cap Y_i = \emptyset$ for $i \neq 0, 1, 3$, $Y_4 \cap Y_i = \emptyset$ for $i \neq 0, 1, 4$, $Y_5 \cap Y_i = \emptyset$ for $i \neq 5$ and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{2, 3, 4, 5\}$ and $i \neq j$. Then $|Z_i| \geq 4$ for $i = 2, 3, 4, 5$. By Claim 4, $\alpha(Z_i) \geq 2$ for $i = 2, 3, 4, 5$. By (1), Z_2, Z_3, Z_4, and Z_5 have Property B, a contradiction. Hence $U_0 \cap U_1 = \emptyset$.

If $U_0 \cap U_3 \neq \emptyset$. By (1), $Y_1 \cap Y_i = \emptyset$ for $i \neq 0, 1, 3$, $Y_2 \cap Y_i = \emptyset$ for $i \neq 2, 5$, $Y_4 \cap Y_i = \emptyset$ for $i \neq 4$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 6$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 2, 4, 6\}$ and $i \neq j$. Then $|Z_i| \geq 4$ for $i = 1, 2, 4, 6$. By Claim 4, $\alpha(Z_i) \geq 2$ for $i = 1, 2, 4, 6$. By (1), Z_1, Z_2, Z_4, and Z_6 have Property B, a contradiction. Hence $U_0 \cap U_3 = \emptyset$. By symmetry, $U_0 \cap U_4 = \emptyset$.

If $U_1 \cap U_3 \neq \emptyset$. By $U_0 \cap U_1 = \emptyset$, $U_0 \cap U_3 \neq \emptyset$ and (1), Y_0, Y_2, Y_4, and Y_5 have Property A, then $|Z_i| \geq 6$ and $\alpha(Z_i) \geq 2$ for $i = 0, 2, 4, 5$. By (1), Z_0, Z_2, Z_4, and Z_5 have Property B, a contradiction. Hence $U_1 \cap U_3 = \emptyset$. By symmetry, $U_1 \cap U_4 = \emptyset$.

If $U_3 \cap U_4 \neq \emptyset$. By (1), $Y_i \cap Y_j = \emptyset$ for $i = 0, 1, 6$ and $j \neq i$, and $Y_2 \cap Y_i = \emptyset$ for $i \neq 2, 5$. For the same reason, $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{0, 1, 2, 6\}$ and $i \neq j$. Then $|Z_i| \geq 5$ for $i = 0, 1, 2, 6$. By Claim 4, $\alpha(Z_i) \geq 2$ for $i = 0, 1, 2, 6$. By (1), Z_0, Z_1, Z_2, and Z_6 have Property B, a contradiction. Thus $U_3 \cap U_4 = \emptyset$.

By (1), $E(U_i, U_j) = \emptyset$ for $i, j \in \{0, 1, 3, 4\}$ and $i \neq j$. For each $i = 0, 1, 3, 4$, let u_i be any vertex in U_i and let $V_i = N_U(u_i)$. Then $|V_i| \geq 4$. By Claim 4, $\alpha(V_i) \geq 2$ for $i = 0, 1, 3, 4$. By (1), V_0, V_1, V_3, and V_4 have Property B, a contradiction. ■

Claim 7. *G contains no* K_4.

Proof. Suppose, to the contrary, that G contains a K_4. By (3), $|U_i| \geq 4$. If there are U_i and U_j with $i \neq j$ such that $U_i \cap U_j \neq \emptyset$, we assume, without loss of generality, that $U_3 \cap U_4 \neq \emptyset$. By (3), $|Y_i| \geq 3$. By Claim 5, $(Y_3 \cup Y_4) \cap (Y_1 \cup Y_2 \cup Y_5) = \emptyset$. By Claim 6, $Y_5 \cap (Y_1 \cup Y_2) = \emptyset$. Let $z_i \in Y_i$ for $i = 1, 2, 3, 5$. Since $|Y_i| \geq 3$, we may choose z_1 such that $z_1 \neq z_2$. Set $A = \{z_1, z_2, z_3, z_5\}$, $Y' = Y - A$ and $Z'_i = N_{Y'}(z_i)$ for $i = 1, 2, 3, 5$. By (1), we have $\Delta(G[A]) \leq 1$. Then $|Z'_i| \geq 4$ for $i = 1, 2, 3, 5$. By Claim 5, $\alpha(Z'_i) \geq 2$ for $i = 1, 2, 3, 5$. By (1), Z'_1, Z'_2, Z'_3, and Z'_5 have Property B, a contradiction. Hence $U_i \cap U_j = \emptyset$ for $1 \leq i < j \leq 4$.

For each $i = 1, 2, 3, 4$, let u_i be any vertex in U_i. Set $T = \{u_1, u_2, u_3, u_4\}$, $U' = U - T$ and $N_{U'}(u_i) = V_i$ for $1 \leq i \leq 4$. If $G[T]$ contains $2K_2$ as a subgraph, then G contains a C_8. Hence we assume that $G[T]$ contains no $2K_2$. If $\Delta(G[T]) = 3$, then $G[T] = K_{1,3}$. By symmetry, we assume that $d_{G[T]}(u_1) = 3$. Let $V'_1 = V_1 \cup \{v_1\}$ and $V'_i = V_i$ for $i = 2, 3, 4$. Then $|V'_i| \geq 4$ for $i = 1, 2, 3, 4$. By Claim 4, $\alpha(V'_i) \geq 2$ for $i = 1, 2, 3, 4$. By (1), V'_1, V'_2, V'_3, and V'_4 have Property B, a contradiction. If $\Delta(G[T]) = 2$, then $G[T] = P_3 \cup \{v_4\}$ or $G[T] = K_3 \cup \{v_4\}$. In this case $|V_i| \geq 4$ and $\alpha(V_i) \geq 2$ for $1 \leq i \leq 4$. By (1), V_1, V_2, V_3, and V_4 have Property B, a contradiction.

Now we have $|E(G[T])| \leq 1$. So $|V_i| \geq 5$ for $1 \leq i \leq 4$. By Claim 5, $\alpha(V_i) \geq 2$ for $i = 1, 2, 3, 4$. By (1), $E(V_i, V_j) = \emptyset$ for $1 \leq i < j \leq 4$. If $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq 4$, then V_1, V_2, V_3, V_4 have Property B, a contradiction. So there exist V_i and V_j, say V_1 and V_2, such that $V_1 \cap V_2 \neq \emptyset$. If $V_1 \cap V_2 \cap V_3 \neq \emptyset$, let $Z = V_1 \cap V_2 \cap V_3$ and $V'_i = V_i - Z$ for $i = 1, 2, 3, 4$. By (1) we have $|Z| = 1$. So $|V'_i| \geq 4$ for $1 \leq i \leq 4$. By Claim 5, $\alpha(V'_i) \geq 2$ for $1 \leq i \leq 4$. By (1), V'_1, V'_2, V'_3, and V'_4 have Property B, a contradiction. So $V_1 \cap V_2 \cap V_3 = \emptyset$. By symmetry, $V_1 \cap V_2 \cap V_4 = \emptyset$. Set $V'_i = V_i \setminus (V_1 \cap V_2)$ for $i = 1, 2$. By (1), $V_1 \cap V_2$ is an independent set, $E(V'_i, V_1 \cap V_2) = \emptyset$ for $i = 1, 2$, $(V_1 \cup V_2) \cap (V_3 \cup V_4) = \emptyset$, and $E(V_i, V_j) = \emptyset$ for $1 \leq i < j \leq 4$. So we can get $\alpha(V_1 \cup V_2) = \alpha(V'_1) + \alpha(V'_2) + |V_1 \cap V_2| \geq 4$. By symmetry, $\alpha(V_3 \cup V_4) \geq 4$, which implies that $\alpha(V_1 \cup V_2 \cup V_3 \cup V_4) \geq 8$, a contradiction. ■

Let H_1 and H_2 be the graphs in Fig. 1. Let $I = \{1, \ldots, 6\}$, $S = \{v_i \mid i \in I\}$. Set $U = V(G) - S$ and $U_i = N_U(v_i)$ for $1 \in I$. It is clear that $|U_i| \geq 2$, for $i \in I$. If $U_i \cap U_j \neq \emptyset$ for some $i, j \in I$, let $v_7 \in U_i \cap U_j$. Set $I' = I \cup \{7\}$,

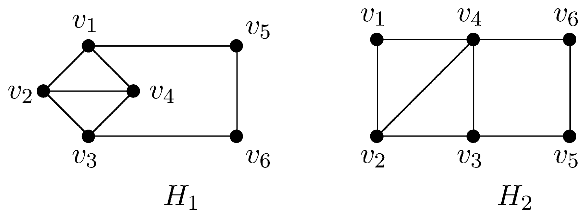

Fig. 1.

$X = S \cup \{v_7\}$, $Y = V(G) - X$ and $Y_i = N_Y(v_i)$ for $i \in I'$. By (3), $Y_i \neq \emptyset$ for $i \in I'$. For each $i \in I'$, let z_i be any vertex in Y_i and let $Z_i = N_Y(z_i)$.

Claim 8. *G contains no* H_1.

Proof. Suppose, to the contrary, that G contains H_1.

If $U_1 \cap U_2 \neq \emptyset$. By (1), $Y_4 \cap Y_i = \emptyset$ for $i \neq 1, 4$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 1, 2, 6$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 7$ and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{4, 6, 7\}$ and $i \neq j$. Then $|Z_i| \geq 5$ for $i = 4, 7$ and $|Z_6| \geq 4$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 4, 7$ and $\alpha(Z_6) \geq 2$. By (1), Z_4, Z_6, and Z_7 have Property B, a contradiction. Thus $U_1 \cap U_2 = \emptyset$. By symmetry, $U_1 \cap U_4 = U_3 \cap U_2 = U_3 \cap U_4 = \emptyset$.

If $U_2 \cap U_5 \neq \emptyset$. By (1), $Y_1 \cap Y_i = \emptyset$ for $i \neq 1, 5$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 6$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 7$ and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 6, 7\}$ and $i \neq j$. Then $|Z_i| \geq 5$ for $i = 1, 6, 7$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 1, 6, 7$. By (1), Z_1, Z_6, and Z_7 have Property B, a contradiction. Thus $U_2 \cap U_5 = \emptyset$. By symmetry, $U_2 \cap U_6 = U_4 \cap U_5 = U_4 \cap U_6 = \emptyset$.

If $U_2 \cap U_4 \neq \emptyset$. By (1) and $U_6 \cap (U_2 \cup U_4) = \emptyset$, we have $Y_1 \cap Y_i = \emptyset$ for $i \neq 1, 3, 6$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 1, 6$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 7$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 6, 7\}$ and $i \neq j$. By Claim 7, $v_1v_7 \notin E(G)$. By (3), $Y_1 \geq 2$, so we can choose z_1, such that $z_1 \neq z_6$. Then by the argument above, $|Z_i| \geq 5$ for $i = 6, 7$ and $|Z_1| \geq 4$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 6, 7$ and $\alpha(Z_1) \geq 2$. By (1), Z_1, Z_6, and Z_7 have Property B, a contradiction. Thus $U_2 \cap U_4 = \emptyset$.

If $U_1 \cap U_5 \neq \emptyset$. By (1), $Y_2 \cap Y_i = \emptyset$ for $i \neq 2, 5$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 1, 6$, and $Y_7 \cap Y_i = \emptyset$ for $i \neq 3, 7$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{2, 6, 7\}$ and $i \neq j$. Then $|Z_i| \geq 5$ for $i = 2, 6, 7$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 2, 6, 7$. By (1), Z_2, Z_6, and Z_7 have Property B, a contradiction. Thus $U_1 \cap U_5 = \emptyset$. By symmetry, $U_3 \cap U_6 = \emptyset$.

If $U_1 \cap U_3 \neq \emptyset$. By $U_2 \cap U_i = \emptyset$ for $i = 1, 3, 4, 5, 6$, we have $Y_2 \cap Y_i = \emptyset$ for $i = 1, 3, 4, 5, 6$. By $U_6 \cap (U_2 \cup U_3 \cup U_4) = \emptyset$, we have $Y_6 \cap (Y_2 \cup Y_3 \cup Y_4) = \emptyset$. By (1), $Y_7 \cap Y_i = \emptyset$ for $i \neq 1, 3, 7$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{2, 6, 7\}$ and $i \neq j$. Then $|Z_2| \geq 6$, $|Z_6| \geq 4$ and $|Z_7| \geq 4$. By Claims 3 and 7, $\alpha(Z_2) \geq 3$ and $\alpha(Z_i) \geq 2$ for $i = 6, 7$. By (1), $\{v_4\}$, Z_2, Z_6, and Z_7 have Property B, a contradiction. Thus $U_1 \cap U_3 = \emptyset$.

Thus $U_1 \cap U_i = \emptyset$ for $i \neq 1, 6$, $U_2 \cap U_i = \emptyset$ for $i \neq 2$, and $U_6 \cap U_i = \emptyset$ for $i \neq 1, 5, 6$. Take $u_i \in U_i$ for $i = 1, 2, 6$. Since $|U_i| \geq 2$, we can choose u_1, such that $u_1 \neq u_6$. Let $T = \{u_1, u_2, u_6\}$, $U' = U - T$, $V_i = N_{U'}(u_i)$. By (1), $\Delta(G[A]) \leq 1$. Then $|V_1| \geq 4$, $|V_2| \geq 5$ and $|V_6| \geq 3$. By Claims 3 and 7, $\alpha(V_1) \geq 2$, $\alpha(V_2) \geq 3$ and $\alpha(V_6) \geq 2$. By (1), $\{v_5\}$, V_1, V_2, and V_6 have Property B, a contradiction. ■

Claim 9. *G contains no* H_2.

Proof. Suppose, to the contrary, that G contains H_2.

If $U_1 \cap U_2 \neq \emptyset$. By (1), $Y_5 \cap Y_i = \emptyset$ for $i \neq 2, 4, 5$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 3, 6$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 7$, and $E(Y_i, Y_j) = \emptyset$ for $5 \leq i < j \leq 7$. Then $|Z_i| \geq 5$ for $i = 6, 7$ and $|Z_5| \geq 4$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 6, 7$ and $\alpha(Z_5) \geq 2$. By (1), Z_5, Z_6, and Z_7 have Property B, a contradiction. Thus $U_1 \cap U_2 = \emptyset$.

If $U_1 \cap U_3 \neq \emptyset$. By (1), $Y_2 \cap Y_i = \emptyset$ for $i \neq 2, 3$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 1, 3, 6$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 7$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{2, 6, 7\}$ and $i \neq j$. Then $|Z_i| \geq 5$ for $i = 2, 7$ and $|Z_6| \geq 4$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 2, 7$ and $\alpha(Z_6) \geq 2$. By (1), Z_2, Z_6, and Z_7 have Property B, a contradiction. Thus $U_1 \cap U_3 = \emptyset$.

If $U_1 \cap U_4 \neq \emptyset$. By (1), $Y_5 \cap Y_i = \emptyset$ for $i \neq 1, 2, 4, 5$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 3, 6$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 7$, and $E(Y_i, Y_j) = \emptyset$ for $5 \leq i < j \leq 7$. Then $|Z_i| \geq 5$ for $i = 6, 7$ and $|Z_5| \geq 3$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 6, 7$ and $\alpha(Z_5) \geq 2$. By (1), Z_5, Z_6, and Z_7 have Property B, a contradiction. Thus $U_1 \cap U_2 = \emptyset$.

If $U_1 \cap U_5 \neq \emptyset$. By (1), $Y_2 \cap Y_i = \emptyset$ for $i \neq 2, 4, 5$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 6$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 7$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{2, 6, 7\}$ and $i \neq j$. Then $|Z_i| \geq 6$ for $i = 6, 7$ and $|Z_2| \geq 3$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 6, 7$ and $\alpha(Z_2) \geq 2$. By (1), Z_2, Z_6, and Z_7 have Property B, a contradiction. Thus $U_1 \cap U_5 = \emptyset$.

If $U_2 \cap U_3 \neq \emptyset$. By (1), $Y_1 \cap Y_i = \emptyset$ for $i \neq 1, 3$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 2, 3, 6$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 4, 7$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 6, 7\}$ and $i \neq j$. Then $|Z_i| \geq 5$ for $i = 1, 7$ and $|Z_6| \geq 4$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 1, 7$ and $\alpha(Z_6) \geq 2$. By (1), Z_1, Z_6, and Z_7 have Property B, a contradiction. Thus $U_2 \cap U_3 = \emptyset$.

If $U_3 \cap U_5 \neq \emptyset$. By (1), $Y_2 \cap Y_i = \emptyset$ for $i \neq 2, 4, 5$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 6$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 7$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{2, 6, 7\}$ and $i \neq j$. Then $|Z_i| \geq 5$ for $i = 2, 6, 7$. By Claims 3 and 7, $\alpha(Z_i) \geq 3$ for $i = 2, 6, 7$. By (1), Z_2, Z_6, and Z_7 have Property B, a contradiction. Thus $U_3 \cap U_5 = \emptyset$.

If $U_3 \cap U_4 \neq \emptyset$. By $U_1 \cap U_i = \emptyset$ for $i = 2, 3, 4$, we have $Y_1 \cap Y_i = \emptyset$ for $i = 2, 3, 4$. By (1), $Y_1 \cap Y_i = \emptyset$ for $i = 5, 6, 7$, $Y_6 \cap Y_i = \emptyset$ for $i \neq 3, 4, 5$, $Y_7 \cap Y_i = \emptyset$ for $i \neq 3, 4, 7$, and $E(Y_i, Y_j) = \emptyset$ for $i, j \in \{1, 6, 7\}$ and $i \neq j$. Then $|Z_1| \geq 6$, $|Z_6| \geq 4$ and $|Z_7| \geq 4$. By Claims 3 and 7, $\alpha(Z_1) \geq 3$ and $\alpha(Z_i) \geq 2$ for $i = 6, 7$. By (1), $\{v_2\}$, Z_1, Z_6, and Z_7 have Property B, a contradiction. Thus $U_3 \cap U_4 = \emptyset$.

By the argument as above, $U_1 \cap U_i = \emptyset$ for $i \neq 1, 6$, $U_3 \cap U_i = \emptyset$ for $i \neq 3, 6$, and $U_5 \cap U_i = \emptyset$ for $i \neq 2, 4, 5, 6$. For each $i = 1, 3, 5$, let u_i be any vertex in U_i and let $V_i = N_U(u_i)$. Then $|V_i| \geq 5$ for $i = 1, 3$, and $|V_5| \geq 3$. By Claims 3 and 7, $\alpha(V_i) \geq 3$ for $i = 1, 3$, and $\alpha(V_5) \geq 2$. By (1), V_1, V_3, and V_5 have Property B, a contradiction. ∎

Claim 10. *G contains no* $K_1 + P_4$.

Proof. Suppose, to the contrary, that G contains $K_1 + P_4$. By Claim 5, $U_0 \cap (U_1 \cup U_4) = \emptyset$. By Claim 6, $U_1 \cap U_4 = \emptyset$. By Claim 9, $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_4 = U_3 \cap U_4 = \emptyset$. For each $i = 0, 1, 4$, let u_i be any vertex in U_i. Set $T = \{u_0, u_1, u_4\}$, $U' = U - T$ and $N_{U'}(u_i) = V_i$ for $i = 0, 1, 4$. By (1), $\Delta(G[T]) \leq 1$. By the argument above, $|V_i| \geq 5$ for $i = 1, 4$ and $|V_0| \geq 3$. By Claims 3 and 7, $\alpha(V_i) \geq 3$ for $i = 1, 4$, and $\alpha(V_0) \geq 2$. By (1), V_0, V_1, and V_4 have Property B, a contradiction. ∎

Claim 11. *G contains no* B_3.

Proof. Assume, to the contrary, that G contains a B_3. By Claim 9, $U_3 \cap U_4 = U_3 \cap U_5 = U_4 \cap U_5 = \emptyset$. By Claim 10, $(U_1 \cup U_2) \cap (U_3 \cup U_4 \cup U_5) = \emptyset$. Take $u_i \in Y_i$ for $i = 3, 4, 5$. Set $T = \{u_3, u_4, u_5\}$, $U' = U - T$ and $N_{U'}(u_i) = V_i$ for $i = 3, 4, 5$. By (1), $\Delta(G[T]) \leq 1$. Thus $|V_i| \geq 5$ for $i = 3, 4, 5$. By Claims 3 and 7, $\alpha(V_i) \geq 3$ for $i = 3, 4, 5$. By (1), V_3, V_4, and V_5 have Property B, a contradiction. ∎

Claim 12. *G contains no* W_4^-.

Proof. Suppose, to the contrary, that G contains a W_4^-. It is clear that $|U_i| \geq 3$. By Claim 8, $U_1 \cap U_2 = U_1 \cap U_4 = \emptyset$. By Claim 9, $U_1 \cap U_0 = U_1 \cap U_3 = \emptyset$. By Claim 10, $(U_0 \cup U_3) \cap (U_2 \cup U_4) = \emptyset$. By Claim 11, $U_0 \cap U_3 = \emptyset$. For each $i = 1, 2, 3$, let u_i be any vertex in Y_i. Set $T = \{u_1, u_2, u_3\}$, $U' = U - T$ and $N_{U'}(u_i) = V_i$ for $i = 1, 2, 3$. By (1), $\Delta(G[T]) \leq 1$. Then $|V_i| \geq 5$ for $i = 1, 3$ and $|V_2| \geq 4$. By Claims 3 and 7, $\alpha(V_i) \geq 3$ for $i = 1, 3$, and $\alpha(V_4) \geq 2$. By (1), V_1, V_2, and V_3 have Property B, a contradiction. ∎

Claim 13. *G contains no* B_2.

Proof. Suppose, to the contrary, that G contains a B_2. By Claim 10, $(U_1 \cup U_2) \cap (U_3 \cup U_4) = \emptyset$. By Claim 11, $U_1 \cap U_2 = \emptyset$. By Claim 12, $U_3 \cap U_4 = \emptyset$. By Claim 8, $E(U_3, U_4) = \emptyset$. By Claim 9, $E(U_1 \cup U_2, U_3 \cup U_4) = \emptyset$. For each $i = 2, 3, 4$, let u_i be any vertex in U_i and let $V_i = N_U(u_i)$. Then $|V_i| \geq 6$ for $2 \leq i \leq 4$. By Claims 3 and 7, $\alpha(V_i) \geq 3$ for $2 \leq i \leq 4$. By (1), $E(V_i, V_j) = \emptyset$ for $2 \leq i < j \leq 4$.

If $V_2 \cap V_3 \cap V_4 \neq \emptyset$, then $|V_2 \cap V_3 \cap V_4| = 1$ by (1). Let $V_i' = V_i - (V_2 \cap V_3 \cap V_4)$ for $i = 2, 3, 4$, then $|V_i'| \geq 5$ for $i = 2, 3, 4$. By Claims 3 and 7, $\alpha(V_i') \geq 3$ for $i = 2, 3, 4$. By (1), V_2', V_3', and V_4' have Property B, a contradiction. So $V_2 \cap V_3 \cap V_4 = \emptyset$. If $V_2 \cap V_3 \neq \emptyset$, let $A = V_2 \cap V_3$, and $V_i' = V_i - A$ for $i = 2, 3$. By (1), A is an independent set, and V_2', V_3', V_4, and A have Property B, a contradiction. So $V_2 \cap V_3 = \emptyset$. By symmetry, $V_2 \cap V_4 = \emptyset$. If $V_3 \cap V_4 \neq \emptyset$, let $A = V_3 \cap V_4$, and $V_i' = V_i - A$ for $i = 3, 4$. By (1), A is an independent set, and V_2, V_3', V_4' and A have Property B, a contradiction. Now $V_i \cap V_j = \emptyset$ for $2 \leq i \leq 4$, which implies that $\alpha(\cup_{i=1}^{3} V_i) \geq 9$, a contradiction. ∎

We now begin to prove Theorem 1.

If there is some vertex v, such that $d(v) \leq 15$, then $G' = G - N[v]$ is a graph of order at least 34. By Lemma 3 and Claim 13, $\alpha(G') \geq 7$, which implies that $\alpha(G) \geq 8$, a contradiction. Hence $\delta(G) \geq 16$. Let $v_0 \in V(G)$. Since $d(v_0) \geq 16$, $G[N(v_0)]$ contains no P_3 by Claim 13. Thus, $G[N(v_0)]$ contains only independent edges and independent vertices, which implies that $\alpha(G[N(v_0)]) \geq 8$, a contradiction. Thus, $R(C_8, K_8) \leq 50$. On the other hand, since $7K_7$ contains no C_8 and its complement contains no K_8, we have $R(C_8, K_8) \geq 50$ and hence $R(C_8, K_8) = 50$. ∎

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