Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
European Journal of Combinatorics ▮ (▮▮▮▮) ▮▮▮–▮▮▮
European Journal of Combinatorics
www.elsevier.com/locate/ejc

The Ramsey numbers for stars of even order versus a wheel of order nine☆

Yunqing Zhang, Yaojun Chen, Kemin Zhang

Department of Mathematics, Nanjing University, Nanjing 210093, PR China

Received 13 April 2007; accepted 26 July 2007

Abstract

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest positive integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let S_n denote a star of order n and W_m a wheel of order $m + 1$. In this paper, we show that $R(S_n, W_8) = 2n + 2$ for $n \geq 6$ and $n \equiv 0 \pmod 2$.

1. Introduction

All graphs considered in this paper are finite simple graph without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The *minimum degree*, *maximum degree*, *independence number* and *connectivity* of G are denoted by $\delta(G)$, $\Delta(G)$, $\alpha(G)$ and $\kappa(G)$, respectively. The edge number of a graph G is $e(G)$. Let $V_1, V_2 \subseteq V(G)$. We use $E(V_1, V_2)$ to denote the set of the edges between V_1 and V_2, and $e(V_1, V_2) = |E(V_1, V_2)|$. For $U \subseteq V(G)$, $G[U]$ is the subgraph induced by U in G. A cycle and a path of order n are denoted by C_n and P_n, respectively. We use mG to denote the union of m vertex disjoint G. A *wheel* of order $n + 1$ is $W_n = K_1 + C_n$. A *book* of order $n + 2$ is $B_n = K_2 + \overline{K_n}$. Let $c(G)$ be the *circumference* of G, that is, the length of a longest cycle, and $g(G)$, the *girth*, that is, the length of a shortest cycle. A graph on n vertices is *pancyclic* if it contains cycles of every length l, $3 \leq l \leq n$. A graph is *weakly pancyclic* if it contains cycles of every length from the girth to the circumference. Let C be a cycle. For a given orientation

☆ This project supported by NSFC under grant number 10671090.
E-mail address: yaojunc@nju.edu.cn (Y. Chen).

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2007.07.005

Please cite this article in press as: Y. Zhang, Y. Chen, K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European Journal of Combinatorics (2007), doi:10.1016/j.ejc.2007.07.005

of C, we use u^+ to denote the successor of u and u^- to denote its predecessor. If $A \subset V(C)$ then $A^+ = \{a^+ \mid a \in A\}$ and $A^- = \{a^- \mid a \in A\}$. Let $u, v \in V(G)$ and s, t with $s \le t$ be integers. If G contains a (u, v)-path of order l for each l with $s \le l \le t$, then we say u and v are (s, t)-*connected* in G. A *linear forest* is a forest with maximum degree not more than two. For notations which are not defined here, we follow [3].

For the Ramsey number of a star versus a wheel, Chen et al. determined all values of $R(S_n, W_m)$ for odd m and $n \ge m - 1 \ge 2$, and obtained the following.

Theorem 1 (*Chen et al. [6]*). *$R(S_n, W_m) = 3n - 2$ for m odd and $n \ge m - 1 \ge 2$.*

Obviously, Theorem 1 shows that the Ramsey number $R(S_n, W_m)$ for m odd and $n \ge m-1 \ge 2$ is determined by n. However, it is not the case when m is even. In fact, as pointed in [6], the Ramsey number $R(S_n, W_m)$ for even m and $n \ge m-1 \ge 2$ cannot be determined by n alone and is a function related to both m and n. In the case when m is even, only the values of $R(S_n, W_4)$ and $R(S_n, W_6)$ are known by now, and it seems difficult to calculate the values of $R(S_n, W_m)$.

In [8], Surahmat et al. determined the value for $R(S_n, W_4)$, and got the following.

Theorem 2 (*Surahmat and Baskoro [8]*). *$R(S_n, W_4) = 2n - 1$ for $n \ge 3$ and $n \equiv 1 \pmod 2$ and $R(S_n, W_4) = 2n + 1$ for $n \ge 4$ and $n \equiv 0 \pmod 2$.*

By using induction on n, Chen et al. established the following.

Theorem 3 (*Chen et al. [6]*). *$R(S_n, W_6) = 2n + 1$ for $n \ge 3$.*

In this paper, we consider the value of $R(S_n, W_8)$. Our main result is the following.

Theorem 4. *$R(S_n, W_8) = 2n + 2$ for $n \ge 6$ and $n \equiv 0 \pmod 2$.*

2. Some lemmas

In order to prove Theorem 4, we need the following lemmas.

Lemma 1 (*Bondy [1]*). *Let G be a graph of order n. If $\delta(G) \ge n/2$, then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.*

Lemma 2 (*Brandt et al. [4]*). *Every non-bipartite graph G with $\delta(G) \ge (n + 2)/3$ is weakly pancyclic and has girth 3 or 4.*

Lemma 3 (*Dirac [7]*). *Let G be a 2-connected graph of order $n \ge 3$ with $\delta(G) = \delta$. Then $c(G) \ge \min\{2\delta, n\}$.*

Lemma 4 (*Zhang [9]*). *If G is a Hamiltonian graph of order n and there exists a vertex x such that $d(x) + d(y) \ge n$ for each y not adjacent to x, then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.*

Given a graph G of order n, repeat the following recursive operation as long as possible: For each pair of nonadjacent vertices a and b, if $d(a) + d(b) \ge n + 1$ then add the edge ab to G. We denote by $\mathrm{cl}(G)$ the resulting graph and call it the closure of G.

Lemma 5 (*Bondy and Chvátal [2]*). *A graph G of order $n \ge 3$ is Hamilton-connected if and only if its closure $\mathrm{cl}(G)$ is Hamilton-connected.*

Lemma 6. *If F is a linear forest of order 6, then $\overline{F}$ is $(4, 6)$-connected.*

Please cite this article in press as: Y. Zhang, Y. Chen, K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European Journal of Combinatorics (2007), doi:10.1016/j.ejc.2007.07.005

Proof. Let $u, v \in V(F)$. Since $\mathrm{cl}(\overline{P_6}) = K_7$, by Lemma 5, $\overline{F}$ has a Hamilton (u, v)-path $P = v_1 v_2 \cdots v_6$, where $u = v_1$ and $v = v_6$. If $\overline{F}$ contains no (u, v)-path of order 5, then $v_i v_{i+2} \in E(F)$ for $1 \le i \le 4$. Since $\Delta(F) \le 2$, we have $v_1 v_4 \in E(\overline{F})$, which implies $v_2 v_6 \in E(F)$. In this case, F contains a triangle $v_2 v_4 v_6$, a contradiction. If $\overline{F}$ contains no (u, v)-path of order 4, then $v_i v_{i+3} \in E(F)$ for $1 \le i \le 3$. If $v_1 v_3 \in E(\overline{F})$, then $v_2 v_6, v_4 v_6 \in E(F)$, which contradicts $\Delta(F) \le 2$. Thus by symmetry we have $v_1 v_3, v_4 v_6 \in E(F)$, which implies F contains a C_4, a contradiction. ■

Lemma 7 (*Chen et al.* [5]). *Let G be a connected graph and C a maximal cycle of G. Suppose that $v \in V(G - C)$ and $d_C(v) \ge 2$. Then for any two distinct vertices x, y in $N_C^+(v)$ or $N_C^-(v)$, $xy \notin E(G)$ and $N(x) \cap N(y) \cap V(G - C) = \emptyset$.*

Lemma 8. *Let $G = K_{4,4}$ and $E_0 \subseteq E(G)$. If $G[E_0]$ is a linear forest, then $G - E_0$ contains a C_8.*

Lemma 9. *Let $G = (V_1, V_2)$ be a bipartite graph with $|V_1| \ge 4$ and $4 + k \le |V_2| \le 6 + 2k$, where $k \ge 0$ is an integer. If $d(a) \ge 4 + k$ for each $a \in V_1$, then G contains a C_8.*

Proof. We need only to consider the case in which $|V_1| = 4$. Let $P = v_1 v_2 \cdots v_l$ be a longest path of G. Obviously, $l \le 9$. If $v_1 \in V_1$, then by the maximality of P, we have $N(v_1) \subseteq \{v_i \mid i \equiv 0 \pmod 2\}$. Since $d(v_1) \ge 4 + k$, we have $k = 0$, $l = 8$ and $v_1 v_l \in E(G)$, which implies G contains a C_8. Thus we may assume $v_1 \notin V_1$. By symmetry, $v_l \notin V_1$. In this case, we have $l \equiv 1 \pmod 2$, $\{v_i \mid i \equiv 1 \pmod 2\} \subseteq V_2$ and $\{v_i \mid i \equiv 0 \pmod 2\} \subseteq V_1$. Since $d(a) \ge 4 + k$ for each $a \in V_1$ and $|V_2| \le 6 + 2k$, we have $|N(a_i) \cap N(a_j)| \ge 2$ for $a_i, a_j \in V_1$, which implies $l \ge 5$. By the maximality of P, we have $N(v_2) \cap N(a) \cap (V_2 - V(P)) = \emptyset$ for each $a \in V_1 - V(P)$. If $l \ne 9$, then we may assume $a \in V_1 - V(P)$ since $|V_1| = 4$. By the maximality of P, we have $v_1, v_l \notin N(a)$. Thus we have $d_P(a) \le 2$, which implies $|N(a) \cap (V_2 - V(P))| \ge 2 + k$. If $N_P(a) = \emptyset$, then $|N(a) \cap (V_2 - V(P))| \ge 4 + k$. Noting that $N(v_2) \cap N(a) \cap (V_2 - V(P)) = \emptyset$, we have $d(v_2) \le 2 + k$, a contradiction. Since $l = 5$ or 7, by symmetry we assume $v_3 a \in E(G)$. By the maximality of P, $v_2 v_l \notin E(G)$. Thus, noting that $N(v_2) \cap N(a) \cap (V_2 - V(P)) = \emptyset$, we have $d(v_2) \le 3 + k$, a contradiction. Therefore, $l = 9$.

Let $U = V_2 - V(P)$ and $X = \{v_3, v_5, v_7\}$. If G contains no C_8, then we have $d_U(v_2) + d_U(v_8) \le |U| \le 1 + 2k$ and $v_1 v_8, v_2 v_9 \notin E(G)$. Since $d(v_2) + d(v_8) \ge 8 + 2k$, we assume $d_X(v_2) = 3$ and $d_X(v_8) \ge 2$. For $v_i \in \{v_3, v_5\}$, if $v_i v_8 \in E(G)$, then $v_1, v_9 \notin N(v_{i+1})$ for otherwise G contains a C_8. Since $v_{i+1} v_i \overleftarrow{P} v_2 v_{i+2} \overrightarrow{P} v_8$ is a path of order 7, we have $d_U(v_{i+1}) + d_U(v_8) \le 1 + 2k$ and $v_9 v_{i+1}, v_1 v_{i+1} \notin E(G)$. Thus, noting that $d(v_{i+1}) + d(v_8) \ge 8 + 2k$, we have $X \subseteq N(v_8)$. Now, consider $d(v_4) + d(v_6)$. Since $X \subseteq N(v_2) \cap N(v_8)$, we have $v_1, v_9 \notin N(v_4) \cap N(v_6)$. Noting that $v_4 v_3 v_2 v_5 v_8 v_7 v_6$ is a path of order 7, we have $d_U(v_4) + d_U(v_6) \le 1 + 2k$. Thus, we have $d(v_4) + d(v_6) \le 7 + 2k$, a contradiction. So G contains a C_8. ■

Lemma 10. *Let G be a 2-connected graph of order 11 and $\delta(G) \ge 4$. If $c(G) = 9$ or 10, then G contains a C_8.*

Proof. Let $C = t_1 t_2 \cdots t_l$ be a longest cycle of G and $H = G - C$. If $h \in V(H)$ and $d_C(h) \ge 4$, then by Lemma 7, G contains a bipartite graph G_0 between $\{h\} \cup N_C^+(h)$ and $V(G) - (\{h\} \cup N_C^+(h))$, which satisfies the conditions of Lemma 9, and hence G contains a C_8. If $d_C(h) \le 3$ for any $h \in V(H)$, then we have $l = 9$ and $H = K_2$. Let $E(H) = \{h_1 h_2\}$ and $t_1 h_1 \in E(G)$. By the maximality of C, we have $t_2, t_3, t_8, t_9 \notin N(h_2)$. If G contains no C_8,

Please cite this article in press as: Y. Zhang, Y. Chen, K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European Journal of Combinatorics (2007), doi:10.1016/j.ejc.2007.07.005

then we have $t_5, t_6 \notin N(h_2)$ and $|N(h_2) \cap \{t_1, t_4, t_7\}| \leq 1$, which implies $d_C(h_2) \leq 1$, and hence $d(h_2) \leq 2$, a contradiction. ■

Lemma 11. *Let G be a graph of order at least $n+3$ and $\Delta(G) \leq n-2$. Suppose (U, X) is a partition of $V(G)$ with $|U| = 6$ and $\overline{G}[U]$ is $(5, 6)$-connected. If $\overline{G}$ contains no C_8, then $e(U, X) \geq \min\{5|X| - 5, \frac{9}{2}|X|\}$.*

Proof. Let $x_0 \in X$ and $d_U(x_0) = \min\{d_U(x) \mid x \in X\}$. If $d_U(x_0) \leq 2$, then $\overline{G}[U \cup \{x_0\}]$ is Hamilton-connected, which implies $d_U(x) \geq 5$ for any $x \in X - \{x_0\}$. In this case, $e(U, X) \geq 5(|X| - 1) = 5|X| - 5$. If $d_U(x_0) \geq 3$, we let $X_0 = \{x \mid x \in X \text{ and } d_U(x) \leq 4\}$ and x any vertex in X_0. Since $|X| \geq n-3$ and $\Delta(G) \leq n-2$, there is some $x' \in X$ such that $xx' \notin E(G)$. If $d_U(x') \leq 5$, then noting that $\overline{G}[U]$ is $(5, 6)$-connected, we see that $\overline{G}[U \cup \{x, x'\}]$ contains a C_8, and hence we have $d_U(x') = 6$. If $x_1, x_2 \in X_0$ and there is some vertex $x \in X$ such that $x_1, x_2 \notin N(x)$, then since $\overline{G}[U]$ is $(5, 6)$-connected, $\overline{G}$ contains a C_8, a contradiction. Thus we have $|X_0| \leq \frac{1}{2}|X|$, and hence $e(U, X) \geq (3+6)|X_0| + 5(|X| - 2|X_0|) = 5|X| - |X_0| \geq \frac{9}{2}|X|$. ■

Lemma 12. *Let G be a graph of order $2n+2 \geq 22$ and $\Delta(G) \leq n-2$. Suppose H is a graph of order 7 and $\overline{H}$ is Hamilton-connected. If G contains an induced $K_1 \cup H$, then $\overline{G}$ contains a W_8.*

Proof. Let $v \in V(G) - V(H)$, $N(v) = Q$ and $N_H(v) = \emptyset$. Set $B = V(G) - V(H) - N[v]$. If $b \in B$ and $d_H(b) \leq 5$, then since $\overline{H}$ is Hamilton-connected, $\overline{G}[V(H) \cup \{v, b\}]$ contains a W_8 with the hub v. Hence we may assume that $e(H, B) \geq 6|B|$.

Assume $e(H) \leq 2$. If $q \in Q$ and $d_H(q) \leq 2$, then it is not difficult to see that $\overline{G}[V(H) \cup \{v, q\}]$ contains a W_8 with the hub h for some $h \in V(H)$. Thus we have $d_H(q) \geq 3$ for any $q \in Q$, which implies $e(H, Q) \geq 3|Q|$. In this case, $7(n-2) \geq \sum_{h \in H} d(h) \geq 3|Q| + 6|B| = 3|B| + 3(2n-6) \geq 9n - 30 \geq 7n - 10$, a contradiction. Therefore, we have $e(H) \geq 3$. If $e(H) = 3$, we assume $h_0 \in V(H)$ with $d_H(h_0) = 0$ and $F = H - \{h_0\}$. Since $e(F) = 3$, it is easy to see $\overline{F}$ contains a C_6, which implies $\overline{G}[\{v\} \cup V(F)]$ is Hamilton-connected. Thus, if $q \in Q$ such that $d_H(q) = 0$ or $qh_0 \notin E(G)$ and $d_F(q) \leq 4$, then $\overline{G}[V(H) \cup \{v, q\}]$ contains a W_8 with the hub h_0. Hence we may assume $d_H(q) \geq 1$ and if $qh_0 \notin E(G)$, then $d_H(q) \geq 5$ for any $q \in Q$. If $q', q'' \in Q$ and $q', q'' \notin N(h_0)$, then we have $e(H, Q) \geq |Q| + 8$, which implies $7(n-2) \geq \sum_{h \in H} d(h) \geq |Q| + 8 + 6|B| + 2e(H) \geq 7n - 12$, a contradiction. Thus we have $d_Q(h_0) \geq |Q| - 1$. If $q', q'' \in N_Q(h_0)$ such that $d_H(q') \leq 2$ and $d_H(q'') \leq 2$, then $e(V(F), \{v, h_0, q', q''\}) \leq 2$. Since $e(F) = 3$, F contains some h such that $d_H(h) \leq 1$ and $q', q'' \notin N(h)$. Let $U = V(F) - N[h]$ and $|U| = 4$. By Lemma 8, we see that $\overline{G}[U \cup \{h, v, h_0, q', q''\}]$ contains a W_8 with the hub h. Thus we may assume $e(H, Q) \geq 3(|Q| - 2) + 2$, which implies $7(n-2) \geq \sum_{h \in H} d(h) \geq 3|Q| - 4 + 6|B| + 2e(H) \geq 7n - 8$, a contradiction. Therefore, we have $e(H) \geq 4$.

If $e(H, Q) \geq 2|Q| - 3$, then $7(n-2) \geq \sum_{h \in H} d(h) \geq 2|Q| - 3 + 6|B| + 2e(H) \geq 8n - 23 \geq 7n - 13$, and hence we have $e(H, Q) \leq 2|Q| - 4$.

If $|Q| \leq 2$, then $7(n-2) \geq \sum_{h \in H} d(h) \geq 6|B| \geq 6(2n-8) \geq 7n + 2$, a contradiction. Thus we may assume $q_1, q_2, q_3 \in Q$ such that $d_H(q_1) \leq d_H(q_2) \leq d_H(q_3)$ and $d_H(q_3) \leq d_H(q)$ for any $q \in Q - \{q_1, q_2, q_3\}$. Set $X = \{v, q_1, q_2, q_3\}$. If $d_H(q_3) = 0$, then since $|H| = 7$ and $\overline{H}$ is Hamilton-connected, we have $\delta(H) \leq 2$, which implies $|V(H) - N(h)| \geq 4$ for $h \in H$ and $d_H(h) = \delta(H)$, and hence $\overline{G}$ contains a W_8 with the hub h. Thus we have $d_H(q_3) \geq 1$. If $d_H(q_3) \geq 3$, then we have $e(H, Q) \geq 3|Q| - 6 > 2|Q| - 4$, a contradiction. Hence we have $1 \leq d_H(q_3) \leq 2$. Since $\Delta(G) \leq n-2$, we have $2e(H) \leq 7(n-2) - (e(H, Q) + e(H, B)) \leq$

Please cite this article in press as: Y. Zhang, Y. Chen, K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European Journal of Combinatorics (2007), doi:10.1016/j.ejc.2007.07.005

$7(n-2)-(|Q|-2+6|B|)=7(n-2)-(5|B|+2n-6-2)\le 14$, that is, $e(H)\le 7$. Let $U=\{h \mid h\in V(H)$ and $d_H(h)\le 2\}$. Then we have $|U|\ge 3$. If $d_H(q_2)=0$, then since $d_H(q_3)\le 2$, there is some $u\in U$ such that $d_X(u)=0$. Let $Y\subseteq V(H)-N[u]$ and $|Y|=4$. By Lemma 8, we see that $\overline{G}[X\cup Y\cup\{u\}]$ contains a W_8 with hub u. Thus we may assume $d_H(q_2)\ge 1$. In this case, we have $d_H(q_3)=1$ for otherwise $e(H,Q)\ge 2|Q|-3$, and $e(H,Q)\ge |Q|-1$, which implies $e(H)\le 6$. If $|U|=3$, then $H=3K_1\cup K_4$, which contradicts that $\overline{H}$ is Hamilton-connected. Hence we have $|U|\ge 4$. Define $Q_1=\{q \mid q\in Q$ and $d_H(q)\le 1\}$. Obviously, $|Q_1|\ge 3$. If $d_H(q_1)=0$ or $|N_H(Q_1)|\ge 2$, say $|N_H(X)|\ge 2$, then since $|U|\ge 4$, there is some $u\in U$ such that $d_X(u)=0$. Let $Y\subseteq V(H)-N[u]$ and $|Y|=4$. By Lemma 8, $\overline{G}[X\cup Y\cup\{u\}]$ contains a W_8 with the hub u. Thus we have $|N_H(Q_1)|=1$ and $d_H(q_1)=1$. If $h\in V(H)-N_H(Q_1)$ and $d_H(h)\le 1$, then $\overline{G}$ contains a W_8 with the hub h, and hence we have $d_H(h)=2$ for any $h\in V(H)-N_H(Q_1)$. This implies $H=K_1\cup C_6$ or $K_1\cup 2K_3$. Let $N_H(Q_1)=\{h'\}$, then we have $d_H(h')=0$. Noting that $e(H)=6$, we have $d_H(q)=1$ for any $q\in Q$ and $|Q|=n-2$ for otherwise $\Delta(G)\ge n-1$. In this case, Q contains at least two vertices, say q_1,q_2 such that $q_1q_2\notin E(G)$. Let $h_1\in V(H)-\{h'\}$ and $h_2,h_3,h_4\in V(H)-\{h'\}\cup N_H[h_1]$. Then $vh'h_2q_1q_2h_3q_3h_4$ is a C_8 in $\overline{G}$, and hence $\overline{G}$ contains a W_8 with the hub h_1. ■

Lemma 13. *Let G be a graph of order $2n+2\ge 22$ and $\Delta(G)\le n-2$. Suppose H is a linear forest with $|H|=6$, $e(H)\le 3$ and $H\ne K_1\cup K_2\cup P_3$. If G contains an induced $K_1\cup H$, then $\overline{G}$ contains a W_8.*

Proof. Let $v\in V(G)-V(H)$, $N(v)=Q$ and $N_H(v)=\emptyset$. Set $X=V(G)-V(H)-N[v]$. By Lemma 12, we may assume $d_H(q)\ge 3$ if $e(H)\le 1$ and $d_H(q)\ge 2$ if $e(H)=2$ for any $q\in Q$. By Lemmas 6 and 11, we may assume $e(H,X)\ge 4|X|+2$. Thus we have $\sum_{h\in H} d(h)\ge 3|Q|+4|X|+2\ge 6n-6$ if $e(H)\le 1$ and $\sum_{h\in H} d(h)\ge 2|Q|+4|X|+2+4\ge 6n-10$ if $e(H)=2$, which implies $\Delta(G)\ge n-1$, a contradiction. If $e(H)=3$, then $H=3K_2$ or $2K_1\cup P_4$. By Lemma 12, Q has at most one vertex which has no neighbors in H. If $e(H,Q)\ge 2|Q|-3$, then by Lemmas 6 and 11, we have $\sum_{h\in H} d(h)\ge 2|Q|-3+4|X|+2+6\ge 6n-11$, a contradiction. Thus there exists $q_1,q_2,q_3\in Q$ such that $\sum_{i=1}^{3} d_H(q_i)\le 3$. Let $Y=\{v,q_1,q_2,q_3\}$ and $U=\cup_{i=1}^{3} N_H(q_i)$. If $|U|\ge 2$, then since $\sum_{i=1}^{3} d_H(q_i)\le 3$, there is some $h\in V(H)-U$ such that $d_H(h)\le 1$. Let $U'\subseteq V(H)-N[h]$ and $|U'|=4$. Obviously, the subgraph induced by $E(U',Y)$ is a linear forest, which implies $\overline{G}[\{h\}\cup U'\cup Y]$ contains a W_8 with the hub h by Lemma 8. If $|U|=1$, then there is some $h\in V(H)-U$ such that $N(h)\cap(V(H)-U)=\emptyset$. Since $|V(H)-U\cup\{h\}|=4$ and $E(V(H)-(U\cup\{h\}),Y)=\emptyset$, we see that $\overline{G}$ contains a W_8 with the hub h. ■

3. Proof of Theorem 4

Proof of Theorem 4. Obviously, the graph $K_{n-1}\cup\overline{H}$ shows that $R(S_n,W_8)\ge 2n+2$, where $H=\frac{n-4}{4}K_4\cup K_{3,3}$ if $n\equiv 0 \pmod 4$ and $H=\frac{n+2}{4}K_4$ if $n\equiv 2 \pmod 4$. In the following proof, we need only to show that $R(S_n,W_8)\le 2n+2$.

Let G be a graph of order $2n+2$. Suppose to the contrary that neither G contains an S_n nor $\overline{G}$ contains a W_8.

We first consider the case in which $n\le 8$. Let v_0 be a vertex of degree $\Delta(\overline{G})$. Set $H=\overline{G}[N_{\overline{G}}(v_0)]$, $B=V(G)-N_{\overline{G}}[v_0]$ and $F=\overline{G}[B]$. Since G contains no S_n, we have $\delta(\overline{G})\ge (2n+1)-(n-2)=n+3$. Assume $d_{\overline{G}}(v_0)=n+3+l$, where $l\ge 0$ is an integer. Since $|B|=n-2-l$, we have $\delta(H)\ge (n+3)-[(n-2-l)+1]=4+l$.

Please cite this article in press as: Y. Zhang, Y. Chen, K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European Journal of Combinatorics (2007), doi:10.1016/j.ejc.2007.07.005

Since $\overline{G}$ contains no W_8, we see that H contains no C_8.

If $n = 6$, then $|H| = 9 + l$. If $l \geq 1$ or $l = 0$ and $\delta(H) \geq 5$, then we have $\delta(H) \geq |H|/2$, which implies H contains a C_8 by Lemma 1, a contradiction. If $l = 0$ and $\delta(H) = 4$, then H is connected and $\overline{G}$ is 9-regular. If $\kappa(H) = 1$, say u_0 is a cut-vertex, then it is easy to see that $H = \{u_0\} + 2K_4$. Since $\overline{G}$ is 9-regular, we have $N_{\overline{G}}(u_0) \cap B = \emptyset$. For each $h \in V(H) - \{u_0\}$, since $d_H(h) = 4$ and $d_{\overline{G}}(h) = 9$, we have $B \subseteq N_{\overline{G}}(h)$, which implies $F = 2K_2$ since $\overline{G}$ is 9-regular. Thus $\overline{G} = 3K_2 + 2K_4$, and hence $\overline{G}$ contains a W_8, a contradiction. If $\kappa(H) \geq 2$ and H is bipartite, then $H = K_{4,5}$, a contradiction. If $\kappa(H) \geq 2$ and H is non-bipartite, then by Lemmas 2 and 3, H contains a C_8, a contradiction. Hence $R(S_6, W_8) \leq 14$.

If $n = 8$, then $|H| = 11 + l$. If $l \geq 3$, then we have $\delta(H) \geq |H|/2$, which implies H contains a C_8 by Lemma 1, a contradiction. Thus we have $l \leq 2$. Suppose $l \neq 0$. If $\kappa(H) \geq 2$ and H is bipartite, then since $\delta(H) \geq 4+l$ and $|H| = 11+l$, H contains a C_8 by Lemma 9, a contradiction. If $\kappa(H) \geq 2$ and H is non-bipartite, then since $\delta(H) \geq 4 + l \geq [(11 + l) + 2]/3$, by Lemmas 2 and 3, H contains a C_8, a contradiction. If $\kappa(H) \leq 1$, then it is not difficult to see that H contains a subgraph H_1 such that $H_1 = K_5$ and $d_H(h) = 4 + l$ for each $h \in V(H_1)$. Since $\delta(\overline{G}) \geq n + 3$, we have $B \subseteq N_{\overline{G}}(h)$ for each $h \in V(H_1)$. Thus, H_1 together with v_0 and any three vertices of B produce a W_8 in $\overline{G}$, a contradiction. Therefore we have $l = 0$. If H is disconnected, then H contains a component $H_1 = K_5$. Thus, this H_1 together with v_0 and any three vertices of B produce a W_8 in $\overline{G}$, a contradiction. If $\kappa(H) = 1$, we let v_1 be a cut-vertex of H. Since $\delta(H) \geq 4$, $H - v_1$ contains exactly two components H_1, H_2 such that $|H_1| = |H_2| = 5$ or $|H_1| = 4$ and $|H_2| = 6$. If $|H_1| = 5$, then since $\delta(H_1) \geq 3$ and the number of vertices of odd degree is even, H_1 contains a vertex v such that $V(H_1) \subseteq N_{\overline{G}}[v]$. Obviously, $d_H(v) \leq 5$. Since $\delta(\overline{G}) \geq 11$, we may assume $B' \subseteq N_{\overline{G}}(v) \cap B$ and $|B'| = 5$. For each $h \in N_{H_1}(v)$, we have $|N_{\overline{G}}(h) \cap B'| \geq 4$. Thus $\overline{G}$ contains a W_8 with the hub v by Lemma 9, a contradiction. If $|H_1| = 4$, then $V(H_1) \cup \{v_1\}$ is a clique and $B \subseteq N_{\overline{G}}(h)$ for each $h \in V(H_1)$. Since $\delta(\overline{G}) \geq 11$ and $|H| = 11$, we see that either $N_{\overline{G}}(v_1) \cap B \neq \emptyset$ or F is not an independent set. If $N_{\overline{G}}(v_1) \cap B \neq \emptyset$, say $b_1 \in N_{\overline{G}}(v_1) \cap B$, then H_1 together with v_0, v_1, b_1 and any two vertices of $B - \{b_1\}$ form a W_8 in $\overline{G}$, a contradiction. If F is not an independent set, say $b_1 b_2 \in E(F)$, then H_1 together with v_0, v_1, b_1, b_2 and any vertex of $B - \{b_1, b_2\}$ form a W_8 in $\overline{G}$, a contradiction. If $\kappa(H) \geq 2$, then $c(H) \geq 8$ by Lemma 3. By Lemma 10, $c(H) = 11$, that is, H is Hamiltonian. If $\delta(H) \geq 5$, then by Lemmas 2 and 3, H contains a C_8, a contradiction. Thus we have $\delta(H) = 4$. Let $v \in V(H)$ and $d_H(v) = 4$. Since $\delta(\overline{G}) \geq 11$, we have $B \subseteq N_{\overline{G}}(v)$. If $\Delta(H) \leq 6$, then $|N_{\overline{G}}(u) \cap B| \geq 4$ for each $u \in N_H(v)$. Thus G contains a W_8 with the hub v by Lemma 9, a contradiction. If $\Delta(H) \geq 7$, then noting that $\delta(H) = 4$, H contains a C_8 by Lemma 4, a contradiction. Thus $R(S_8, W_8) \leq 18$.

Now, we consider the case in which $n \geq 10$.

Let I be a maximum independent set of G. If $|I| \leq 2$, then G contains an S_n, and hence we have $|I| \geq 3$. By Lemma 13, we have $|I| \leq 6$ and if $|I| = 6$, then $d_I(v) \geq 3$ for any $v \in V(G) - I$. Suppose $|I| = 6$. Since $\sum_{a \in I} d(a) \leq 6(n - 2)$ and $|V(G) - I| = 2n - 4$, we have $d_I(v) = 3$ for any $v \in V(G) - I$ and $d(a) = n - 2$ for each $a \in I$. Let $a \in I$, $N(a) = Q$ and $X = V(G) - I - N[a]$. Obviously, $|X| = n - 2$. Let $u \in X$. Since G contains no S_n and $d_I(u) = 3$, there exists $v, w \in X - \{u\}$ such that $v, w \notin N(u)$. Noting that $d_I(v) = d_I(w) = 3$, we see that $\overline{G}[I \cup \{u, v, w\} - \{a\}]$ contains a C_8, and hence $\overline{G}$ contains a W_8 with the hub a, a contradiction. Thus we have $3 \leq |I| \leq 5$.

In order to consider the cases when $3 \leq |I| \leq 5$, we need the following claim.

Claim 1. Let $H \in \{K_3 \cup K_4, K_3 \cup B_2, P_3 \cup B_2\}$. If $\alpha(G) = \alpha(H) + 1$, then G contains no induced $K_1 \cup H$.

Please cite this article in press as: Y. Zhang, Y. Chen, K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European Journal of Combinatorics (2007), doi:10.1016/j.ejc.2007.07.005

Proof. Let $v \in V(G) - V(H)$, $d_H(v) = 0$, $N(v) = Q$, $R = V(G) - N[v]$ and $U = R - V(H)$. Assume $V(H) = A \cup B$ with $G[A] = K_3$ or P_3, $G[B] = K_4$ or B_2 and $E(A, B) = \emptyset$. Set $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3, b_4\}$. Choose H such that $e(H, U)$ is as large as possible.

We first show that $e(H, U) \geq 6|U|$. Since $\overline{G}$ contains no W_8, we can see that $\overline{G}[R]$ contains no C_8. Define $X = \{u \mid u \in U, A \subseteq N(u) \text{ and } B \not\subseteq N(u)\}$, $Y = \{u \mid u \in U \text{ and } A \cup B \subseteq N(u)\}$ and $Z = \{u \mid u \in U, B \subseteq N(u) \text{ and } A \not\subseteq N(u)\}$. If there is some vertex $u \in U$ such that $d_A(u) \leq 2$ and $d_B(u) \leq 3$, then since $\alpha(G) = \alpha(H) + 1$, we have $\alpha(G) \geq 4$, and hence $G[B] = B_2$. In this case, since $\overline{H}$ contains an (a, b)-path of order 7 for any $a \in A$ and $b \in B$, we see $\overline{G}[R]$ contains a C_8, a contradiction. Thus, (X, Y, Z) is a partition of U.

If $d_B(u) \leq 2$ for some $u \in U$, say $b_1, b_2 \notin N(u)$, then $a_1b_1ub_2a_2b_3a_3b_4$ is a C_8 in $\overline{G}[R]$. If $xz \notin E(G)$ for some $x \in X$ and $z \in Z$, then since $\overline{H}$ contains an (a, b)-path of order 6 for any $a \in A$ and $b \in B$, we see $\overline{G}[R]$ contains a C_8. Thus we have $d_B(u) \geq 3$ for each $u \in U$ and $X \subseteq N(z)$ for each $z \in Z$. If $Z = \emptyset$, then we have $e(H, U) \geq 6|U|$. Hence we may assume $Z \neq \emptyset$. Define $Z_i = \{z \mid z \in Z \text{ and } d_A(z) = i\}$ for $i = 0, 1, 2$.

Let $z \in Z_0$. If there is some $z' \in Z$ such that $zz' \notin E(G)$, then we have $\alpha(G) \geq 4$, and hence $G[B] = B_2$. Assume without loss of generality that $b_1b_2, z'a_1 \notin E(G)$. Then $a_1z'za_2b_1b_2a_3b_3$ is a C_8 in $\overline{G}[R]$, and thus we have $Z \subseteq N[z]$. Since G contains no S_n, we have $|Q| \leq n - 2$ and $|U| \geq n - 4$. Thus $d_Y(z) \leq |Y| - 1$. If $d_Y(z) = |Y| - 1$, then we must have $|Q| = n - 2$, $|U| = n - 4$ and $d_R(z) = n - 2$. By the choice of H, we have $d_R(b_1) = d_R(b_2) = n - 2$, where $d_B(b_1) = d_B(b_2) = 3$. Assume $d_A(a_1) = 2$. Since $d_Q(a_1) + d_Q(b_3) \leq 2(n - 2) - [(|U| + 1) + 2 + 2] = n - 5$, there exists $q_1, q_2, q_3 \in Q$ such that $q_1, q_2, q_3 \notin N(a_1) \cup N(b_3)$. In this case, $\overline{G}[\{a_1, v, q_1, q_2, q_3, b_1, b_2, b_3, z\}]$ contains a W_8 with the hub a_1, a contradiction. Hence we have $d_Y(z) \leq |Y| - 2$ for any $z \in Z_0$.

Let $z \in Z_1$. If $d_Y(z) = |Y|$, then there exists $z_1 \in Z - \{z\}$ such that $z_1 \notin N(z)$ since $\Delta(G) \leq n - 2$. Assume $a_1z_1, a_2z \notin E(G)$. If $G[B] = B_2$, say $b_1b_2 \notin E(G)$, then $a_1z_1za_2b_1b_2a_3b_3$ is a C_8 in $\overline{G}[R]$, and hence we have $\alpha(G) = 3$. In this case, we have $a_2, a_3 \notin N(z)$ and $a_2, a_3 \in N(z_1)$. If $z_2 \in Z - \{z, z_1\}$ and $z_1z_2 \notin E(G)$, then since $\alpha(G) = 3$, we have $a_2 \notin N(z_2)$ or $a_3 \notin N(z_2)$, which implies $a_1b_1a_2z_2z_1za_3b_2$ or $a_1b_1a_3z_2z_1za_2b_2$ is a C_8 in $\overline{G}[R]$, and hence we have $Z - \{z\} \subseteq N[z_1]$. Since $d(z_1) \leq n - 2$, $z_1 \in Z_2$ and $X \subseteq N(z_1)$, we have $Y \not\subseteq N(z_1)$. Thus there is some $y \in Y$ and $z' \in Z - \{z\}$ such that $y, z \notin N(z')$ if $z \in Z_1$ and $d_Y(z) = |Y|$.

Let $z \in Z_0 \cup Z_1$. Define $N^*(z) = \{y \mid y \in Y \text{ and } yz \notin E(G)\}$ if $d_Y(z) \leq |Y| - 1$ and $N^*(z) = \{y \mid y \in Y \text{ and } y, z \notin N(z') \text{ for some } z' \in Z\}$ if $d_Y(z) = |Y|$. By the argument above, we have $|N^*(z)| \geq 2$ if $z \in Z_0$ and $|N^*(z)| \geq 1$ if $z \in Z_1$. Assume $z_1, z_2 \in Z_0 \cup Z_1$ and $y \in N^*(z_1) \cap N^*(z_2) \neq \emptyset$. If $d_Y(z_1) \leq |Y| - 1$, then there is some $z'_1 \in Z - \{z_1\}$ such that $z_1, z'_1 \notin N(y)$. Thus we can choose two vertices, say $a_1, a_2 \in A$ such that $z_1a_1, z'_1a_2 \notin E(G)$, which implies $a_1z_1yz'_1a_2b_1a_3b_2$ is a C_8 in $\overline{G}[R]$, a contradiction. Hence by symmetry we have $d_Y(z_1) = d_Y(z_2) = |Y|$, and thus $z_1, z_2 \in Z_1$. Assume $z'_i \in Z$ and $z_iz'_i, yz'_i \notin E(G)$ for $i = 1, 2$. Since $z'_1z_2, z'_2z_1 \in E(G)$, we have $z'_1 \neq z'_2$. Since $z_1, z_2 \in Z_1$, we can choose two vertices, say $a_1, a_2 \in A$ such that $z_1a_1, z_2a_2 \notin E(G)$, which implies $a_1z_1z'_1yz'_2z_2a_2b_1$ is a C_8 in $\overline{G}[R]$, a contradiction. Hence we have $N^*(z_1) \cap N^*(z_2) = \emptyset$ for any $z_1, z_2 \in Z_0 \cup Z_1$. Let $Y_0 = \cup_{z \in Z_0} N^*(z)$, $Y_1 = \cup_{z \in Z_1} N^*(z)$ and $Y_2 = Y - Y_0 - Y_1$, then $|Y_0| \geq 2|Z_0|$ and $|Y_1| \geq |Z_1|$. Thus $e(H, U) = e(H, X \cup Y_2 \cup Z_2) + (e(H, Z_0) + e(H, Y_0)) + (e(H, Z_1) + e(H, Y_1)) \geq 6|U|$.

If $|Q| \leq 2$, then $7(n - 2) \geq \sum_{h \in H} d(h) \geq 6|U| \geq 6(2n - 8) \geq 7n + 2$, and hence $|Q| \geq 3$. If $q_1, q_2, q_3 \in Q$ and $d_H(q_1) + d_H(q_2) + d_H(q_3) \leq 1$, then since $|A| = 3$, there is some $a \in A$ such that $q_1, q_2, q_3 \notin N(a)$. By Lemma 8, $\overline{G}[B \cup \{v, q_1, q_2, q_3\}]$ contains a C_8, and hence $\overline{G}$ contains a W_8 with the hub a, a contradiction. Thus we have $e(H, Q) \geq |Q| - 1$, which implies $7(n-2) \geq \sum_{h\in H} d(h) \geq e(H,Q)+e(H,U)+2e(H) \geq (|Q|-1)+6|U|+14 = 5|U|+(2n-6)+13 \geq 5(n-4)+(2n-6)+13 = 7n-13$, a contradiction. ■

We now consider the following three cases separately.

Case 1. $\alpha(G)=3$

If G contains an induced $3K_2$, we assume $U=\{u_i \mid 1\leq i\leq 6\}$ and $E(G[U]) = \{u_1u_2, u_3u_4, u_5u_6\}$. Set $V(G)-U=X$. Since G contains no S_n, we have $e(U,X)\leq 6(n-3)$. Since $\alpha(G)=3$, we have $d_U(x)\geq 2$ for each $x\in X$ and if $d_U(x)=2$, then $G[N_U(x)]=K_2$. Since $|X|=2n-4$ and $e(U,X)\leq 6(n-3)$, X contains at least four vertices, say x_i $(1\leq i\leq 4)$ such that $d_U(x_i)=2$. This implies G contains an induced $2K_2\cup K_4$. Assume $Y=\{u_i \mid 1\leq i\leq 8\}$ and $E(G[Y])=\{u_1u_2,u_3u_4\}\cup\{u_iu_j \mid 5\leq i<j\leq 8\}$. Set $V(G)-Y=Z$. Since G contains no S_n, we have $e(Y,Z)\leq 8(n-2)-16=8n-32$. Since $|Z|=2n-6$, it follows that Z contains at least four vertices, say z_i $(1\leq i\leq 4)$ such that $d_Y(z_i)\leq 3$. Since $\alpha(G)=3$, we have $|N(z_i)\cap\{u_5,u_6,u_7,u_8\}|\leq 1$ for $1\leq i\leq 4$ and either $u_1,u_2\in N(z_i)$ or $u_3,u_4\in N(z_i)$. Assume without loss of generality that $u_1,u_2\in N(z_i)$ for $i=1,2$. By Claim 1, we have $|N(z_i)\cap\{u_5,u_6,u_7,u_8\}|=1$ for $i=1,2$. By Lemma 8, $\overline{G}[Y\cup\{z_1,z_2\}-\{u_4\}]$ contains a W_8 with the hub u_3, a contradiction. Therefore, G contains no induced $3K_2$.

Since G contains no S_n, $V(G)-I$ contains a vertex v such that $d_I(v)=1$, which implies G contains an induced $2K_1\cup K_2$. Let $G_0=2K_1\cup K_2$. For the same reason, $V(G)-V(G_0)$ contains a vertex v such that $d_{G_0}(v)=1$, which implies G contains an induced $K_1\cup 2K_2$ since $\alpha(G)=3$. Let $U=\{u_i \mid 1\leq i\leq 4\}$ and $E(G[U\cup\{u_0\}])=\{u_1u_2,u_3u_4\}$. Set $N(u_0)=X$ and $Y=V(G)-U-N[u_0]$. Since G contains no induced $3K_2$, we have $e(U,X)\geq |X|$. If $d_U(y)\geq 3$ for each $y\in Y$, then $4(n-2)\geq\sum_{i=1}^{4} d(u_i)=e(U,X)+e(U,Y)+2e(G[U])\geq 4n-1$, and hence there is some $u_5\in Y$ such that $d_U(u_5)\leq 2$. Since $\alpha(G)=3$, we may assume without loss of generality that $N_U(u_5)=\{u_3,u_4\}$. Let $A=\{u_i \mid 0\leq i\leq 5\}$ and $B=V(G)-A$. Obviously, $G[A]=K_1\cup K_2\cup K_3$. Since $\alpha(G)=3$ and G contains no induced $3K_2$, we have $d_A(b)\geq 2$ for each $b\in B$. Set $B_0=\{b \mid b\in B \text{ and } d_A(b)=2\}$. Since $\sum_{i=0}^{5} d_B(u_i)\leq 6(n-2)-8=6n-20$ and $3|B|=6n-12$, we have $|B_0|\geq 8$. If $b_1,b_2\in B_0-N(u_0)$, then since $\alpha(G)=3$, we have $N_A(b_1)=N_A(b_2)=\{u_1,u_2\}$ and $b_1b_2\in E(G)$, which contradicts Claim 1. Thus we have $d_{B_0}(u_0)\geq 7$. Since G contains no induced $3K_2$, we have $N_A(b)\subseteq\{u_0,u_1,u_2\}$ for any $b\in N_{B_0}(u_0)$. Assume without loss of generality that $b_i\in N_{B_0}(u_0)$ for $1\leq i\leq 3$ and $N_A(b_i)=\{u_0,u_1\}$. Since $\alpha(G)=3$, we have $b_ib_j\in E(G)$ for $1\leq i<j\leq 3$, which contradicts Claim 1.

Case 2. $\alpha(G)=4$

If G has an induced $2K_1\cup K_2\cup K_4$, we let $V(H)=X\cup Y$, $X=\{x_1,x_2,x_3,x_4\}$, $E(G[X])=\{x_3x_4\}$, $G[Y]=K_4$ and $E(X,Y)=\emptyset$. Set $Z=V(G)-V(H)$. By Lemma 13, $d_H(z)\geq 2$ for any $z\in Z$. Let $Z_0=\{z \mid z\in Z \text{ and } d_H(z)\leq 3\}$. Since $\Delta(G)\leq n-2$, we have $e(H,Z)\leq 8(n-2)-14=8n-30$, which implies $|Z_0|\geq 3$. Let $z\in Z_0$. Since $\alpha(G)=4$, we have $d_Y(z)\leq 2$. If $x_1z\notin E(G)$, then $\overline{G}[V(H)\cup\{z\}]$ contains a W_8 with the hub x_1 by Lemma 8, and hence we have $x_1,x_2\in N(z)$ for any $z\in Z_0$. Since $|Z_0|\geq 3$, Z_0 contains two vertices, say z_1,z_2, such that $z_1,z_2\notin N(x_3)$ or $z_1,z_2\notin N(x_4)$, and hence $\overline{G}$ contains a W_8 with the hub x_3 or x_4 by Lemma 8, a contradiction. Therefore G contains no induced $2K_1\cup K_2\cup K_4$.

If G contains an induced $K_1\cup K_2\cup P_4$, we assume $U=\{u_i \mid 1\leq i\leq 6\}$ and $E(G[U\cup\{u_0\}])=\{u_iu_{i+1} \mid i=1,3,4,5\}$. Set $N(u_0)=Q$ and $X=V(G)-U-N[u_0]$. By Lemma 11, $e(U,X)\geq 4|X|+2$. If $e(U,Q)\geq 2|Q|-4$, then $6(n-2)\geq\sum_{i=1}^{6} d(u_i)\geq 2|Q|-4+4|X|+2+8=2|X|+2(2n-5)+6\geq 6n-10$, a contradiction. Thus there exists

$q_1, q_2, q_3 \in Q$ such that $\sum_{i=1}^{3} d_U(q_i) \leq 3$. Let $Y = \{u_0, q_1, q_2, q_3\}$ and $Z = \cup_{i=1}^{3} N_U(q_i)$. If $|Z| \geq 2$ or $\sum_{i=1}^{3} d_U(q_i) \leq 2$, then there exists $u \in U - \{u_4, u_5\}$ such that $u \notin Z$. By Lemma 8, $\overline{G}[(U - N(u)) \cup Y]$ contains a W_8 with the hub u, a contradiction. Thus we have $\sum_{i=1}^{3} d_U(q_i) = 3$ and $|Z| = 1$. If $u_3, u_6 \notin Z$, then there is some $u \in U - \{u_4, u_5\}$ such that $u \notin Z$ and $E(U - N[u], Y) = \emptyset$, and hence $\overline{G}$ contains a W_8 with the hub u, a contradiction. Thus by symmetry we may assume $Z = \{u_6\}$. Since $\alpha(G) = 4$, we have $q_i q_j \in E(G)$ for $1 \leq i < j \leq 3$, which implies G contains an induced $2K_1 \cup K_2 \cup K_4$, a contradiction. Hence G contains no induced $K_1 \cup K_2 \cup P_4$.

If G has an induced $2K_1 \cup 2K_2$, we let $U = \{u_i \mid 1 \leq i \leq 5\}$ and $E(G[U \cup \{u_0\}]) = \{u_2u_3, u_4u_5\}$. Set $X = V(G) - U \cup \{u_0\}$, $N(u_0) = Y$ and $X - Y = Z$. Since $\alpha(G) = 4$ and G contains no induced $K_1 \cup 3K_2$ by Lemma 13, we have $d_U(z) \geq 2$ for any $z \in Z$. Define $Z_i = \{z \mid z \in Z \text{ and } d_U(z) = i\}$ for $2 \leq i \leq 5$. Let $z \in Z_3$. Since $\Delta(G) \leq n - 2$, we have $|Z| \geq n - 2$, and hence there exists $z', z'' \in Z - N[z]$. If $\{z', z''\} \cap Z_5 = \emptyset$, then $z', z'' \in Z_4$ for otherwise $\overline{G}[U \cup \{z, z', z''\}]$ contains a C_8 since $\overline{G}[U] = W_4$ is Hamilton-connected, which implies $\overline{G}$ contains a W_8 with the hub u_0, a contradiction. For the same reason, we have $N_{\overline{G}}(z') \cap Z_5 \neq \emptyset$. Let $N^*(z) = N_{\overline{G}}(z) \cap Z_5$ if $N_{\overline{G}}(z) \cap Z_5 \neq \emptyset$ and $N^*(z) = \{x \mid x \in Z_5 \text{ and } z, x \notin N(x') \text{ for some } x' \in Z_4\}$ if $N_{\overline{G}}(z) \cap Z_5 = \emptyset$. By the argument above, $N^*(z) \neq \emptyset$ for any $z \in Z_3$. If $z_1, z_2 \in Z_3$ and $z_0 \in N^*(z_1) \cap N^*(z_2)$, then $\overline{G}[Z]$ contains a (z_1, z_2)-path of order k with $3 \leq k \leq 5$. Note that $\overline{G}[U]$ is $(3, 5)$-connected, we see that $\overline{G}$ contains a W_8 with the hub u_0, and hence $N^*(z_1) \cap N^*(z_2) = \emptyset$, which implies $|Z_3| \leq |Z_5|$. Therefore we have $e(U, Z) \geq 4|Z| - 2|Z_2|$. By Lemma 13, $e(U, Y) \geq |Y|$. Since G contains no S_n, we have $5(n-2) \geq \sum_{i=1}^{5} d(u_i) \geq |Y| + 4|Z| - 2|Z_2| + 4 = 3|Z| + (2n-4) - 2|Z_2| + 4 \geq 5n - 6 - 2|Z_2|$, and hence $|Z_2| \geq 2$. Because G contains no induced $K_1 \cup K_2 \cup P_4$ and $\alpha(G) = 4$, $N_U(z) = \{u_2, u_3\}$ or $\{u_4, u_5\}$ for any $z \in Z_2$. Note that G contains no induced $2K_1 \cup K_2 \cup K_4$ and $\alpha(G) = 4$, there exists $z_1, z_2 \in Z_2$ such that $N_U(z_1) = \{u_2, u_3\}$ and $N_U(z_2) = \{u_4, u_5\}$. In this case, $\mathrm{cl}(\overline{G}[U \cup \{z_1, z_2\}]) = K_7$. By Lemma 5, $\overline{G}[U \cup \{z_1, z_2\}]$ is Hamilton-connected, which contradicts Lemma 12. Thus G contains no induced $2K_1 \cup 2K_2$.

If G has an induced $3K_1 \cup K_3$, we let $U = \{u_i \mid 1 \leq i \leq 6\}$ and $E(G[U]) = \{u_4u_5, u_5u_6, u_4u_6\}$. Set $X = V(G) - U$. Since $\alpha(G) = 4$ and G contains no induced $2K_1 \cup 2K_2$, we have $d_U(x) \geq 2$ for each $x \in X$. Let $X_0 = \{x \mid x \in X \text{ and } d_U(x) = 2\}$. Since $\sum_{u \in U} d(u) \leq 6(n-2)$ and $|X| = 2n - 4$, we have $|X_0| \geq 6$. Let $x \in X_0$. Note that $\alpha(G) = 4$ and G contains no induced $2K_1 \cup 2K_2$, we have $N(x) \subseteq \{u_1, u_2, u_3\}$. Thus, since $|X_0| \geq 6$, there exists $x_1, x_2 \in X_0$ such that $N_U(x_1) = N_U(x_2)$. Assume without loss of generality that $N_U(x_1) = N_U(x_2) = \{u_2, u_3\}$. By Claim 1, we have $x_1x_2 \notin E(G)$. In this case, $\mathrm{cl}(\overline{G}[U \cup \{x_1, x_2\} - \{u_1\}]) = K_7$. By Lemma 5, $\overline{G}[U \cup \{x_1, x_2\} - \{u_1\}]$ is Hamilton-connected, which contradicts Lemma 12. Thus G contains no induced $3K_1 \cup K_3$.

Let $I = \{u_0, u_1, u_2, u_3\}$, $V(G) - I = X$ and $X_1 = \{x \mid x \in X \text{ and } d_I(x) = 1\}$. Since $|X| = 2n - 2$ and $\Delta(G) \leq n - 2$, we have $|X_1| \geq 4$. If $|X_1| \geq 5$ or $d_{X_1}(u_i) \geq 2$ for some i with $0 \leq i \leq 3$, then G contains an induced $3K_1 \cup K_3$ since $\alpha(G) = 4$, a contradiction. Thus we have $|X_1| = 4$ and $d_{X_1}(u_i) = 1$ for $0 \leq i \leq 3$, which implies $d_I(x) = 2$ for any $x \in X - X_1$ and $d(u_i) = n - 2$ for $0 \leq i \leq 3$. Let $N(u_0) = Y$ and $Z = X - Y$, then $|Z| = n$. Assume $Z_0 = \{v_i \mid 1 \leq i \leq 3\} \subseteq X_1$ and $u_iv_i \in E(G)$. Set $Z_{ij} = \{z \mid z \in Z \text{ and } N_U(z) = \{u_i, u_j\}\}$ for $1 \leq i < j \leq 3$. By the arguments above, we see that $(Z_0, Z_{12}, Z_{23}, Z_{13})$ is a partition of Z. If $z \in Z - Z_0$ and $d_{Z_0}(z) = 0$, then $\mathrm{cl}(\overline{G}[Z_0 \cup I \cup \{z\} - \{u_0\}]) = K_7$. By Lemma 5, $\overline{G}[Z_0 \cup I \cup \{z\} - \{u_0\}]$ is Hamilton-connected, which contradicts Lemma 12. Thus $d_{Z_0}(z) \geq 1$ for any $z \in Z - Z_0$. Since G contains no induced $2K_1 \cup 2K_2$, we have $G[Z_0] = K_3$. Since

Please cite this article in press as: Y. Zhang, Y. Chen, K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European Journal of Combinatorics (2007), doi:10.1016/j.ejc.2007.07.005

$|Z| = n$, there exists $u \in Z$ such that $v_1u \notin E(G)$. Obviously, $u \notin Z_0$. Since $d_I(u) = 2$, $|Z| = n$ and $d_{Z_0}(u) \geq 1$, there exists $v \in Z - Z_0 \cup \{u\}$ such that $uv \notin E(G)$. If $v \in Z_{12} \cup Z_{13}$, then $v_1uvu_3v_2u_1v_3u_2$ or $v_1uvu_2v_3u_1v_2u_3$ is a C_8 in $\overline{G} - N[u_0]$, a contradiction. If $v \in Z_{23}$, then $v_2, v_3 \in N(v)$ for otherwise $v_1u_2v_3u_1u_3v_2vu$ or $v_1u_3v_2u_1u_2v_3vu$ is a C_8 in $\overline{G} - N[u_0]$, and hence there exists $w \in Z - Z_0 \cup \{u, v\}$ such that $wv \notin E(G)$. In this case, $v_1u_2v_3u_1u_3wvu$ or $v_1u_3v_2u_1u_2wvu$ or $v_1u_3u_2v_3u_1wvu$ is a C_8 in $\overline{G} - N[u_0]$, also a contradiction.

Case 3. $\alpha(G) = 5$

If G has an induced $2K_1 \cup K_2 \cup P_3$ or $2K_1 \cup P_5$, we let $H \in \{K_1 \cup K_2 \cup P_3, K_1 \cup P_5\}$, $v \in V(G) - V(H)$ and $N_H(v) = \emptyset$. Set $N(v) = Q$ and $X = V(G) - V(H) - N[v]$. Let $h_0 \in V(H)$ and $d_H(h_0) = 0$. If $q \in Q$, then by Lemmas 5 and 12, $d_H(q) \geq 1$ and if $d_H(q) = 1$, then $N_H(q) = \{h_0\}$. If $q_i \in Q$ and $d_H(q_i) = 1$ for $1 \leq i \leq 3$, then we may assume $q_1q_2 \in E(G)$ since $\alpha(G) = 5$, which contradicts Claim 1. Thus we have $e(H, Q) \geq 2|Q| - 2$. By Lemma 11, we have $6(n-2) \geq \sum_{h\in H} d(h) \geq e(H, Q) + e(H, X) + 2e(H) \geq 2|Q| - 2 + 4|X| + 2 + 6 \geq 6n - 10$, a contradiction. Thus G contains no induced $2K_1 \cup K_2 \cup P_3$ and $2K_1 \cup P_5$.

If G has an induced $4K_1 \cup K_2$, we let $U = \{u_i \mid 1 \leq i \leq 6\}$ and $E(G[U]) = \{u_5u_6\}$. Set $X = V(G) - U$. By Lemma 13, $d_U(x) \geq 2$ for any $x \in X$. Since $|X| = 2n - 4$ and $\sum_{u\in U} d_X(u) \leq 6(n-2) - 2$, X contains at least two vertices, say x_1, x_2 such that $d_U(x_1) = d_U(x_2) = 2$. By Lemma 13, G contains no induced $3K_1 \cup P_4$. Thus noting that G contains no induced $2K_1 \cup K_2 \cup P_3$, we have $N_U(x_1) = N_U(x_2) = \{u_5, u_6\}$. Since $\alpha(G) = 5$, we have $x_1x_2 \in E(G)$. Now, let $U' = U \cup \{x_1, x_2\}$ and $X' = V(G) - U'$. Since $\sum_{u\in U'} d(u) \leq 8(n-2)$, $e(G[U']) = 6$ and $|X'| = 2n - 6$, X' contains a vertex x such that $d_{U'}(x) \leq 3$. Since $\alpha(G) = 5$, we have $|N(x) \cap \{u_5, u_6, x_1, x_2\}| \leq 2$. By Lemma 8, $\overline{G}$ contains a W_8 with the hub u_i for some $u_i \in U - \{u_5, u_6\}$, a contradiction. Hence G contains an induced $4K_1 \cup K_2$ is impossible.

If G has an induced $3K_1 \cup P_3$, we let $U = \{u_i \mid 1 \leq i \leq 6\}$ and $E(G[U]) = \{u_4u_5, u_5u_6\}$. Set $X = V(G) - U$. Since $\alpha(G) = 5$ and G contains no induced $4K_1 \cup K_2$, we have $d_U(x) \geq 2$ for any $x \in X$. Let $X_0 = \{x \mid x \in X \text{ and } d_U(x) = 2\}$. Since $e(G[U]) = 2$, $|X| = 2n - 4$ and $\Delta(G) \leq n-2$, we have $|X_0| \geq 4$. Since G contains no induced $2K_1 \cup P_5$ and $4K_1 \cup K_2$, we have $N_U(x) \subseteq \{u_1, u_2, u_3\}$ or $\{u_4, u_5, u_6\}$ for any $x \in X_0$. Let $x_1 \in X_0$. If $N_U(x_1) \subseteq \{u_4, u_5, u_6\}$, then $N_U(x_1) = \{u_4, u_6\}$ since G contains no induced $4K_1 \cup K_2$. Let $x_2 \in X_0 - \{x_1\}$. By Lemmas 5 and 12, we have $N_U(x_2) \subseteq \{u_4, u_5, u_6\}$, and hence $N_U(x_2) = \{u_4, u_6\}$. Since $\alpha(G) = 5$, we have $x_1x_2 \in E(G)$, which contradicts that G contains no induced $4K_1 \cup K_2$. Thus we have $N_U(x) \subseteq \{u_1, u_2, u_3\}$ for each $x \in X_0$. Noting that $|X_0| \geq 4$, there exists $x_1, x_2 \in X_0$ such that $N_U(x_1) = N_U(x_2)$. Assume $N_U(x_1) = N_U(x_2) = \{u_2, u_3\}$. By Lemmas 5 and 12, we have $x_1x_2 \in E(G)$, which contradicts Claim 1. Thus G contains an induced $3K_1 \cup P_3$ is also impossible.

On the other hand, since $\Delta(G) \leq n-2$, $|I| = 5$ and $|V(G) - I| = 2n-3$, $V(G) - I$ contains a vertex v such that $d_I(v) \leq 2$, which implies G contains an induced $4K_1 \cup K_2$ or $3K_1 \cup P_3$, a contradiction.

By now, we have shown $R(S_n, W_8) \leq 2n + 2$. Therefore, we have $R(S_n, W_8) = 2n + 2$ for $n \geq 6$ and $n \equiv 0 \pmod 2$. The proof of Theorem 4 is completed. ■

References

[1] J.A. Bondy, Pancyclic graphs, Journal of Combinatorial Theory, Series B 11 (1971) 80–84.
[2] J.A. Bondy, V. Chvátal, A method in graph theory, Discrete Mathematics 15 (1976) 111–135.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, 1976, Elsevier, New York.

[4] S. Brandt, R.J. Faudree, W. Goddard, Weakly pancyclic graphs, Journal of Graph Theory 27 (1998) 141–176.
[5] Y.J. Chen, F. Tian, B. Wei, Degree sums and path-factors in graphs, Graphs and Combinatorics 17 (2001) 61–71.
[6] Y.J. Chen, Y.Q. Zhang, K.M. Zhang, The Ramsey numbers of stars versus wheels, European Journal of Combinatorics 25 (2004) 1067–1075.
[7] G.A. Dirac, Some theorems on abstract graphs, Proceedings of the London Mathematical Society 2 (1952) 69–81.
[8] Surahmat, E.T. Baskoro, On the Ramsey number of path or star versus W_4 or W_5, in: Proceedings of the 12th Australasian Workshop on Combinatorial Algorithms, Bandung, Indonesia, 14–17 July 2001, pp. 174–179.
[9] S.M. Zhang, Pansyslism and bipancyclism of hamiltonian graphs, Journal of Combinatorial Theory, Series B 60 (1994) 159–168.

Please cite this article in press as: Y. Zhang, Y. Chen, K. Zhang, The Ramsey numbers for stars of even order versus a wheel of order nine, European Journal of Combinatorics (2007), doi:10.1016/j.ejc.2007.07.005