

A New Assignment Model and its Algorithm ^{1,2}

Song Zeng-min

(Department of Mathematics

Southeast University

Nanjing, 210018, People's Republic of China)

Zhang Ke-min

(Department of Mathematics, Nanjing University,

Nanjing, 210008, People's Republic of China)

Abstract. In [1], [2], there are many assignment models. This paper gives a new assignment model and an algorithm for solving this problem.

In a certain company, n workers A_1, \dots, A_n are available for m jobs B_1, \dots, B_m , the effectiveness of the workers in these various jobs may be distinct; we wish to take an assignment that maximises the total effectiveness of the workers. The problem of finding such an assignment is known as the *optimal assignment problem*. This problem is equivalent to that of finding a maximum-weight matching in a weighted complete bipartite graph; we shall refer to such a matching as an *optimal matching*. We suppose that $m = n$.

It is clear that if the effectiveness of every worker in these various jobs is greater than 0, then the optimal matching must be a perfect matching. Otherwise the optimal matching is not necessarily a perfect matching; that is, it is possible that there exists an assignment that maximises the total effectiveness without needing n workers. In this case, one is interested in an assignment that maximises the total effectiveness and minimizes the number of workers. This problem is equivalent to finding a maximum-weight matching M such that $|M|$ is as small as possible. We shall refer to such a matching as a *minimum cardinality optimal matching*. In the following, we prove several Theorems, and then present a good algorithm for finding a minimum cardinality optimal matching in a weighted complete bipartite graph.

Let $G = (X, Y, W)$ be a weighted complete bipartite graph. As in [1, p.86], a *feasible vertex labelling* is a real-valued function l on the vertex set $X \cup Y$ such that for all $x \in X$ and $y \in Y$

$$l(x) + l(y) \geq w(xy). \quad (1)$$

¹The project supported by National Natural Science Foundation of China.

²This paper was accepted to appear in the Proceedings of the Second International Conference on Combinatorial Mathematics and Computing (Ars Combinatorial, vol. 26 A/B), but was omitted due to an error of the proceedings editor.

The real number $l(v)$ is called the label of the vertex v . No matter what the edge weights are, there always exists a feasible vertex labelling.

Let $E_l = \{xy \mid l(x) + l(y) = w(xy)\}$, where l is feasible vertex labelling. The spanning subgraph of G with edge set E_l is denoted by G_l .

Theorem 1. *Let l be a feasible vertex labelling of G . If G_l contains a perfect matching M , then*

(a) M is an optimal matching of G ;

(b) Any optimal matching M^* of G is contained in G_l .

Proof: According to Theorem 5.5 in [1] we only need to prove (b). Let M^* be any optimal matching and M any perfect matching of G_l . It follows from (a) that

$$w(M^*) = w(M) \quad (2)$$

If M^* is not contained in G_l , let $M^* = E_1 \cup E_2$, where $E_1 \subseteq E \setminus E_l$, $E_2 \subseteq E_l$ and E_1 is nonempty. It follows from [1] that

$$w(M^*) = \sum_{e \in M^*} w(e) = \sum_{e \in E_1} w(e) + \sum_{e \in E_2} w(e) < \sum_{v \in V} l(v) \quad (3)$$

Since M is a perfect matching of G ,

$$w(M) = \sum_{e \in M} w(e) = \sum_{v \in V} l(v). \quad (4)$$

By (3) and (4), we have $w(M^*) < w(M)$. This contradicts (2), hence M^* is contained in G_l . #

Let M be an optimal matching of G and $P = x_1y_1 \dots x_ky_k$ an M -alternating path; if $x_1y_1, x_ky_k \in M$ and

$$\sum_{e \in E(P) \setminus M} w(e) = \sum_{e \in E(P) \cap M} w(e),$$

then such a path P is known as an M -adjusting path. Clearly, $M' = M \Delta E(P)$ is also an optimal matching of G and $|M'| = |M| - 1$, where $A \Delta B$ denotes the symmetric difference of A and B .

Theorem 2. *Let M be an optimal matching of G . Then M is a minimum cardinality optimal matching of G if and only if G contains no M -adjusting path.*

proof: Let M be an optimal matching in G , and suppose that G contains an M -adjusting path P . By the definition, we may obtain another optimal matching M' and $|M'| = |M| - 1$. Thus M is not a minimum cardinality optimal matching in G .

Conversely, suppose that M is not a minimum cardinality optimal matching in G . Let M' be a minimum cardinality optimal matching; then

$$|M'| < |M|. \quad (5)$$

Let $H = G[M \Delta M']$. Each vertex of H has degree either one or two in H . Thus each component of H is either an even cycle or a path with edges alternately in M' and M . For each component H_1 of H ,

$$\sum_{e \in E(H_1) \cap M} w(e) = \sum_{e \in E(H_1) \cap M'} w(e) \quad (6)$$

since both M and M' are the optimal matching. By (5), the number of edges of M , which H contains, is greater than that of M' , and there exists some path component P of H which starts and ends edges of M . By (6) and the definition of M-adjusting path, the path is an M-adjusting path in G . #

Theorem 1 provides the basis of a good algorithm for finding a minimum cardinality optimal matching in a bipartite weighted graph. If a feasible vertex labeling l is found such that G_l contains a perfect matching M , then by theorem 1, this matching is optimal and any optimal matching in G is contained in G_l . By theorem 2, we start with an arbitrary perfect matching M of G_l . If G_l contains no M-adjusting path, then M is. If not, we choose an M-adjusting path P in G_l . $\hat{M} = M \Delta E(P)$ is another optimal matching which satisfies $|\hat{M}| < |M|$. So it remains to determine whether G contains an M-adjusting path, and to find one if one exists.

THEOREM 3. *If G_l contains an M-adjusting path, then there exists an edge with weight 0 belongs to G_l .*

Proof: Let $P = x_1 y_1 \dots x_k y_k$ be an M-adjusting path of G_l ; then

$$\sum_{e \in M \cap \hat{E}(P)} w(e) = \sum_{e \in E(P) \setminus M} w(e) \quad (7)$$

According to the definition of G_l ,

$$\begin{aligned} l(x_1) + l(y_k) &= l(x_1) + l(y_k) + \sum_{i=1}^{k-1} (l(x_{1+i}) + l(y_i)) - \sum_{i=1}^{k-1} (l(x_{i+1}) + l(y_i)) \\ &= \sum_{i=1}^k w(x_i y_i) - \sum_{i=1}^{k-1} w(y_i x_{i+1}), \end{aligned}$$

from (7) we have $l(x_1) + l(y_k) = 0$. Also from (1) and $w \geq 0$, it follows that

$$0 \leq w(x_1 y_k) \leq l(x_1) + l(y_k) = 0.$$

Thus, $x_1 y_k \in E_l$ and $w(x_1 y_k) = 0$. #

Let $P = x_1 y_1 \dots x_k y_k$ be an M-alternating path, and $x_1 y_k, x_k y_k \in M$, then P is called an (x_1, y_k) -path.

Theorem 4. Let x_1, y_k be M -saturated, $x_1 y_k \in E_t$ and $w(x_1 y_k) = 0$, then

(a) G_t contains an M -adjusting path if and only if $G_t - \{x_1 y_1, x_k y_k\}$ contains an M -augmenting path from y_1 to x_k ;

(b) If G_t contains no M -adjusting (x_1, y_k) -path then $x_1 y_k \notin E(P)$ for any M -adjusting path P of G_t .

Proof: (a) Necessity is clear. Now we prove the sufficiency as follows.

Let $P_1 = y_1 x_2 y_2 \dots x_k$ be an M -augmenting path in $G_t - \{x_1 y_1, x_k y_k\}$. Thus $P = P_1 \cup \{x_1 y_1, x_k y_k\}$ is a (x_1, y_k) -path in G_t . Since $x_1 y_k \in E$ and $w(x_1 y_k) = 0$ we have that $l(x_1) + l(y_k) = 0$. With a proof similar to that of Theorem 3, it is easily obtained that P is an M -adjusting path.

(b) If G_t contains an M -adjusting path P starting at $x(\neq x_1)$ and ending at $y(\neq y_k)$, and $x_1 y_k \in E(P)$, by the proof of Theorem 3, $w(x_1 y) = 0$. Note that $x_1 y_k \notin M$, and we have an M -adjusting (x_1, y_k) -path from (y_k, y) -path in P to x to (x, x_1) -path in P . This is a contradiction.

By Theorem 3 and 4, we may get an algorithm for finding an M -adjusting path. By this algorithm, either G_t contains no M -adjusting path, or an M -adjusting path in G_t is obtained.

Algorithm 1

Step 0: Let $G_0 = G_t[V(M)]$, that is, G_0 is a subgraph of G_t induced by the ends of edges in M .

Step 1: If G_0 contains no edge with weight 0. By Theorem 3, G_0 contains no M -adjusting path and hence, by Theorem 2, M is minimum, in this case, stop. Otherwise, let $x_1 y_k \in E(G_0)$ and $w(x_1 y_k) = 0$, go to step 2.

Step 2: Let $e_1 = x_1 y_1 \in M$ and $e_k = x_k y_k \in M$, by the Hungarian method [1, p. 82], we find an M -augmenting path from y_1 to x_k in $G_0 - \{e_1, e_k\}$. If $G_0 - \{e_1, e_k\}$ contains no such path, then replace G_0 by $G_0 - \{x_1 y_k\}$ and go to step 1. Otherwise let P be an M -augmenting path, then $P_1 = P \cup \{e_1, e_k\}$ is an M -adjusting path.
Stop.

We see that Algorithm 1 is a good algorithm.

By Algorithm 1, it is not difficult to give an algorithm for finding a minimum cardinality optimal matching.

Algorithm 2

Step 1: By the Kuhn-Munkres algorithm [1, p. 87] we find a feasible vertex labelling such that G_t contains perfect matching. Let M be any optimal matching in G_t , go to step 2.

Step 2: By Algorithm 1, we find an M-adjusting path P_1 in G_t . If it exists, then replace M by $M \Delta E(P_1)$ and go to step 1. Otherwise, M is a minimum cardinality optimal matching, stop.

It is easy to see that Algorithm 2 is a good algorithm too.

References

- [1] J.A. Bondy and U.S.R. Murty; *Graph Theory with Applications*, Amer. Elsevier (1976), New York.
- [2] L. Lovasz and M.D. Plummer, *Matching Theory*, North-Holland, Elsevier (1986).