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Note

New upper bound formulas with parameters for Ramsey numbers☆

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Abstract

In this paper, we obtain some new results $R(5, 12) \leqslant 848$, $R(5, 14) \leqslant 1461$, etc., and we obtain new upper bound formulas for Ramsey numbers with parameters.

Keywords: Ramsey number; Upper bound

For two given graphs G_1, G_2, the Ramsey number $R(G_1, G_2)$ is the smallest positive integer $p+1$ such that for any graph G of order $p+1$ either G contains G_1 or G^c contains G_2, where G^c is the complement of G. A graph H of order p is called a $(G_1, G_2; p)$-*Ramsey graph* if H does not contain G_1 and H^c does not contain G_2. Let $R(K_m, K_n) =: R(m, n)$ and $(K_m, K_n; p)$-*Ramsey graph* $=:$ $(m, n; p)$-*Ramsey graph*. When an edge e is removed from G, we denote the graph by $G - e$. Let d_i be the degree of vertex i in G of order p, and let $\bar{d}_i = p - 1 - d_i$, where $1 \leqslant i \leqslant p$. And let $f(K_r)$ ($g(K_r)$, resp.) denote the number of K_r in G (G^c, resp.).

In the following we always assume that $G_1 = K_m$ or $K_m - e$, $G_2 = K_n$ or $K_n - e$, $G_1^{m-i} = K_{m-i}$ or $K_{m-i} - e$, $G_2^{n-i} = K_{n-i}$ or $K_{n-i} - e$, $m \geqslant 4$ and $n \geqslant 4$.

Lemma 1 (*Goodman [1]*). *For any graph G of order p, we denote i as its vertex, d_i as the degree of vertex i, $\bar{d}_i = p - 1 - d_i$, $1 \leqslant i \leqslant p$. Then we have*

$$f(K_3) + g(K_3) = \binom{p}{3} - \frac{1}{2} \sum_{i=1}^{p} d_i \bar{d}_i.$$

Lemma 2 (*Huang and Zhang [3]*). *For any $(G_1, G_2; p)$-Ramsey graph G of order p, the following inequalities*

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must hold:

$$(s+1)f(K_{s+1})\leqslant f(K_s)[R(G_1^{m-s},G_2)-1],$$
$$(t+1)g(K_{t+1})\leqslant g(K_t)[R(G_1,G_2^{n-t})-1].$$

Theorem 1. *Let* $R(G_1^{m-2},G_2)\leqslant\alpha+1$, $R(G_1,G_2^{n-2})\leqslant\beta+1$, $R(G_1^{m-1},G_2)\leqslant\gamma+1$, $R(G_1,G_2^{n-1})\leqslant\delta+1$ *and* $t>0$. *Let* $A=2\gamma-2-\frac{1}{3}(4\alpha+2\beta)$, $B=(\alpha+\beta+2)^2+\frac{1}{3}(\beta-\alpha)^2$ *and* $F(t)=\alpha+\beta+4-t+\sqrt{\frac{4}{3}t^2+2At+B}$. *Then*

$$R(G_1,G_2)\leqslant F(t). \tag{1}$$

In particular, when $4B-3A^2>0$, *and* $t_0=\frac{3}{4}(\sqrt{4B-3A^2}-A)>0$, *then*

$$R(G_1,G_2)\leqslant F(t_0). \tag{2}$$

Proof. Let $p=R(G_1,G_2)-1$. For any $(G_1,G_2;p)$-Ramsey graph G, by Lemma 2, we have $3f(K_3)\leqslant\frac{1}{2}\alpha\sum_{i=1}^{p}d_i$ and $3g(K_3)\leqslant\frac{1}{2}\beta\sum_{i=1}^{p}\bar{d}_i$. Combining these two inequalities and Lemma 1, then

$$\begin{aligned}p(p-1)(p-2-\alpha)&\leqslant\sum_{i=1}^{p}(p-1-d_i)(3d_i+\beta-\alpha)\\&\leqslant\sum_{i=1}^{p}\{-3\bar{d}_i^2+(3p-3+\beta-\alpha+t)\bar{d}_i-t(p-1)+t\gamma\}\\&\leqslant\sum_{i=1}^{p}\left\{\frac{1}{12}(3p-3+\beta-\alpha+t)^2-t(p-1)+t\gamma\right\}.\end{aligned}$$

Thus $R(G_1,G_2)\leqslant F(t)$.

From the definition of $F(t)$, when $4B-3A^2>0$ and $t_0=\frac{3}{4}(\sqrt{4B-3A^2}-A)>0$, we have

$$F'(t_0)=0,\quad F''(t_0)>0.$$

Hence we have $R(G_1,G_2)\leqslant F(t_0)$. The proof of theorem is completed. □

Noting the symmetry of γ and δ, we have the following corollary immediately.

Corollary 1. *Under the assumption of Theorem* 1, *let* $t>0$, $C=2\delta-2-\frac{1}{3}(2\alpha+4\beta)$, $D=(\alpha+\beta+2)^2+\frac{1}{3}(\alpha-\beta)^2$ *and* $G(t)=\alpha+\beta+4-t+\sqrt{\frac{4}{3}t^2+2Ct+D}$. *If* $4D-3C^2>0$ *and*; $t^*=\frac{3}{4}(\sqrt{4D-3C^2}-C)>0$, *then we have*:

$$R(G_1,G_2)\leqslant G(t^*).$$

Now we obtain another new upper bound formula with parameters x, y.

Theorem 2. *Let* $m\geqslant4$, $n\geqslant4$, $R(m-2,n)\leqslant\alpha+1$, $R(m,n-2)\leqslant\beta+1$ *and parameter* $x\in(0,3)$. *And let*

$$f(x,y)=A+\sqrt{A^2-B},\quad g(x,y)=A-\sqrt{A^2-B},$$
$$A=\frac{3(y+\alpha-\beta)-2(1+\alpha)x}{9-4x},\quad B=\frac{(3-x)(y+\alpha-\beta)^2+xy^2}{(3-x)(9-4x)}.$$

Then

(a) $R(m,n)\geqslant2+f(x,y)$ *or* $R(m,n)\leqslant2+g(x,y)$ *if* $0<x<\frac{9}{4}$;
(b) $R(m,n)\leqslant2+f(x,y)$ *if* $x\in\left(\frac{9}{4},3\right)$;
(c) $R(m,n)\leqslant\alpha+\beta+4+\frac{2}{3}\sqrt{(\alpha+2\beta+3)(2\alpha+\beta+3)+(\beta-\alpha)^2}$ *if* $x=\frac{9}{4}$.

Especially, when $m=n$, *we obtain K. Walker's formula once again*

$$R(n,n)\leqslant 4R(n-2,n)+2.$$

Proof. Let $p=R(m,n)-1$. Then by using the analogous arguments of Theorem 1, the following inequalities must hold in $(m,n;p)$-Ramsey graph G:

$$\begin{aligned}p(p-1)(p-2-\alpha) &\leqslant \sum_{i=1}^{p}\{-3\bar{d}_i^2+(3p-3+\beta-\alpha)\bar{d}_i\}\\ &= \sum_{i=1}^{p}\{-x\bar{d}_i^2+(3p-3+\beta-\alpha-y)\bar{d}_i-(3-x)\bar{d}_i^2+y\bar{d}_i\}\\ &\leqslant \frac{1}{4x}(3p-3+\beta-\alpha-y)^2p+\frac{y^2p}{4(3-x)}.\end{aligned}$$

Thus, we have $(9-4x)(3-x)(p-1)^2-2(3-x)\{3(y+\alpha-\beta)-2(1+\alpha)x\}(p-1)+xy^2+(3-x)(y+\alpha-\beta)^2\geqslant 0$.

(1) When $0<x<\frac{9}{4}$, (a) follows immediately.
(2) When $\frac{9}{4}<x<3$, since $B<0$, $g(x,y)<0$. Note that in this case $(9-4x)(3-x)<0$. Hence $R(m,n)\leqslant 2+f(x,y)$.
(3) When $x=\frac{9}{4}$, we have

$$R(m,n)\leqslant 2+\frac{4y^2-2(\beta-\alpha)y+(\beta-\alpha)^2}{6y-3\alpha-6\beta-9}=:2+f(y).$$

It is easy to check that when $y_0=\frac{1}{2}\left(\alpha+2\beta+3+\sqrt{(\alpha+2\beta+3)(2\alpha+\beta+3)}\right)$,

$$\min f(y)=f(y_0)=\alpha+\beta+4+\tfrac{2}{3}\sqrt{(\alpha+2\beta+3)(2\alpha+\beta+3)+(\beta-\alpha)^2}.$$

Hence (c) follows. □

It is not difficult to generalize the results to $R(G_1,G_2)$ for $G_1=K_m$ or K_m-e, and $G_2=K_n$ or K_n-e. Hence using (c) of the generalized Theorem 2, taking $\alpha=20$, $\beta=35$, we have $R(K_6-e,K_6)\leqslant 116$ once more, which appears in [2].

Noting the symmetry of α and β, we have the following corollary immediately.

Corollary 2. *Under the hypotheses of Theorem* 2, *let* $F(x,y)=C+\sqrt{C^2-D}$, $G(x,y)=C-\sqrt{C^2-D}$, $C=(3(y+\beta-\alpha)-2(1+\beta)x)/(9-4x)$ *and* $D=(y+\beta-\alpha)^2/(9-4x)+xy^2/\{(3-x)(9-4x)\}$. *Thus we have*:

(1) *If* $0<x<\frac{9}{4}$, *then* $R(m,n)\geqslant 2+F(x,y)$ *or* $R(m,n)\leqslant 2+G(x,y)$.
(2) *If* $\frac{9}{4}<x<3$, *then* $R(m,n)\leqslant 2+F(x,y)$.

Note that there is the well-known formula:

$$R(m,n)\leqslant R(m-1,n)+R(m,n-1), \tag{3}$$

and its generalized formula in [3]:

$$R(G_1,G_2)\leqslant R(G_1^{m-1},G_2)+R(G_1,G_2^{n-1}), \tag{4}$$

where $G_1=K_m$ or K_m-e, $G_2=K_n$ or K_n-e.

Up to date upper and lower bounds on Ramsey numbers are listed in [5]. Using these tables and (3), (4), 18 new upper bounds of $R(m,n)$ obtained by (1) are shown in Table 1, where $(-,-,-;-)=(\alpha,\beta,\gamma;t_0)$, and the number with $*$ is obtained by (3).

Table 1

n	m				
	5	6	7	8	9
11	633*	1804*	4553 (632, 1712, 1803; 1756.4)	10630 (1803, 3582, 4552; 2794.2)	22325 (4552, 6587, 10629; 1162.1)
12	848 (58, 441, 237; 735.3)	2566 (237, 1170, 847; 1679.1)	6954 (847, 2825, 2565; 3488.4)	16944 (2565, 6089, 6953; 5585.5)	39025 (6953, 12676, 16943; 9110.7)
13	1139*	3705*	10581 (1138, 4552, 3704; 4215.9)	27490 (3704, 10629, 10580; 12059.2)	64871 (10580, 22324, 27489; 17928.8)
14	1461 (77, 847, 348; 1520.4)	5033 (348, 2565, 1460; 4118.6)	15263 (1460, 6953, 5032; 10099.1)	41525 (5032, 16943, 15262; 21087.03)	89203 (15262, 39024, 41524; 41657.1)
15	1878*	6911*	22116 (1877, 10580, 6910; 16179.1)	63620 (6910, 27489, 22115; 45750.3)	

Remarks. (1) Theorems 1 and 2 can be generalized by using the ideas in [3,4]. (2) Taking $(\alpha, \beta, \gamma, t_0) = (33, 66, 87, 45.9)$, we have $R(K_6 - e, K_7) \leqslant 202$ once more, which appears in [2].

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