第 27 卷 南京大学学报 Vol. 27
1991 年 10 月 Journal of Nanjing University Oct. 1991

CYCLES IN DIGRAPHS—A SURVEY*

Zhang Ke Min *Song Zeng Min*
(NanjingUniversity) (SoutheastUniversity)

ABSTRACT

The main subjects of this survey paper are Hamiltonian cycles, the longest cycles, 2-cyclability and girth in digraphs or oriented graphs; various types of pancyclicity, generalized cycles in tournaments. Several unsolved problems and a bibliograph are included.

Key Words: Digraph, Tournament, Cycle, Survey

In graph theory, cycle is such a fundamental concept that the some simple properties of them, as cycle axiomatic system, become one of the equivalent axiomatic systems in matroid theory. Undirected graphs, compared with digraphs, without the confinement of orientation of edges, are easier to study. Therefore, most of the early results on cycles belong to undirected graphs. After 70′s, as the study of graph theory going on and the accumulation of research techniques, so people turned their attentions on digraphs progressively. Today, there are a great amount of literatures on direceted cycles and the amount is increasing rapidly. Hence, it is difficult, even impossible, to list all the results on the topics in a paper. In this survey, the collection of results are according to the following rules. a) Almost all the results and the unsolved conjectures with little progress in [*BeT*] are excluded, and in this sence, this paper is a continuation and a complement of [*BeT*]. b) For the same question, if there are several consecutive results, the last and the best one is collected. c) The matter in the survey is confined to the subjects mentioned in the abstract. d) There exist many such results that the holding of their conclusions requires one or several special classes of digraphs to be excluded. In order not to make the paper too long, the definitions of these digraphs are deleted, for which the reader is refered to the related literatures.

Tournaments is a special classes of digraphs. But because of their speciality and their rich contents, which is more profound than digraphs, we treat them as an independent section parallel with digraphs. Also our treatment is convenient.

---

* 收稿日期: 1991-06-12
The project supported by NSFC.

We use standard terminology in [*BeT,BoM*], but we specify the most important definitions and notations:

A **digraph** D=(V,A), where V is the vertex set of D and A is a set of ordered pairs xy of vertices called arcs from x to y. We always assume that D is strict, i.e. there are no loops and no multiple arcs. Set $|V|=n$, $|A|=\varepsilon$. The subdigraph $P_k=(V_1,A_1)$ of D is called a **dipath**, simply called a **path** in no ambiguous case, if $V_1=\{x_0,x_1,...,x_k\}$ and $A_1=\{x_{i-1}x_i: i=1,...,k\}$. Denote the path by $P_k=x_0x_1...x_k$ where k is called the length of P_k. If $x_kx_1\in A$ also, the digraph P_k plus the arc x_kx_1 is called (k+1)−cycle, denote as $C_{k+1}=x_0x_1...x_kx_0$. Particularly, P_{n-1} and C_n are called Hamiltonian path and Hamiltonian cycle respectively.

An **oriented** graph is a digraph without 1−, 2−cycles and a **tournament** is an oriented complete graph. A **k−partite tournament** is an orientation of complete k−partite graph. Specially, denote a bipartite tournament as $(V_1,V_2;\ A)$ where V_1, V_2 is a partition of its vertex set. A digraph is **symmetric** if every arc is contained in a 2−cycle. If G is an undirected graph, we denote by G^* the symmetric digraph associated with G. Let $\overleftarrow{D}$ denote the converse digraph of D.

A digraph D is **strong**, if for any two vertices x and y, D contains a dipath from x to y. D is **strong k−connected** if the deletion of fewer than k vertices always results in a strong digraph. Similarly, The concept of **k−arc−connected digraph** can be difined analogously. A component of a digraph D is a maximal strong subdigraph. The components of D can be labelled by $D_1, D_2,...,D_k$ such that no vertex of D_j dominates a vertex of D_i if $j\leqslant i$, where D_1, D_k are called the initial and terminal components respectively and the other components are the intermediate components. For digraph D, let $N_D^+(x)=\{y|xy\in A\}$, $N_D^-(x)=\{y|yx\in A\}$, $d_D^-(x)=|N_D^-(x)|$, $d_D^+(x)=|N_D^+(x)|$; $h_D^-=\min_{x\in V} d_D^-(x)$, $h_D^+=\min_{x\in V} d_D^+(x)$, $h(D)=\min\{h_D^+,h_D^-\}$; $d_D(x)=d_D^+(x)+d_D^-(x)$, $\delta(D)=\min_{x\in V} d_D(x)$. If no confusion arise, the subscript D can be omitted from above notations. A digraph is **k−diregular** if $d^+(x)=d^-(x)=k$ and **k−regular** if $d(x)=k$ for every vertex x of D. For any real number x, [x] denotes the integer part of x and {x} is the smallest integer no less than x.

§ 1. Digraphs.

1.1 Hamiltonian cycles and pancyclicities.

(1) Conditions on the degrees.

Up to date a generalized sufficient condition for the existence of Hamiltonian cycle in a digraph was obtained by Meyniel ([*Mey*], in 1973).

Theorem (Meyniel) Let D be a strong digraph with n vertices. If for any two non-adjacent vertices x and y, $d(x)+d(y) \geqslant 2n-1$, then D is Hamiltonian.

An efficient algorithm with $O(n^4)$ steps for finding a Hamiltonian cycle in a digraph satisfying the hypothesis of Meyniel's Theorem was obtained by M. Minoux ([*BeT*]).

We say that a digraph D of order n satisfies the **condition (C_i)** if for any two non-adjacent vertices x and y, $d(x)+d(y) \geqslant 2n-2+i$.

Clearly, the above Theorem can be rewrited as: a strong digraph satisfying condition (C_1) is Hamiltonian.

A digraph D is **pancyclic** if D contains r-cycles $(2 \leqslant r \leqslant n)$. If in addition, for any vertex x (arc e) of D, D contains r-cycles passing x (e) $(2 \leqslant r \leqslant n)$, D is called **vertex (arc) pancyclic**. For oriented graphs, of course $r \neq 2$, it must be relaxed $r=2$ in mentioned definitions.

Theorem 1.1.1[*Be2*] If a strong digraph D of order $n \geqslant 3$ satisfies condition (C_1), then D is pancyclic or D is a tournament or else n is even and D is isomorphic to one of $K^*_{\frac{n}{2},\frac{n}{2}}$, $K'^*_{\frac{n}{2},\frac{n}{2}}$, and D_0.

In Theorem 1.1.1, $K'^*_{\frac{n}{2},\frac{n}{2}}$ is $K^*_{\frac{n}{2},\frac{n}{2}}$ minus one arc and D_0 are defined as follows. For given r, add the arcs $x_jx_i(n \geqslant j \geqslant i+1,\ j \neq i+r)$ to the path $x_1x_2...x_n$ and then delete some arcs such that keeping the condition (C_1).

Obviously, $K'^*_{\frac{n}{2},\frac{n}{2}}$ and D_0 don't satisfy (C_2). Hence, Theorem 1.1.1 extends Theorem 2.2.2 of [*BeT*]. We can also ask that whether or not a strong digraph satisfying condition (C_0) or (C_{-1}) with some more exceptions is pancyclic and that for what i, a strong digraph satisfying (C_i) has vertex pancyclitity.

Besides condition (C_i), people can use $\delta(D)$ to describe the conditions for a digraph having Hamiltonian cycles.

Theorem 1.1.2[*Da2*] If a digraph D of order n satisfies $\delta(D) \geqslant n-1$, $h(D) \geqslant \left[\frac{n-1}{2}\right]$, then D is pancyclic except a few classes of digraphs.

Theorem 1.1.3[*Qin*] If a digraph D of order n with $\delta(D) \geqslant n$. Let $x \in V(D)$, if $D-x$ is strong, then x is contained in cycles of all possible lengths from 2 to n excluding 3, unless $D \simeq K^*_{\frac{n}{2},\frac{n}{2}}$.

For oriented graphs, there are some further results. First, the well-known Woodall condition([*Woo*]) is improved to

Theorem 1.1.4 [*So10*] An oriented graph D with $n(\geqslant 9)$ vertices and $\delta(D) \geqslant n-1$ is pancyclic if for any two vertices x,y, $xy \notin A$ implies $d^+(x)+d^-(y) \geqslant n-3$.

Theorem 1.1.5[*Th3*] If an oriented graph D of order n satisfies $h(D) \geqslant n/2-\sqrt{n/1000}$, then D is Hamiltonian.

For smaller n, the following theorems are better.

Theorem 1.1.6[*WaC*] An oriented graph D of order n is Hamiltonian if $h(D) \geqslant k \geqslant 5$ and $n \leqslant 2k+5$.

Theorem 1.1.7[*Sol*] If a strong oriented graph D satisfies $h(D)=k \geqslant 6$. Then D is Hamiltonian or contains a path of length at least $2k+3$.

In [*Wo2*], all non-hamiltonian and non-hamiltonian connected digraphs with given h^-, h^+ and maximum size are descibed.

(2). Conditions on the number of arcs.

The following theorem improves Theorem 1.3.2 of [*BeT*].

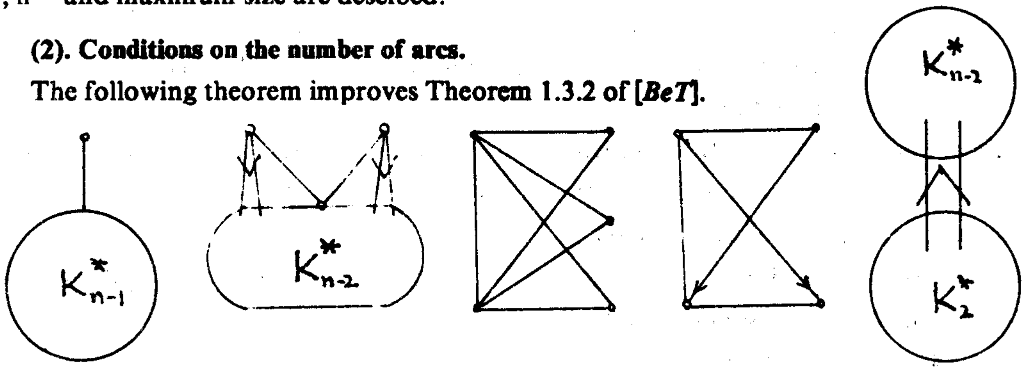


Fig 1. (the edge without orientetion indicats a 2-cycle)

Theorem 1.1.8[*Wol,BeT*] Let D be a digraph with n vertices. If $\varepsilon \geqslant (n-1)(n-2)+2$, and D is not isomorphic to the digraphs in Figure 1 and their converse or the digraphs containing a sourse or a sink. Then a) D is Hamiltonian, b) D is pancyclic unless $D \simeq C_4^*$.

With additional condition $h(D)=r$, [*AFG*] has

Theorem 1.1.9[*AFG*] Let D be a digraph with n vertices, $h(D)=r$. (a). If $\varepsilon \geqslant n^2-(r+2)n+(r+1)^2+1$, then D is Hamiltonian; (b). If $\varepsilon \geqslant n^2-(r+1)n+r(r+1)+1$, then D is Hamiltonian-connected, i.e. for any two vertices x,y of D, D contains a Hamiltonian dipath from x to y.

Let n,l,r be integers with $6 \leqslant 2r+2 \leqslant l < n$. Let $D_1(n,l,r)$ be the following digraph of order n. Whose vertex set consists of two disjoint set E and F with $|E|=l-r-1$, $|F|=n-l+r+1$. The arc set consists of all possible arcs xy, where $x,y \in E$ or $x \in E$, $y \in F$ or $x \in F$, $y \in E'$ where E' is a r-subset of E. Obviously, $D_1(n,l,r)$ contains no cycle of length at least l. This implies Theorem 1.1.9 is, in a sence, best possible.

For oriented graph, there exists a further result and a conjecture:

Theorem 1.1.10 [*FaO*] A strong 2-connected oriented graph D is Hamiltonian if $\delta(D) \geqslant C_n^2-2$.

Conjecture 1.1.11 [*FaO*] A strong k-connected oriented graph D is Hamiltonian if $\delta(D) > C_n^2 - C_{k+2}^2$.

Theorem 1.1.10 implies that the conjecture is true for $k=2$.

(3) Other conditions.

If D is a digraph, we may difine three kinds of independence numbers as follows: $\alpha_1(D)$ ($\alpha_2(D)$; $\alpha_3(D)$ resp.) is the maximum cardinal of a vertex subset B of D such that D[B] has no arc (no cycle, no 2–cycle resp.). Clearly, $\alpha_1(D) \leqslant \alpha_2(D) \leqslant \alpha_3(D)$. We notice that the Chvatal–Erdos condition: $\alpha(G) \leqslant \kappa(G)$ about an undirected graph impling a Hamiltonian cycle and there exists an counterexample ([*Th2*]) about 3–connected non–Hamiltonian digraph D with $\alpha_3(D)=3$. Thus the Chvatal–Erdos condition can not be generalized for digraphs. So, the following conjecture may be true.

Conjecture 1.1.12[*JaO*]. For any integer $a \geqslant 1$, there exists a least integer $f_3(a)$ such that every strong $f_3(a)$–connected digraph D with $\alpha_3(D) \leqslant a$ is Hamiltonian.

Theorem 1.1.13[*Ja3*] Let D be a strong k–connected digraph with $\alpha_3(D) \leqslant a$. If $k \geqslant 2^a(a+2)!$, then D is Hamiltonian.

This implies $f_3(a) \leqslant 2^a(a+2)!$, i.e. Conjecture 1.1.12 is true. As for the precise value of $f_3(a)$, up to now we only know that $f_3(1)=1$, $f_3(2)=3$, $f_3(3)=4$.

Consequently, the following holds:

Conjecture 1.1.14[*Ja3*] For any integer $a \geqslant 1$, there exists a least integer $f_i(a)$ such that any strong $f_i(a)$–connected digraph with $\alpha_i(D) \leqslant a$ is Hamiltonian $(i=1,2)$.

[*He4*] asserts $f_1(1)=f_2(1)=1$, $f_2(2)=3$.

1.2 The longest cycles and 2–cyclability.

For the difficulty of searching into the sufficient conditions of a digraph to be Hamiltonian and the existence of non–Hamiltonian digraph, therefore the longest cycles in a digraph have received more attention. This topic has been studied extensively.

(1) Condition on the degrees.

Theorem 1.2.1[*He3,He4*] If a strong digraph (a strong oriented graph resp.) D with n vertices satisfies the condition (C_{3-2h}) ((C_{1-2h}) resp.), where $1 \leqslant h \leqslant n-1$, then (a) D contains a cycle of length $\left\{\frac{n-1}{h}\right\}+1$. (b) For $h \geqslant 2$, D contains a path of length at least $\left\{\frac{n-1}{h}\right\}+\left\{\frac{n-2}{h}\right\}$. (c) The vertex set of D can be covered by at most h cycles.

For $h=1$, the Theorem is Meyniel's Theorem of § 1.1. So it is an extension of Meyniel's Theorem in the sense of longest cycles.

Let $D_2(n,q)$ be the following digraph with n vertices, where $n=qh+r+1$, $0 \leqslant r \leqslant q-1$. The vertex set of $D_2(n,q)$ consists of the disjoint union of q sets $X_i (1 \leqslant i \leqslant q)$ and a vertex z, where r of which are of cardinality h+1 and q–r of cardinality h. The arcs of $D_2(n,q)$ are all arcs of the form xy with $x \in X_i$, $y \in X_j (1 \leqslant i \leqslant j \leqslant q)$ and all the possible arcs between z and $X_j (1 \leqslant i \leqslant q)$. $D_2(n,q)$ implies that Theorem 1.2.1 is, in a sense, best possible.

Theorem 1.2.2[*So3*] Let D be a digraph with n vertices. If $h^-(D) \geqslant k$, $h^+(D) \geqslant h$ and $n \geqslant h+k+1$, then D contains a cycle of length at least k+h or a path of length at least k+h+1 unless D is isomorphic to the union of $m(\geqslant 2)$ disjoint copies $D_1,\ldots,D_m$ of K^*_{k+1} except one vertex in common.

Bermond and Thomassen conjectured independently that if a strong 2-connected digraph D with at least 2k vertices and $h(D) \geq k$, then D contains a cycle of length at least 2k. Theorem 1.2.2 implies that the Conjecture holds for $n = 2k+1$. For $n \geq 2k+2$, although a conterexample was shown by [*Th2*], the following result holds:

Theorem 1.2.3[*So4*] Let D be a strong digraph of order n with $h(D) \geq k$, $n = 2k+2$. Then D contains a cycle of length at least 2k unless D is isomorphic to one of Figure 2 or a spanning subgraph of the union of K^*_{k+1}, K^*_{k+2} with only one vertex in common. It is, in a sense, best possible.

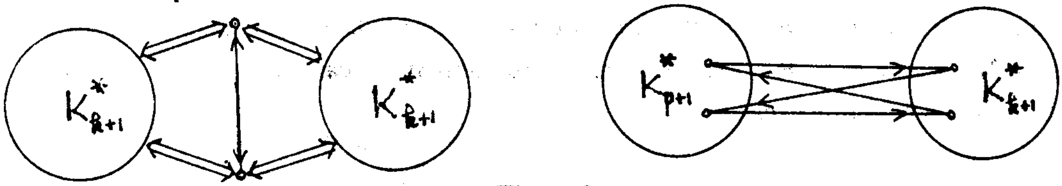


Figure 2

In [*SoZ*] it is also proved that a digraph D satisfying the conditions of Theorem 1.2.3 contains a cycle of length at least 2k+1, unless D is one of 8 exceptions.

For oriented graph, the following results hold.

Theorem 1.2.4[*So9*] If D is an oriented graph with $h^-(D) = k \geq 1$, $h^+(D) = h \geq 3$,

1) If $n \leq k+h+2$, then D is hamiltonian;

2) If $n \geq k+h+3$, then D contains a cycle of length at least h+k+2 or a path of length at least h+k+3.

Theorem 1.2.5[*So9*] If D is an oriented graph with $\delta(D) = r$. If for any $uv \notin A(D)$, $d^+(u)+d^-(v) \geq r-1$, then D contains a cycle of length at least r+1 or a path of length at least r+1.

Conjecture 1.2.6[*Ja2*] Every 2-connected oriented graph with $h^-(D) = k$ contains either a hamiltonian cycle, or else a cycle of length at least 2k+2.

This Conjecture may be, in a sense, best possible.

As for bipartite digraphs, the following further results are best possible in a sense.

Theorem 1.2.7[*AmM*] Let $D = (X,Y;A)$ be bipartite digraph with $h(D) \geq k$. If $a = |X|$, $b = |Y|$ and $a \leq 2k-1$, then D has a cycle of length 2a unless D is one of two exceptions.

Theorem 1.2.8[*Zh7*] Let D be a bipartite oriented graph with $h^+(D) \geq h$, $h^-(D) \geq k$. Then D contains a cycle of length at least 2(h+k) or a path of length at least 2(h+k)+3.

By Theorem 1.2.8, the following conjecturs may be true.

Conjecture 1.2.9[*Zh7*] A bipartite oriented graph with $h^+(D) \geq h$, $h^-(D) \geq k$ contains a cycle of length at least 2(h+k).

Conjecture 1.2.10[*Zh7*] Let D be a bipartite oriented graph. If $h^+(D) \geq k$, then D contains a Hamiltonian cycle or a cycle of length at least 4k.

Clearly, if the above two conjectures hold, they are, in a sense, best possible.

In digraph, there is an interesting fact different from undirected graph. In a 2-connected undirected graph, any two vertices are contained in a common cycle. But not so for strong 2−connected digraphs. Thus, the following concept and problem was deduced.

D has **k−cyclability** if any k vertices of the digraph D are contained in a common cycle.

Problem 1.2.11[*BeL*] Whether or not does there exist such an integer k that any strong k−connected digraph D has 2−cyclability?

For $k \leqslant 5$, [*BeT*] gives counterexamples. Hence if the problem is true, then it must be $k \geqslant 6$. Similarly B. Jackson conjectured that a strong 3−connected oriented graph has 2−cyclability. It is regrettable that the conjecture is not true. The counterexample is shown in Figure 3, where R_1, R_2 are 7−diregular tournaments. Clearly u and v in the Figure can't belong to a common cycle.

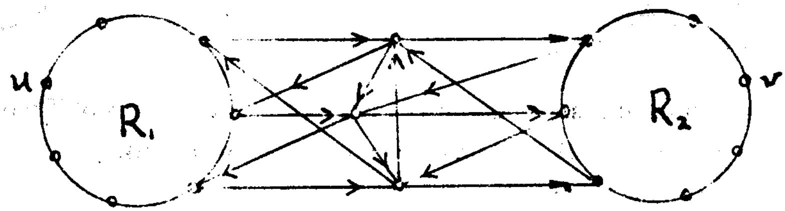


Figure 3

Nevertheless, by additive conditions on independent number, the following result holds:

Theorem 1.2.12[*JaO*] Let D be a strong k−connected digraph. Then D has 2−cyclability if D satisfies one of the following conditions:

(i) $k \geqslant 2$, $\alpha_2(D)=1$; (ii) $k \geqslant 3$, $\alpha_1(D) \leqslant 2$; (iii) $k \geqslant 15$, $\alpha_1(D) \leqslant 3$.

Specially, when $\alpha_1(D)=1$, $k \geqslant 1$, also D is Hamiltonian.

There remains:

Conjecture 1.2.13[*JaO*] Given any integer $m>1$, there exists an integer g(m) such that every g(m)−connected digraph D with $\alpha_1 \leqslant m$ has 2−cyclability.

Using also condition on degrees and on the number of arcs, the following theorem holds:

Theorem 1.2.14[*So4*] Let D be a digraph of order n with $h(D)=k$ and $n \leqslant 2k+1+i$ $(i=0,1)$. Then, with some exceptions, D has 2−cyclability and any two vertices of D belong to a cycle of length at least $n-1-2i$.

Theorem 1.2.15[*Man*] Let D be a strong digraph with n vertices. If $h(D) \geqslant k \geqslant 1$, $\varepsilon(D) \geqslant n^2-n(k+3)+k^2+3k+5$, then D has 2−cyclability.

It leads that:

Conjecture 1.2.16[*Man*] Let D be a (m−1)−connected digraph of n vertices such that $h(D) \geqslant k \geqslant m-1 \geqslant 1$ and $\varepsilon(D) \geqslant n^2-n(k+3)+k^2+3k+m+3$. Then D has k−cyclability.

(2) Conditions on the number of arcs.

Many theorems relating to the conditions on the number of arcs that insure a digraph to contain a k−cycle were shown in § 2.2 of [*BeT*]. Only the conjecture there is repeated.

Conjecture 1.2.17[*BeT*] Let D be a strong digraph of order $n = q(k-2)+r+1$. If D satisfies one of the following conditions, then D contains a cycle of length at least k.

(i) $\varepsilon(D) > \psi(n,k) = (k-1)n-2k+4$ if $k \leqslant n \leqslant 2k-4$;

(ii) $\varepsilon(D) > \phi(n,k) = \binom{n}{2}+(n-1)-(k-2-r)q(q-1)/2-rq(q+1)/2$ if $n \geqslant 2k-4$.

Here $\psi(n,k)$, $\phi(n,k)$ are the numbers of arcs of digraphs $D_1(n,k,1)$, $D_2(n,k-2)$ respcctively. Hence it is best possible, if the Conjecture is true.

[*BeT*] points out that the Conjecture is true for $n=2k-4$ and for $k \leqslant 5$. Besides when $k=n,n-1$, (i) is true; when D is strong oriented graph, (ii) is true.

Theorem 1.2.18[*HeS*] Let D be a strong digraph with n vertices. If $k=2$, $n \geqslant 13$ or $k \geqslant 3$, $n \geqslant k^2+2k+4$ and $\varepsilon(D) \geqslant n^2-(k+3)n+2k+4$. Then D contains a cycle of length at least $n-k$ unless $D \simeq D_1(n,n-k,1)$.

By additive conditions on half-degree, the following further results hold:

Theorem 1.2.19[*HeS*] Let D be a strong digraph of order $n(\geqslant 15$, rerp.) with $h(D) \geqslant 2$, $\varepsilon(D) \geqslant n^2-5n+13$ ($\geqslant n^2-6n+16$, resp.). Then D contains a cycle of length at least $n-1$ ($n-2$, resp.).

Theorem 1.2.18 implies Conjecture 1.2.17 (i) is true for sufficient large n. From Theorem 1.2.19, [*HeS*] proposes the following more generalized conjecture than 1.2.17.

Conjecture 1.2.20 [*HeS*] Let k,r be two integers and $r \geqslant 1$. Then there exists such a function $f(k,r)$ with the following property. If D is a strong digraph of order n and $n \geqslant f(k,r)$, $h(D) \geqslant r$, $\varepsilon(D) > |A(D_1(n,n-k, r))| = n^2-(k+r+2)n+(k+r+1)(r+1)+1$, then D contains a cycle of length at least $n-k$.

As a partial solution to Conjecture 1.2.20, Theorem 1.1.9, 1.2.18, and 1.2.19 imply $f(0,r)=0$; $f(2,1)=13$; $f(k,1) \leqslant k^2+2k+4$ for $k \geqslant 3$; $f(1,2)=0$; $f(2,2) \leqslant 15$ respectively. By [*BGH*], $f(1,1)=6$; by [*So2*], $f(1,3)=0$ and by [*So4,8, SoZ*], the Conjecture holds for $n \leqslant 2r+3$ and $k=1$.

Let $n=kh+r$, $k+1 \geqslant r \geqslant 2$ and $w(n,k)=\binom{n}{2}-(k-r+1)\binom{h}{2}-(r-2)\binom{h+1}{2}$. For oricnted graphs, the corresponding results below hold.

Theorem 1.2.21 [*He2*] Let D be a strong oriented graph of order n with $\varepsilon(D) > w(n,k)$. Then D contains a cycle of length at least $k+1$.

Theorem 1.2.22 [*Qin*] Let D be a digraph of order n ($\geqslant 4$) with $\varepsilon(D) \geqslant (n-1)(n-2)+2$, then every vertex of D is contained in cycles of all possible lengths from 3 to n, except D isomorphic to some special digrphs which are described.

1.3 The shortest cycle and its girth.

In digraphs, besides the longest cycles, shortest cycle and its length——the girth present one aspect of cycles. First, for diregular digraphs it has:

Conjecture 1.3.1[*BCW*] k−diregular digraph D with n vertices has the girth at most $\{n/k\}$.

The Conjecture holds for $k=2$ ([*BCW*]) and 3 ([*Ber*]).

Theorem 1.3.2 [*Hal*] Let D be a k−diregular digraph of order n with $k \geqslant 4$. Then its girth is at most $\{n/4\}$.

Theorem 1.3.2 implies that Conjecture 1.3.1 is also true for $k=4$.

The following conjecture is a generalization of Conjecture 1.3.1.

Conjecture 1.3.3 [*CaH*] Let D be a digraph of order n with $h^+(D)=k$.

Then its girth is at most $\{n/k\}$.

Notice that the digraph $D=(V,A)$ where $V=\{1,2,\ldots,n\}$, $A=\{i$ dominates $i+1, i+2, \ldots, i+k \pmod n)\ i=1,2,\ldots,n\}$. It cannot be improved if the above Conjecture is true.

Conjecture 1.3.3 holds for $2 \leqslant k \leqslant 5$ ([*CaH,Ha2,HoR*]). Besides the following holds:

Theorem 1.3.4 [*Nis*] If D is a digraph of order n with $h^+(D)=k$, then its girth is not greater than $\mathrm{Min}\{n/k+304, 2n/(k+1)\}$.

A conjecture related with girth was proposed in [*JaO*].

Conjecture 1.3.5 [*JaO*] Let D be a digraph with n vertices. If $h^+(D) \geqslant k$, then D contains cycles $C_1, C_2, \ldots, C_k$ such that $\left|\bigcup_{i<j} C_i \cap C_j\right| \leqslant 1$ for all $j \leqslant k$.

Obviously, if the girth of D is g, then $\left|\bigcup_{i\leqslant k} C_i\right| \geqslant k(g-1)+1$, thus $n \geqslant k(g-1)$, i.e., $n>k>g-1$. Therefore Conjecture 1.3.5 implies 1.3.3. Conjecture 1.3.5 is true for $k=2$ ([*Th6*]).

R. Haggkvist suggested the following problem:

Given a digraph D and a condition implying that D is Hamiltonian, for any r−subset S of D, find an upper bound on the length of minimum cycle containing S. [*Fra*] has:

Theorem 1.3.6 [*Fra*] Let D be a digraph (an oriented graph resp.) with n vertices. Such that for any two vertices x and y, $xy \notin A(D)$ implies $d^+(x)+d^-(y) \geqslant n$ ($d^+(x)+d^-(y) \geqslant n-2$ resp.). Let S be any r−subset of V(D). Then S is contained in such a cycle of length at most $\min\{n,2r\}$ in D.

One could expect that Theorem 1.3.6 might be still valid under the weak condition−−Meyniel's condition for Hamiltonian cycle. It is to be regretted that it doesn't hold. The counterexample is shown in Figure 4.

Where for $p>1$, the $S_p x S_p x S_p$ is a tripartite complete symmetric digraph. Add a vertex x, then the arcs between x and $S_p x S_p x S_p \setminus \{y\}$ are shown in Figure 4, and there is no arc joining x and y. It is easy to check that the mentioned digraph satisfies the condition C_1, but x and y are contained in no cycle of length 4 or less.

The other topic related with the girth is the enumeration of the shortest cycles.

Theorem 1.3.7 [*All,Th6*] Let D be a digraph of order n with girth g. If $g \geqslant 4$ or $n \leqslant 2g$, then D contains at most 2^{n-g} distinct cycles of length g.

Theorem 2.4.1 of [*BeT*] is a result related with the girth by the condition on the number of arcs which is no longer repeated here.

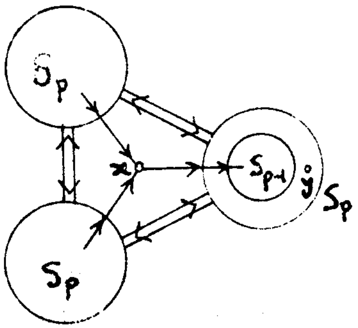


Figure 4

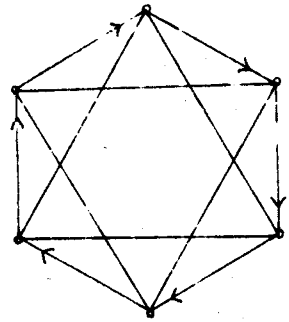


Figure 5

1.4 The relation among cycle and others.

First, we comment on the decomposition of a digraph into cycles. In § 4.1 of [*BeT*], it is pointed out that K_m^*, $K^*_{\frac{n}{2},\frac{n}{2}}$ can be partition into Hamiltonian cycles respectively. Since a digraph usually can't have such a strong conclusion, the following conjecture may be true.

Conjecture 1.4.1 [*Ber*] Let D be a digraph with $n(\geqslant 5)$ vertices. If $h(D)\geqslant n/2$, then D contains two arc-disjoint Hamiltonian cycles.

Theorem 1.4.2 [*SoZ*] Let D be a digraph with $n\ (\geqslant 5)$ vertices. If $h(D)\geqslant n/2$, then D contains two arc-disjoint cycles C_1 and C_2 where C_1 is a Hamiltonian cycle and C_2 is a cycle of length at least $n-1$ unless D is isomorphic to Figure 5 ~~(see P.10)~~.

The theorem generalized the result of [*Zh3*].

Let D be a digraph. The cycle number $C_y(D)$ is the cardinality of a minimum cycle decomposition of D, and $C_y(D)=\infty$ if D can't be partition into cycles. Obviously, $C_y(D)<\infty$ if and only if D is euler digraph. B. Jackson conjectured that $C_y(D)\leqslant \{n/2\}$ for euler oriented graph D, And W. Bienia and H. Meyniel conjectured that $C_y(D)\leqslant n$ for euler digraph D.

They are disproved by some counterexamples in [*Den*], even for strong 2-connected digraph. So the following conjectures may be true.

Conjecture 1.4.3 [*Den*] Let D be an euler digraph of order $n\geqslant 2$. Then $C_y(D)\leqslant \{8n/3\}-3$.

Conjecture 1.4.4 [*Den*] Let D be a symmetric digraph of order $n\geqslant 2$. Then $C_y(D)\leqslant 2n-3$.

Also, B. Jackson suggested that:

Conjecture 1.4.5 [*Den*] For any euler oriented graph D of order n, then $C_y(D) \leq \{2n/3\}$; If D has 2-cyclability in addition then $C(D) \leq \{n/2\}$.

In the following we only consider disjoint cycles in digraphs.

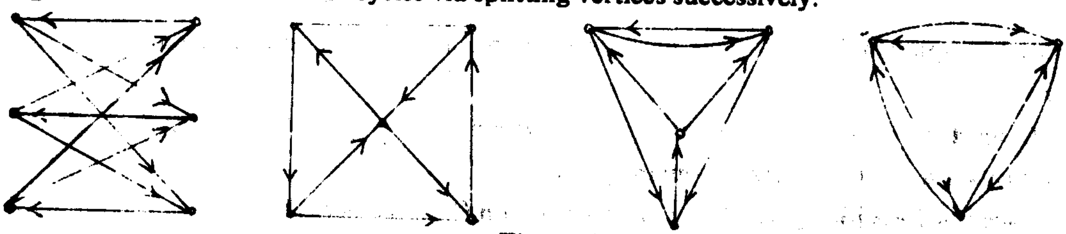
Theorem 1.4.6 [*Th4*] For any positive integer k, there exists a least positive intrger f(k) such that any digraph D with $h^+(D) \geq f(k)$ has k disjoint cycles.

Hence, the following conjecture may be true.

Conjecture 1.4.7 [*Th4*] $f(k) = 2k+1$.

[*Th4*] proves that the Conjecture holds for $k = 2$.

Let u be a vertex of digraph D. We call that the result digraph by splitting vertex u in D, say D_1, is obtained as follows: First replace u by two vertices u_1, u_2 and an arc u_1u_2; and each arc of D with head u by a new arc with head u_1; and each arc of D with tail u by a new arc with tail u_2, and then reserve all other arcs of D. Let C_k be a cycle of length k. From C_k^* by successive splitting vertices, the result digraph is called **weak k-double-cycle**. Figure 6 are weak 3-double-cycles via splitting vertices successively.



Figuer 6

Now we study the characterization of a digraph in which any two cycles always meet.

Theorem 1.4.8 [*Th6*] Every cycle of a digraph D contains a given vertex iff any two cycles of D always meet, and D isn't isomorphic to any subdivision of the digraphs in Figure 6.

Theorem 1.4.9 [*All*] Let D be a digraph with n vertices. If there is no vertex joint with all cycles of D, then D contains a cycle of length at most $2n/3$.

Theorem 1.4.10 [*Th10*] In a digraph D, every cycle contains a given arc iff any two cycles of D have at least two vertices in common, and D is not isomorphic to any subdivision of the digraph in Figure 6(d), or any subdivision of $\tilde{C}_k$ ($k \geq 2$), where $\tilde{C}_k$ is a digraph obtained by replacing every arc of k-cycle by two parallel arcs.

A digraph D is **randomly k-cyclic** ($k \geq 3$), if every path P of length at most $k-1$ can be extended to a k-cycle containing P. This concept is an extension of randomly Hamiltonian of [*BeT*]. The randomly k-cyclic digraphs, for $k = 3,4,5$ ([*COR*]) and for $k \geq 6$ ([*EMR*]), were characterized, so this problem is completely solved. A digraph D has **odd cycle property** if there exists an arc subset S such that any cycle of D contains an odd number of arcs of S. Otherwise, D is **even** if D has not odd cycle property. It is equivalent to the following definition: If any subdivision of D always contains a cycle of even length, then D is even. For any closed directed walk W of D, obviously it can be decomposed into the union of cycles where the union is understood to be in the sence of multiset. We call that the digraph has **a unique parity property** if for every closed walk W in D, $A(W)=\bigcup_{i=1}^{k} A(C_i)$ and $A(W)\bigcup_{j=1}^{l} A(C_j)$ imply that $k\equiv l \pmod 2$, where C_i and D_j are cycles in D ($i=1, 2, \ldots, k$; $j=1, 2, \ldots, l$). Its property is depicted by the following theorem.

Theorem 1.4.11 [*SeT*] A digraph D is even iff it contains a weak odd double cycle.

Theorem 1.4.12 [*MaS*] A digraph D has the odd cycle property iff D has the unique parity property.

§ 2 Tournaments

Tournaments is a special kind of digraphs. It has complete, irreflexive and antisymmetric properties. Just by these very good properties, it deduces a very rich and deep going results.

2.1 Hamilton cyclys and Kelly's conjecture.

Many equivalent statements about Hamiltonian cycles in tournaments are well known.

Theorem 2.1.1 Let T be a tournament, the following five statements are equivalent:

(i) there exists a Hamiltonian cycle in T;

(ii) T is strong (P. Camion);

(iii) T has vertex pancyclicity (J. W. Moon);

(iv) For any proper subset $V_1\subset V(D)$, $(V_1, V(D)\setminus V_1)\neq\phi$;

(v) T has a dipath from R to S, where $R=\{x|d^-(x)=\Delta^-\}$, $S=\{x|d^+(x)=\Delta^+\}$ ([*Wal*]).

The further study on tournaments may be the conditions insuring two or more Hamiltonian cycles and Kelly's conjecture related to this problem.

Kelly's conjecture: The arcs of a diregular tournament T can be partitioned into $(n-1)/2$ Hamiltonian cycles.

This Conjecture has been verified for tournaments of order $n\leqslant 9$ by [*SeT*]. R. Haggkvist proved that ([*Th8*]) the Conjecture is true for sufficiently large n. The remains have the following results: ([*Zhl*], [*WaC*]) prove that there are two (three) arc disjoint Hamiltonian cycles, when $n\geqslant 5$ ($n\geqslant 15$). These results will be deduced by the following theorem, when n is sufficiently large.

Theorem 2.1.2 [*Th3*] Let T be a diregular or almost diregular tournament of order n, then T has $[\sqrt{n/1000}]$ arc–disjoint Hamiltonian cycles.

By the way, Conjecture 4.1.9 of [*BeT*], which is stronger than Kelly's conjecture, has been disproved in [*Den*]. And besides Conjecture 1.4.5, the following conjecture is weaker than Kelly's conjecture:

Conjecture 2.1.3 [*Th3*] For any $\varepsilon > 1$, almost all tournaments of order n have $[(1/2-\varepsilon)n]$ arc−disjoint Hamiltonian cycles.

If this problem is transformed into covering the arc set of a tournament by Hamiltonian cycles, then

Theorem 2.1.4 [*Th7*] The arc set of a diregular tournament of order n is covered by at most 12n Hamiltonian cycles.

From the point of view of connectivity we can also get some conditions for insuring Hamiltonian cycles. First, [*Th1*] pointed out that: "every strong 4−connected tournament is Hamiltonian connected and in a strong 3−connected tournament, every arc is contained in a Hamiltonian cycles". These results are very useful for studying tournaments with high strong connectivity.

Theorem 2.1.5 [*Th5*] There exists a function f(k): N→ N such that every strong f(k)−connected tournament has a Hamiltonian cycle containing given k pairwise nonadjacent arcs.

For f(k), we only know that f(1) = 3. On the other hand, there also is:

Theorem 2.1.6 [*FrT*] Let T be a strong k−connected tournament, I is any k−1 arc set in T. Then there is a Hamiltonian cycle in T−I.

Clearly, the above result is best possible. Note that if we substitute the strong k−connected condition by degree conditions $h(T) \geqslant k$ in a tournament T, it can't insure the existence of a Hamiltonian cycle in T−I; And if we substitute k arc−connected condition for strong k−connected condition in Theorem 2.1.5 and 2.1.6, the conclusions are not true either. The counterexample as follows: Let T_1, T_2 be k−arc−connected tournaments with one common vertex v, and joined by all the possible arcs from T_1-v to T_2-v, then add two new vertices x,y with arc xy. The remains arcs are all the possible arcs of the form uv with $u \in T_1$, $v \in \{x,y\}$ or $u \in \{x,y\}, v \in T_2-v$. Clearly,the resulting tournament is k arc−connected, but its every Hamiltonian cycle must contain xy.

The results on Hamiltonian cycle in k−partite tournaments are few less than ordinary tournaments, even if we consider the condition of existence of Hamiltonian cycles.

The bipartite tournament T = (V ,V ; A) is denoted by $T(r_1, r_2, r_3, r_4)$, containing of four vertex−set $B_i (i = 1, 2, 3, 4)$ and $|B_i| = r_i$, and possible arcs from B_i to B_{i+1} $(i = 1, 2, 3)$ and from B_4 to B_1. Usually $T(r, r, r, r)$, $T(r+1, r+1, r, r)$ are called F_{4r}, F_{4r+2} respectively.

Theorem 2.1.7[*Wa4*] Let $T = (V_1, V_2; A)$ be a strong bipartite tournament. If for any $u,v \in V_1 \cup V_2$ and $uv \notin A$, $d^+(u)+d^-(v) \geqslant k$, then T contains a cycle of length at least $2\min\{k+1, |V_1|, |V_2|\}$ unless n is even and T is isomorphic to one of $T(k_1, k_2, k_3, k_4)$ for any $k_i \geqslant k/2$, i = 1,2,3.

Theorem 2.1.8[*LWZ, Wa6*] Let T be a n× n bipartite tournament. If $T = (V_1, V_2; A)$ satisfies the condition $d^-(u)+d^+(v) \geqslant n-2 \geqslant 4$ whenever $uv \in A$, then T is Hamiltonian except two exceptional graphs.

Conjecture 2.1.9 There exists a Hamiltonian cycle in every diregular k−partite tournament.

This Conjecture is true for $k=2$ ([BeT, WH1]). Furthermore there is an analogous result about almost diregular balance bipartite tournament $T \simeq F_{4r+2}$ ([So3], [Wh1]).

Theorem 2.1.10 [*Zh4*] Every diregular k–partite tournament of order n contains a cycle with the length at least $n-1$. Thus it contains a Hamiltonian dipath.

Usually, for a strong k–partite tournament ($k \geq 2$), it doesn't contain a Hamiltonian cycle. So the following results are still interesting.

Theorem 2.1.11 [*Bon*] Let T be a strong k–partite tournament ($k \geq 3$). For any r, $3 \leq r \leq k$, there is r–cycle in T. If in addition, each part of T is of size at least 2, then T contains a h–cycle for some $h \geq k+1$.

By the above Theorem, the following conjecture may holds:

Conjecture 2.1.12 [*Bon*] For $k \geq 5$, every strong k–partite tournament T with each part being of size at least 2 contains a (k+1)–cycle.

About this Conjecture, [*Gut*] proved that T contains (k+1)– or (k+2)–cycle. But [*BaP*] gets counterexamples for any k, and proved the following modified version.

Theorem 2.1.13 [*BaP*] Let T be a strong k–partite tournament. If T contains a k–cycle which visits at most $k-1$ parts of T, then T also contains a (k+1)–cycle.

For k–partite tournemant, it also has the following interesting property:

Theorem 2.1.14 [*GoO*] Let v be a vertex in a strong k–partite tournament T, then v lies on a cycle that contains vertices from exactly m parts ($3 \leq m \leq k$).

A bipartite version of Kelly's Conjecture is proposed in [*BeT*] that "every diregular bipartite tournament can be partitioned into n / 4 Hamiltonian cycles". Clearly, this Conjecture implies that every diregular bipartite tournament is arc–Hamiltonian, which will be obtained by Theorem 2.2.5 in the next section. A possible generalization of this Conjecture is the following.

Conjecture 2.1.15[*AmM*] Let $D=(V_1,V_2; A)$ be a bipartite oriented graph with $h(D) \geq k$, where $2k=|V_1| \leq |V_2|$. Then D has exactly k pairwise arc–disjoit cycles of length 2k.

2.2 Pancyclicities and complementary cyclicities in tournaments.

Let $D=(V,A)$ be a digraph of order n. D is said to be **arc k–cyclic** if each arc of A is contained in a cycle of length k. Particularly, arc n–cyclic is often called **arc–Hamiltonian cyclic**. An arc e of A is said to be m quasi–arc–pancyclic if it is contained in cycles of all lengths l, $m \leq l \leq n$. D is said to be **m quasi–arc–pancyclic** if each arc of A is m quasi–pancyclic. Especially, a 2(3) quasi–arc–pancyclic digraph is often called to be **arc–pancyclic** in digraphs (in oriented graphs), which was defined before.

As a result of Theorem 2.1.1, the study of arc–pancyclic is a main direction in this area, since B. Alspach introduced the concept of arc–pancyclic. And some deep results were obtained.

Lemma 2.2.1 [*Th1*] Let v_1 and v_2 be distinct vertices of a tournament T. Then T has a

Hamiltonian dipath from v_1 to v_2 iff T contains an acyclic spanning subgraph D such that for any vertex $v\in D$, D has a dipath from v_1 to v_2 containing v.

This is a very useful Lemma. The effect of the Lemma in the study of pancyclicity is no less than the effect of Camion's result in Theorem 2.1.1 in the study of Hamiltonian cycle.

Theorem 2.2.2 [*WZ2*] If T is a tournament of order n which is arc 3-cyclic and arc n-cyclic, then T is arc-pancyclic.

In the following, Theorem 2.2.2 was improved to

Theorem 2.2.3 [*TWZ*] Except two special classes of tournaments (in which only one arc is not pancyclic), every arc 3-cyclic tournament is arc-pancyclic.

Many known results are corollaries of Theorem 2.2.3. For example:

Corollary 1 [*Als*] Every diregular tournament is arc-pancyclic.

Corollary 2 [*Zh2*] Let T be an arc 3-cyclic tournament of order n with $h(T)\geqslant k$ and $n\leqslant 4k-3$. Then T is arc-pancyclic.

By Theorem 5.5 of [*ReB*], almost all tournaments have arc 3-pancyclicity. So, by the Theorem 2.2.3, almost all tournaments are arc-pancyclic. But the tournaments which are not arc-pancyclic may be quasi-arc-pancyclic. In [*ZhT*],[*ZTC*] and [*HZL*],they got some sufficient conditions by the score list of a tournament. We say that a tournament of order n satisfies a condition O(n,q) if for any arc v_1v_2, $d^+(v_2)+d^-(v_1)\geqslant n-q$, where q is a constant. [*ZhT*] proved that a tournament T of order n is 4 quasi-arc-pancyclic, if T satisfies the condition O(n,q),$q\leqslant 2$ and $n\geqslant 7$; [*ZTC*] also proved that T is 5 quasi-arc-pancyclic, if T satisfies the condition O(n,q) with $n\geqslant 3q+3$. We can also depict the m quasi-arc-pancyclicity by **the irregularity** $i(T)=\underset{v\in V}{Max}\left|d^-(v)-d^+(v)\right|$, which is different from the condition O(n,q). Using this concept, Theorem 3.1.5 of [*BeT*] gave a condition of 4 quasi-arc-pancyclicity.

From Theorem 2.2.2 and 2.2.3, it is easy to deduce the following problem. What is the action of arc 3-cyclicity, arc Hamiltonian cyclicity on arc-pancyclicity respectively [*WZ1*] and [*WZ3*] point out that an arc 3-cyclic tournament T is only arc k-cyclic where $k\leqslant[(n+1)/2]+2$. There exists a counterexample with $k\geqslant[(n+1)/2]+3$. On the other hand, [Zhu] and [*Wa3*] point out that an arc Hamilton-cyclic tournament with $n\geqslant 8$ is arc (n-1)-, (n-2)-cyclic. Except a special class of tournament,it is arc (n-3)-cyclic too.

Similarly, we may consider pancyclic, vertex-pancyclic, arc-pancyclic k-partite tournaments. So far, the known results on these topics are few, and mainly on bipartite tournaments. [*BeL*] discussed even-pancyclicity of bipartite tournaments. Then [*Zh6*] obtained.

Theorem 2.2.4 [*Zh6*] A $\frac{n}{2}\times\frac{n}{2}$bipartite tournament T is vertex even-pancyclic$\rightleftharpoons$T contains a Hamiltonian cycle and $T\not\cong F_{4r}\rightleftharpoons$T contains a Hamiltonian cycle and a 6-cycle.

For the arc-pancyclicity of bipartite tournaments, there exists the following result:

Theorem 2.2.5 [*WH1*] A diregular bipartite tournament $T \not\cong F_{4r}$ is even arc-pancyclic.

Besides pancyclicity, K. B. Reid [*Re2*] discussed a problem, originally posed by Bollobas, as follows:

Problem 2.2.6 Let m be a positive integer. What is the least integer g(m) so that all but a finite number of g(m)-connected tournaments contain m vertex-disjoint cycles that include all vertices?

Clearly, g(1) = 1. Reid [*Re1*] proved that g(2) = 2. For $m \geq 3$, the problem remains open. Reid [*Re2*] proved that $m \leq g(m) \leq 3m-4$. [*So6*] pose a further problem.

Let $n_1, n_2, \ldots, n_m$, be m positive integers. If they satisfy

(A) $n_i \geq 3, i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} n_i = n$,

we say that they are a solution of (A).

Problem 2.2.7 Let m be a positive integer. What is the least integer f(m) so that, for any solution of (A), all but a finite number of f(m)-connected tournaments contain m vertex-disjoint cycles of lengths $n_1, n_2, \ldots, n_m$?

Clearly, f(1) = g(1) and $g(m) \leq f(m)$. [*So6*] makes the following conjecture.

Conjecture 2.2.8 For all m, g(m) = f(m).

[*So6*] proved that f(2) = 2. This implies the Conjecture holds for m = 1 or 2.

A digraph D has **pan-complementary cycles property** if for any k, $3 \leq k \leq [n/2]$, D contains two complementary cycles (two vertex disjoint cycles which include all vertices of D) C_k and C_{n-k}.

For bipartite tournaments, we obtain.

Theorem 2.2.9 [*ZW1, ZSW*] Let $T = (V_1, V_2; A)$ be a k-diregular bipartite tournament with $k \geq 2$. Then for any $uv \in A$ and $w \in V_1 \cup V_2 \setminus \{u,v\}$, there exists a pair of complementary cycles C_4 and C_{4k-4} with $uv \in C_4$ and $w \in C_{4k-4}$, unless $T \simeq F_{4k}$.

[*So5*] also proved that every almost diregular bipartite tournament has two complementary cycles.

A bipartite tournament T with order n has **even pan-complementary cycles property (even pan-complemenyary cycles containing a pair of given vertices, even pan-complementary cycles containing a given arc** resp.) if for each k $(2 \leq k \leq n/4)$ (any given distinct vertices $u, v \in T$, any given arc uv in T resp.). T contains a pair of complementary cycles C_{2k} and C_{n-2k} (with $u \in C_{2k}$, $v \in C_{n-2k}$; with $uv \in C_{2k}$ resp.).

[*ZMS*] also suggests the following three conjectures:

Conjecture 2.2.10 [*ZMS*] If T is a k-diregular bipartite tournament with $k \geq 2$ and T is not isomorphic to F_{4r}, then T has even pan-complementary cycles property.

Conjecture 2.2. 11[*ZMS*] If T is a k diregular-bipartite tournament with $k \geq 2$ and T is not isomorphic to F_{4r} or some other special digraphs, then T has even pan-complementary cycles containing a pair of given vertices.

Conjecture 2.2.12 [*ZMS*] If T is a k diregular bipartite tournament with $k \geq 2$ and T is not isomorphic to F_{4r} or some other special digraphs, then T has even pan-complementary cycles containing a given arc.

Evidently, Conjecture 2.2.11 and 2.2.12 are stronger than Conjecture 2.2.10, Conjecture 2.2.11 and 2.2.12 don't imply each other. Theorem 2.2.9 is some results in support of Conjecture 2.2.10—2.2.12.

2.3 Generalized paths and generalized cycles.

Let$(\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a k-tuple of 1's and -1's, D be a digraph.

An $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-generalized path (V_1, A_1), as **simply a generalized path** and denoted by GP_k, is a subdigraph of D, where $V_1 = \{x_0, x_1, \ldots, x_k\}$, $A_1 = \{x_{i-1}x_i$ if $\alpha_i = 1$; $x_i x_{i-1}$ if $\alpha_i = -1$, $i = 1, 2, \ldots, k\}$ and k is called the length of GP_k. Especially, we call GP_{n-1} a generalized Hamiltonian path. Similarly, **an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-generalized cycle** (V_2, A_2), as simply a **generalized cycle** and denoted by GC_{k+1}, is a subdigraph of D, where $V_2 = \{x_0, x_1, \ldots, x_k\}$, $A_2 = \{x_{i-1}x_i$ if $\alpha_i = 1$; $x_i x_{i-1}$ if $\alpha_i = -1$, $i = 0, 1, 2, \ldots, k$, and let $x_{-1} \equiv x_k\}$. Especially, we call GC_n a generalized Hamiltonian cycle. Clearly, a $(1, 1, \ldots, 1)$- or $(-1, -1, \ldots, -1)$- GP_k or GC_k is an ordinary dipath or a directed cycle in D. We may consider 2^k distinct types of GP_k and $2^k / k$ distinct types of GC_k. Among of them, a $(1, -1, 1, -1, \ldots)$-GP_k or GC_k is called an AD-path or an AD-cycle; a $(1, -1, -1, \ldots, -1)$-GC_k is called a (1)-GC_k. Especially, let e be a given arc in D, **a bypass of e** or **an anticycle of e** is a (1)-GC_k if $\alpha_1 = 1$ corresponds to e. Up to date, almost all the literatures on GP_k and GC_k is about these two special classes.

As the ordinary directed cycles for variant types of generalized cycles, pan-generalized cyclicity, vertex pan-generalized cyclicity, arc pan-generalized cyclicity were similarly defined. Especially, we point out that a digraph D is **m quasi-arc-antipancyclic** if for any r $(m \leqslant r \leqslant n)$, D is arc-r-anticyclic. A 3 quasi-arc-antipancyclic digraph is usually called **arc-antipancyclic**. D is **(l,m)-quasi-strong path-connected** if D is both l quasi-arc-pancyclic and m quasi-arc-antipancyclic. A (3,3)-quasi-strong path-connected digraph is usually called **completely strong pan-connected**. [*ZWZ*] proved the existence of completely strong path-connected tournament in construction.

Theorem 2.3.1 [*ZWZ*] A tournament T of order n is completely strong path-connected iff T is arc 3-, arc n-cyclic and arc 3-, arc n-anticyclic.

The following theorem improved the above result.

Theorem 2.3.2 [*Zh5*] A tournament T is completely strong path-connected iff T is arc 3-cyclic and arc 3-anticyclic, except that T is isomorphic to T_0, where T_0 is a special class of tournaments.

By § 9 of [*ReB*], almost all tournaments are arc 3-anticyclic. And in § 2.2, we point out that almost all tournaments are arc 3-cyclic.

So, by Theorem 2.3.2, almost all tournaments are completely strong path-connected. But on the other hand, note that there is at least one tournament with a given score list that has no arc 3−anticyclicity. Hence there are infinite tournaments, including some diregular tournaments, which are not arc 3−anticyclic. But among of them, some may be (m,l)−quasi−strong path connected. So the following theorems are still interesting.

Theorem 2.3.3 [*ZhT*] If a tournament T of order n (≥ 10), satisfies the condition O(n,2), then T is (4,4)−quasi−strong path −connected.

Theorem 2.3.4 [*Tao,Gua,Da1*] If a tournament T of order n with $i(T) \leq k$, and $n \geq 5k+4$, $k \geq 4$ or $n \geq 5k+5+2(1-\mathrm{sign}\ k)$, then T is (4,4)−quasi−strong path connected.

From Theorem 2.3.1 and 2.3.2, [*Zh5*] suggested a conjecture that a tournament T is completely strong path−connected iff T is arc 3−anticyclic and arc Hamilton anticyclic. [Wa2] showed a counterexample for $n \geq 91$, and hence disproved the conjecture. On the other hand, [*Wa2*] also shown a class of tournaments of order n each of which is Hamiltonian cyclic and arc Hamiltonian anticyclic but not arc (n−1)−anticyclic. Therefore it is easy to suggest the following problem:

Problem 2.3.5 If a tournament T is arc 3− and arc Hamilton anticyclic, what is the smallest number m such that T is m quasi−arc antipancyclic?

[*WH2, He1*] got that $m \leq n-2$.

The results about arc antipancyclicity and (l,m)−quasi−strong path−connected properties is few in other classes of digraphs. The following results may be worthy pointing out:

Theorem 2.3.6 [*Wa5*] A diregular bipartite tournament T with $T \not\cong F_{4r}$ is even arc antipancyclic.

By the condition (C_1), the analogous result with Theorem 1.1.1 was obtained by [*BW3*].

Theorem 2.3.7 [*BW3*] Let D be a digraph of order n (≥ 3). If D satisfies the condition (C_2), then D is pananticyclic, except that D is isomorphic to T_3^* or T_5^* or $K^*_{\frac{n}{2},\frac{n}{2}}$, for even n.

About the study of vertex pananticyclicity of digraphs etc, up to date, it seems that there is no literatures in this area.

Since 1971, the concepts of an AD−path and an AD−cycle was first introduced by B. Grunbaum ([*Gru*]). And he proved that every tournament has an ADH−path, except the three rotational tournaments R(1), R(1,2) and R(1,2,4). [*Ro2*] proved further that in every tournament T with $n \geq 9$, there is an ADH−path starting at any vertex. There are some courterexamples if $n \leq 8$. By the microcomputer, we exhausted all possible tournaments for $n \leq 8$. 7412 nonisomorphic tournaments are checked. And we got 27 tournaments which are counterexamples. In other words, we have:

Theorem 2.3.8 Let T be tournament of order n. Except 27 special tournaments ($n \leq 8$), there is an ADH−path starting at any vertex in T.

[*Ro2*] further suggested the following conjecture of path.

Conjecture 2.3.9 [*Ro2*] There exists a number N such that every tournament T with $n \geq$

N contains all types of GP_{n-1}.

The following are two theorems in support of Conjecture 2.3.9.

Theorem 2.3.10 [*Tho,Zh4*] Every tournament T of order n contains all types of GP_{n-2}.

A **block** in a generalized path GP_k is a maximal dipath (maximal with respect to inclusion), and denoted by B_i. Clearly every GP_k is uniquely determined by its blocks B_1, $B_2,\ldots,B_r$.

Theorem 2.3.11 [*ARR*] If $GP_{n-1}=B_1B_2\ldots B_r$, $|B_i|\geqslant i+1$, then every tournament contains GP_{n-1}.

[*Gru*] showed examples for $n\leqslant 8$ which don't contain any ADH−cycles. Hence

Conjecture 2.3.12 [*Ro1*] Every tournament T with $n=2k\geqslant 10$ has an ADH−cycle.

First [*Ro1*] proved this Conjecture for $n=2k\geqslant 28$. And then [*Pet*] proved this Conjecture with $n=2k\geqslant 16$. So it remains open only for three cases $n=10,12,14$.

[*Ro1*] deduced conjecture of cycle from Conjecture 2.3.12 as follows:

Conjecture 2.3.13 [*Ro1*] There is a number N such that $n\geqslant N$ implies that every tournament T contains all types of GC_k with two possible exceptions: $(1,\ldots,1)-GC_k$ and $(-1,\ldots,-1)-GC_k$.

[*Tho*] proved Conjecture 2.3.13 and got $N\geqslant 2^{128}(\approx 3.4\times 10^{38})$. The result has only theoretical meaning, since this estimation is much bigger than the real which may be $N=9$. So, the following theorems are still interesting.

Let D(n,p) be a $(1,\ldots,1,-1,\ldots,-1)-GC$ with $p-1$ consecutive -1's.

Theorem 2.3.14 [*BW2*] Every tournament T with $n\geqslant 3$ contains D(n,p), $p=2,3,\ldots,n$, except 5 especial classes of tournament T with $n\leqslant 6$.

Theorem 2.3.15 [*So7*] Le $T_1,T_2,\ldots,T_m$ be dicomponents of tournament T with $n\geqslant 5$. Then T is arc pan−(1)−generalized cyclic iff T satisfies one of the following conditions: (i) except some especial classes of tournaments, $m=1$ (ii) $2\leqslant m\leqslant 3$, and every arc in T_i is contained in some Hamilton dipath in T_i.

For bipartite tournaments, we have

Theorem 2.3.16[*WZL*] Let T be a bipartite tournament. If $h(T)\geqslant k(\geqslant 3)$, then T contains either an AD−cycle or an AD−path of the length at least 4k, except $T\simeq T(k_1,k_2,k_3,k_4)$ where $k\leqslant k_1,k_3\leqslant 2k-1$, $k\leqslant k_2,k_4$.

Corollary[*WZL, Qin*] Every diregular bipartite tournament contains an ADH−cycle except $T\simeq F_{4k}$.

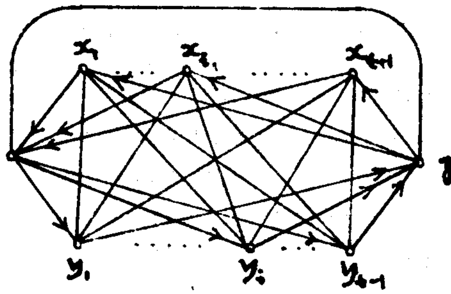
We turn our attention to generalized cycles in digraph. [*Gra*] suggests the following conjecture:

Conjecture 2.3.17 [*Gra*] Let D be a strict digraph with $n=2k\geqslant 4$. If D satisfies the condition $h(D)\geqslant n/2$, then D contains an ADH−cycle.

The following theorem seems to support this Conjecture.

Theorem 2.3.18 [*Gra*] Let D be a strict digraph with $n=2k\geqslant 4$. If D satisfies the conditions $h^+(D)\geqslant 2n/3+\sqrt{n\log n}$ and $h^-(D)\geqslant 2n/3$, then D contains an ADH−cycle.

It is much to be regretted that this Conjecture was disproved by [*Cai*]. The counterexample (Fig 7) as following:



But for D(n,2), [*Be1*] has

Theorem 2.3.19 [*Be1*] Let D be a digraph with order n. If D satisfies one of the following conditions, then D contains a D(n,2).

(i) $d^+(x)+d^-(y)<n \Rightarrow xy\in A(D)$ (Woodall condition)

(ii) $\delta(D)\geqslant n$ (Ghouila–Houri condition)

(iii) D is strong 2–connected with $\delta(D)\geqslant n-1$, and $D\not\cong D_0$

where D_0 is a special digraph with odd n as follows: $V(D_0)=A\cup B$, $A\cap B=\phi$, A is an independent set with $|A|=(n+1)/2$, $|B|=(n-1)/2$, between A and B join all possible arcs.

About the condition on the number of arcs, [*BW1*] suggested the following conjecture.

Conjecture 2.3.20 [*BW1*] A strong digraph D of order n with $\varepsilon(D)\geqslant(n-1)(n-2)+2$ contains all types of GC_n except (a)–(d) in Figure 1 and their converse.

[*HST*] proved that Conjecture 2.3.20 is true if $\varepsilon\geqslant(n-1)(n-2)+3$. [*Wo1*] further proved that Conjecture 2.3.20 is also true, except several special classes of digraphs.

Finally, it would be worthy pointing out that there is such a class of digraphs which doesn't contain any AD–cycle.

Conjecture 2.3.21 [*ApH*] For every planar graph G, there is an orientation of G such that the reault digraph doesn't contain any AD–cycle.

[*GrM*] proved that Conjecture 2.3.21 is true if G is a bipartite planar graph or an outerplanar graph.

Let t(n) ($t_k(n)$) the maximal number of arcs of all possible digraphs (k–partite digraphs) with n vertices which don't contain any AD–cycle, then

Theorem 2.3.22 [*GJP,JaP*]

(1) $[\frac{16}{5}(n-1)]\leqslant t(n)\leqslant\frac{7}{2}(n-1)$ when $n\geqslant 5$, the lower bound can be reached;

(2) $t_2(n)=2(n-1)$ when $n\geqslant 2$;

(3) $t_3(n)=3(n-1)$ when $n\geqslant 3$ and n is odd;

$3(n-1)-1\leqslant t_3(n)\leqslant 3(n-1)$ when $n\geqslant 4$ and n is even;

(4) $t_4(n)=3(n-1)$ when $n\geqslant 3$;

(5) $[16(n-1)/5]\leqslant t_5(n)\leqslant t_6(n)\leqslant 10(n-1)/3$ when $n\geqslant 5$;

(6) $t_k(n)\leqslant\begin{cases}4(1-1 \\ (k+1))(n-1) & k \text{ is odd,} \\ 4(1-1 \\ k)(n-1) & k \text{ is even}\end{cases}$

2.4 The enumeration of some special cycles and others

The enumeration of 3–cycles in a tournament is a solved problem, it only depends on the score list of the tournament. But if $k \geqslant 4$, the enumeration of k–cycles in a tournament is still open. Let C(T,4) denote the number of 4–cycles in a tournament T. Although the upper bound of the number of 4–cycles in a tournament of order n is obtained ([*BeT*]), the upper bound of C(T,4) over all tournaments with a given score list is still interesting. [*AlT*] has:

Theorem 2.4.1 [*AlT*] If $(s_1, s_2, \ldots, s_n)$ is a score list of a tournament T of order n, then

$$\binom{n}{4} - max\{S_1, S_2\} \geqslant C(T,4) \geqslant \binom{n}{4} - min\{S_1 - \sum_{i=1}^{n}(s_i)^*, S_2 - \sum_{i=1}^{n}(n-s_i-1)^*\},$$

where $S_1 = \sum_{i=1}^{n}\binom{s_i}{3}$, $S_2 = \sum_{i=1}^{n}\binom{n-s_i-1}{3}$, and $(m)^* = \begin{cases} m(m^2-1)/24 & m \text{ is odd}, \\ (m^3-4m)/24 & m \text{ is even}. \end{cases}$

By Theorem 2.4.1, it is easy to get the upper bound of the number of 4–cycles in a tournament of order n once again.

For 4–cycles in a random bipartite tournament, it also has:

Theorem 2.4.2 [*BFK*] Let $T_{m,n}$ be a random bipartite tournament. Let $M_{m,n}(p)$ and $\sigma^2_{m,n}(p)$ denote the expectation and the variance of 4–cycle in $T_{m,n}$ respectively. The

$$M_{m,n}(p) = 2\binom{m}{2}\binom{m}{2}p^2q^2, \quad \text{where } q = 1-p,$$

$$\sigma^2_{m,n}(p) = 2\binom{m}{2}\binom{m}{2}p^2q^2[1+2pq(mn-m-n)-2p^2q^2(2m-3)(2n-3)].$$

We call that a tournament T has the **property** $\mathfrak{K}_m$ ($\mathfrak{K}'_m$ resp.) if any arc in T is exactly contained in k(in k or k+1 resp.) m–cycles, where k is a positive constant. A tournament T is **doubly–regular** if all pairs of vertices jointly dominate the same number of vertices. Many propertices of doubly–regular tournaments were shown in § 8 of [*Row*]. The following is a new one.

Theorem 2.4.3 [*Row*] A tournament T is doubly–regular $\rightleftharpoons$ T satisfies the property $\mathfrak{K}_3$ $\rightleftharpoons$ T satisfies the property $\mathfrak{K}_4$ and $\mathfrak{K}_5$ or T is 3–cycle $\rightleftharpoons$ The degree of the minimal polynomial of adjacency matrix of T is three.

For the enumeration of 4–cycles in diregular tournaments, it has:

Theorem 2.4.4 [*Tab*] The lower bound of the number of 4–cycles in a diregular tournament with $n = 2k+1$ vertices is

$$\begin{cases} (2k+1)(k^3-k)/8 & if\, k \text{ is odd}, \\ (2k+1)k^3/8 & if\, k \text{ is even}, \end{cases}$$

and these bounds are reached by the tournaments which satisfy the property $\mathfrak{K}_3$(if k is odd) or $\mathfrak{K}'_3$(if k is even).

In [*Th7*], the author also estimated the lower bound of the number of Hamiltonian cycles in diregular tournaments.

The following conjecture is related to the enumeration of cycles in digraphs.

Adam's Conjecture. Any digraph containing a cycle has an arc whose reversal decrease the total number of cycles.

Counterexamples to Adám's Conjecture were obtained independently by [*Th8*] and [*DGN*]. But the Conjecture is still open for the classes of strict digraph, also for tournaments.

Acknowledgement

We would like to thank Qin Yu Sheng for his help in preparing the final manuscript.

References

[*All*]] Allender, E.W.,On the number of cycles possible in digraphs with large girth. Dis. Appl. Math. 10 (1985), 211–225.

[*Als*]] Alspach,B., Cycles of each length in regular tournaments. Canad. Math. Bull. 10 (1967), 283–286.

[*ARR*]] Alspach,B., K.B. Reid, and D.P. Roselle, Bypass in asymmetric digraphs. J. Combin. Theory B 17 (1974), 11–18.

[*AlR*] Alspach,B. and M.Rosenfeld, Realization of certain genera– lized path in tournaments, Dis. Math. 34 (1981), 199–202.

[*AlT*]] Alspach, B. and C. Tabib, A note on the number of 4–circuits in tournaments. Annals of Dis. Math. 12 (1982) 13–19.

[*AFG*] Amer, D., I.Fourmer and A.Germa, Some conditions for digraphs to be hamiltonian. Ann. Dis. Math. 20 (1984), 37–41.

[*AmM*] Amer, D. and Y. Manoussakis, Cyles and paths of many lengths in bipartite digraphs. J. Combin. Theory B 50(1990) 254–254.

[*ApH*] Appel, K. and W. Haken, Every planar map is four colorable. Bull. Amer. Math. Soc. 82 (1976), 711–712.

[*BaP*] Balakrishna,R. and P. Paulraja, Note on the existence of directed (k+1)–cycles in diconnected complete k–partite digraphs. J. Graph Theory 8 (1984), 423–426.

[*BCW*] Behzad, M., G. Chartrand and C.E.Wall, On minimal regular digraphs with given girth. Fund Math. 69 (1970), 227–231.

[*BeL*] Beineke, L. W. and C. H. C. Little, Cycles in bipartite tournaments. J. Combin. Theory B 32 (1982), 140–145.

[*Be1*] Benhocine, A., On the existence of a specified cycle in digraphs with constraints on degree. J. Graph Theory 8 (1984) 101–107.

[*Be2*] Benhocine,A., Pancyclism and meynic's conditions. Dis. Math. 58 (1986) 113–120.

[*BW1*] Benhocine, A. and A. P. Wojda, On the existence of D(n,p) in digraphs., Annals Dis. Math. 17 (1983), 47–52.

[BW2] Benhocine, A. and A.P. Wojda, On the existence of specified cycles in a tournaments. J. Graph Theory 7 (1983) 469–473.

[BW3] Benhocine, A. and A.P. Wojda, Bypasses in digraphs. Ars combinatoria 16 (1983) 85–94.

[Ber] Bermond, J. C., 1–graphs regulars de girth donne. Cahiers du CERO Bruxelles 17 (1975), 123–135.

[BGH] Bermond, J. C., A. Germa, M.C.Heydemann and D. Sotteau, Chemins et circuits dans les graphes orientes. Ann. Dis. Math. 8(1980) 293–309.

[BeL] Bermond, J. C. and L. Lovasz, Problem 3, Recent advances in graph theory. Proc. Coll. Prague Acad. Prague (1975), 541.

[BeT] Bermond, J.C. and C. Thomassen, Cycles in digraphs–A survey. J. Graph Theory 5, (1981), 1–43.

[Ber] Berge, C., Graphs (2th revised edition). North–Holland, Amsterdam, New York, Oxford (1985).

[BFK] Bollabas, B., O. Frank and M.Karonski, On 4–cycle in random bipartitie tournaments. J. Graph Theory 7 (1983), 183–194.

[Bon] Bondy, J.A., Diconnected orientations and a conjecture of Las Vergnas. J. London Math.Soc. 14 (1976), 277–282.

[BoM] Bondy, J.A. and U.S.R. Murty, Graph theory with its applications. The Macmillan Press LTD (1976).

[Cai] Cai Mai Cheng, A conterexample to a conjecture of Grant. Dis. Math 44 (1983), 111.

[CaH] Caccetta, L. and R. Haggkvist, On minimal digraph with given girth. Congressus Numerantium XXI (1978), 181–187.

[Cha] Chakroun, N. These 3me Cycle. Universite de Parise–Sud (1986).

[COR] Chartrand, G.,R. Oellermann and S. Ruiz, Randomly n–cyclic digraphs. Graphs and Combin. 1 (1985), 29–40.

[ChZ] Chen zhi–bo and Zhang Fu–ji, Bounds of the longest directed cycle length for minimal strong digraphs, Dis. Math. 68 (1987),9–13.

[DGN] Dambit, J. J., E. J. Grinberg and J. Nesetril, On the number of cycles in directed graphs. Comment. Math. Univ. Cardin in press.

[Da1] Darbinyan, S. Kh., Bypasses in tournaments. Akad. Nauk Armyan. SSR Dokl 66 (1978),138–141.

[Da2] Darbinyan, S. Kh., Special contours of directed graphs. Akad. Nauk Armyan. SSR Dokl 84 (1987), 51–55.

[Den] Dean, N., What is the smallest number of dicycles in a dicycle decomposition of an eulerian digraphs. J.Graph Theory 10 (1986), 299–308.

[EMR] Egawa, Y., T. Miyamoto and S. Ruiz, On randomly n–cyclic digraphs. Graphs and combinatorics 3 (1987), 227–238.

[FaO] Favaron, O. and O. Ordaz, A sufficient condition for oriented graphs to be hamiltonian. Dis. Math. 58 (1986), 243–252.

[Fra] Fraisse, P., Circuits including a given set of vertices. J. Graph Theory 10 (1986), 553–557.

[FrT] Fraisse, P. and C. Thomassen, Hamiltonian dicycles avoiding prescribed arc in tournaments. Graphs and Combinatorics 3 (1987) 239–250.

[GoO] Goddard, W. D. and O. R. Oellermann, On the cycle structure of multipartite tournaments. Proc. of the sixth international conference on the theory and applications of graphs at Western Michigan University (1988).

[God] Goddyn, L., A counterexample to a conjecture about oriented graphs. Annals of Dis. Math. 27 (1985), 235–236.

[Gra] Grant, D. D., Antidirected hamiltonian cycles in digraps. Ars Combin. 10 (1980), 205–209.

[GJP] Grant, D. D., F. Jaegar and C. Payan, On digraphs without antidirected cycles, J. Graph Theory 6 (1982), 133–138.

[GrM] Grant, D.D. and K.L.Mcavaney, Antidirected cycles in planar graphs. Ars Combin. 19A (1985), 143–147.

[Gru] Grunbaum,B., Antidirected hamiltonian paths in tournaments. J. Combin. Theory B 11 (1971), 249–257.

[Gua] Guan Shi Rong, Remarks on the best possible condition that a tournament is strongly panconnected. J. of Guangxi University (Natural science edition) 1 (1984), 21–26.

[Gut] Gutin, G. M., Cycles in strong n–partite tournaments. Vestsi Akad. Nauk BSSR Ser Fiz–Mat Navak 5 (1984), 105–106.

[HaT] Haggkvist, R. and C. Thomassen, On pancyclic digraphs. J. Combin. Theory B 20 (1976), 20–40.

[Ha1] Hamidoune, Y.O., A note on the girth of digraphs. Combinatorica 2 (1982),143–147.

[Ha2] Hamidoune, Y.O., A note on minimal directed graphs with given girth. J. Combin Theory B43 (1987), 343–348.

[He1] He Zhenbang, The arc–k–anticyclic property in tournaments, J. Nanjing University (Nature sciences edition) 26(3) (1990) 392–396.

[He2] Heydemann, M.C., Cycle in strong oriented graphs. Dis. Math. 38 (1982), 185–190.

[He3] Heydemann, M.C., Degrees and cycles in digraphs. Dis. Math. 41 (1982), 241–251.

[He4] Heydemann, M.C., Minimum number of circuits covering the vertices of a strong digraph. Annals of Dis. Math. 27 (1985), 287–296.

[HeS] Heydemann, M.C. and D. Sotteau, Numbers of arcs and cycles in digraphs. Dis. Math. 52 (1984), 199–207.

[HST] Heydemann, M.C., D.Sotteau and C.Thomassen, Orientations of hamiltonian cycles in digraph. Ars Combin 14 (1982), 3–8.

[HoR] Hoang, C.T. and B. Read, A note on short cycles in digraphs. Dis. Math. 66 (1987), 103–107.

[HZL] Hu Guan Zhang, Zhu Yong Jin and Liu Zheng Hong, A Sufficient condition for the arc–pancyclicity of a tournament. J. Sys. Science Math. Scin. 2 (1982), 207–219.

[Ja1] Jackson, B., Hamilton cycles in regular 2–connected graph. J. Combin. Theory B 29 (1980), 27–46.

[Ja2] Jackson B., Long paths and cycles in oriented graphs, J. Graph theory 5(1981) 145–157.

[Ja3] Jackson, B., A Chvatal–Erdos Condition for hamilton cycles in digraphs. J. Combin. Theory B 43 (1987), 245–252.

[JaO] Jackson, B. and O.Ordaz, Chvatal–Erdos conditions for 2– cyclability in di graphs. Ars Combin. 25(1988), 39–49.

[JaP] Jaeger, F. and C. Payan, On graphs without antidirected cycles and related topices. Rescarch Repoert No 135, Laboratoire d Informatique et Mathematiques Appliques de Grenable.

[Lew] Lewin, M., On maximal circuits in directed graph. J. Combin. Theory B 18 (1975), 175–179.

[LWZ] Li Gui Rong, Wang Jian Zhong, Zhang Ke Min and Song Zeng Min, Hamiltonian propertics for bipartite tournaments. Chinese J. Math. in press.

[MaS] Manber, R. and Shao Jia Yu, On digraphs with the odd cycle property. J. Graph Theory 10 (1986), 155–165.

[Man] Manoussakis, Y., k–linked and k–cyclic digraphs. J. Combin. Theory B 48(2) (1990) 216–226.

[Mey] Meyniel, H., Une condition suffisante d'existence d'un circuit hamiltonion dans un graphe oriente. J. Combin. Theory B 14 (1973), 137–147.

[Nas] Nash–Williams, C.St.J.A., Hamiliton circuits in graphs and digraphs, The many facets of graph theory. Springer Verlag Lecture Notes 110 (1969),237–243.

[Nis Nishimura, T., Short cycle in digraphs, Dis. Math. 72(1988), 295–298.

[Pet] Petrovic Vojislav, Antidirected hamiltonian circuits tournaments. Graph Theory, Novi Sad (1983), 259–269.

[Qin] Qin Yu Sheng, Thesis, Nanjing University(1990).

[Re1] Reid, K.B., Three problems on tournaments, Graph Theory and its Applications: East and West. Annals of the New York Academy of Sciences 576(1989), 466–473.

[Re2] Reid,K.B., Two Completmentary Circuits in two–connected tournaments. Annals of Dis. Math. 27 (1985), 321–334.

[ReB] Reid, K.B. and L.W.Beineke, Tournaments. in Selected Topics in Graph Theory, Edited by L.W. Beineke and K.J.Wilson, Academic press (1978), 169–204.

[Ro1] Rosenfeld, M., Antidirected hamiltonian circuits in tournaments. J. Combin. Theoery B 16 (1974), 234–242.

[Ro2] Rosenfeld,M., Antidirect hamiltonian paths in tournaments. J. Combin. Theory B12 (1972), 93–99.

[Row] Rowlinson,R., On 4–cycles and 5–cycles in regular tournaments. Bull. Londen Math. Soc. 18, (1986), 135–139.

[SeT] Seymour, P. and C. Thomassen, Characterization of even directed graphs. J. Combin. Theory B 42 (1987), 36–45.

[So1] Song Zeng Min, Long paths and cycles in oriented graphs. J. Nanjing Inst. Tech. 16(5) (1986), 102–108.

[So2] Song Zeng Min, Number of arcs and cycles in digraphs. J. Nanjing Inst. Tech. 17(2) (1987), 122–125.

[*So3*] Song Zeng Min, The longest path or cycle in digraph. J. Nanjing Inst. Tech. 17(4) (1987), 133–137.

[*So4*] Song Zeng Min, About some cyclic properties in digraphs. J. Nanjing Inst. Tech. 17(5) (1987), 60–69.

[*So5*] Song Zeng Min, Complementary cycles in bipartite tournaments. J. Nanjing Inst. Tech. 18(5) (1988), 32–38.

[*So6*] Song Zeng Min, The Complementary cycles of all lengths in 2–Connected tournaments, J. Combin. Theory B, in press.

[*So7*] Song Zeng Min, Arc 1–antidirected cycles in tournaments. to submited J. of Southeast University.

[*So8*] Song Zeng Min, The longest cycle and number of arcs in digraph. J. of Southeast University, 19(3) (1989), 74–80..

[*So9*] Song Zeng Min, Long paths and cycles in oriented graphs, J. of Nanjing University, 25(1989) 365–370.

[*So10*] Song Zeng Min, Pancyclic oriented graphs, to appear in J. Graphs Theory.

[*SoZ*] Song Zeng Min, Zhang Zhong Pu, Circumference in digraphs and Nash–Williams's Conjecture Forbidding subgraph on the longest cycle, J. of Nanjing University, 1991.

[*Tab*] Tabib, C., The number of 4–cycles in regular tournaments. Utilitas Math. 22 (1982), 315–322.

[*Tao*] Tao Pei Hua, The optimal bound for a tournament having arc– pan bypass property. J. of Guangxi University (Natural science edition) 1 (1983), 138–141.

[*Tho*] Thomason, A., Paths and cycles in tournaments. Trans Amer. Math. Soc. 296 (1986), 167–180.

[*Th1*] Thomassen, C., Hamiltonian–connected tournaments. J. Combin. Theory B 28 (1980), 142–163.

[*Th2*] Thomassen, C., Long cycles in digraphs. Proc. London Math. Soc. 42 (1981), 231–251.

[*Th3*] Thomassen, C., Edge–disjoint hamiltonian paths and cycles in tournaments. Proc. London Math. Soc. (3) 45 (1982), 151– 168.

[*Th4*] Thomassen, C., Disjoint Cycles in digraphs. Combinatorica 3 (1983), 393–396.

[*Th5*] Thomassen, C., Connectivity in tournaments. in Graph Theory and Combinatorics (B. Bollabas ed.) London Acad. Press (1984), 305–313.

[*Th6*] Thomassen, C., The 2–linkage problem for cyclic digraphs. Dis. Math. 55 (1985), 73–87.

[*Th7*] Thomassen, C., Hamilton circuits in regular tournaments. Annals of Dis. Math. 27 (1985), 159–162.

[*Th8*] Thomassen, C., Reflections on graph theory. J. Graph Theory 10 (1986), 309–324.

[*Th9*] Thomassen, C., Counterexamples to Adam's conjecture on arc reversals in directed graphs. J. Combin. Theory B 42 (1987), 128–130.

[*Th10*] Thomassen, C., On digraphs with no two disjoint directed cycles. Combinatorica 7(1) (1987), 145–150.

[*TWZ*] Tian Feng, Wu Zheng Sheng and Zhang Cun Quan, Cycles of each length in tournaments. J. Combin. Theory B 33 (1982), 245–255.

[WaC] Wang Jian Fan and Cai Chen, Hamiltonian circuits of oriented graphs, Chinese J. of Oper. Res. 1(1) (1982), 64–65.

[Wa1] Wang Jian Zhong, A necessary and sufficient condition for Hamiltonian tournaments. J. of Qufu Normal College (Natural science edition) 1(1982), 14.

[Wa2] Wang Jian Zhong, A kind of strong (p–1)–path–connected tournament which has not strong (p–2)–path–connectivity. Hunan Annals of Math. 1(1983), 1–5.

[Wa3] Wang Jian Zhong, On a property of arc–hamiltonian tournaments. Acta Math. Appl. Sincia 7(4) (1984), 413–422.

[Wa4] Wang Jian Zhong, Long cycles in bipartite tournaments. Dis. Math. in press.

[Wa5] Wang Jian Zhong, On Alspach problem in bipartite tournaments , Chinese Science Bull, 9(1989) 716.

[Wa6] Wang Jian Zhong, A sufficient condition for hamiltonian cycles in bipartite tournaments , Australasian J. of Combin. in press.

[WH1] Wang Jian Zhong and He Shu Quang, On arc pancyclicity of regular bipartite tournaments, Chinese Science Bull. 1 (1987), 76.

[WH2] Wang Jian Zhong and He Zhen Bang, On P property of tournaments, J. of Taiyuan Institute of Mechanices 2(1988) 28–36.

[WZL] Wang Jian Zhong, Zhang Ke Min and Li Gui Rong, On antidirected cycles and paths in bipartite tournaments. to submit Appl. Math. ——A. J. of China Universities.

[Wo1] Wojda, A. P., Orientations of hamiltonian cycles in digraphs. J. Graph Theory 10 (1986) 211–218.

[Wo2] Wojda, A. P., Extremal non–hamiltonian or non–hamiltonian connected digraphs. Ars Combin. 25C(1988) 235–240.

[Woo] Woodall, D. R., Sufficient conditions for circuits in graphs, Proc. London Math. Soc. 24 (1972),739–755.

[WZ1] Wu Zheng Sheng, Zhang Ke Min and Zou Yuan, k–arc–cyclicity of $T_n$graph. J. of Nanjing Normal College (Natural science edition) 1(1981), 1–3.

[WZ2] Wu Zheng Sheng, Zhang Ke Min and Zou Yuan, A necessary and sufficient condition for arc–pancyclicity of tournaments, Scientia Sinica (A) 25 (1982), 249–254.

[WZ3] Wu Zheng Sheng, Zhang Ke Min and Zou Yuan, A kind of counterexamples on arc–pancyclic tournaments, Acta Math. Appl. Sinica 6(1) (1983), 47–49.

[Yan] Yang Chang Guan, On a conjecture about arc–antipancyclicity of tournament. J. of Nanjing Normal University (Natural science edition) 3(1987), 10–13.

[Zh1] Zhang Cun Quan, Every regular tournament having two arc– disjoint Hamilton cycles. J. of Qufu Normal Collage (Natural science edition), Speial issue Oper. Res. (1980), 70–81.

[Zh2] Zhang Cun Quan, On arc–pancyclicity of a class of tourna– ments. Scientia Sinica 9 (1981), 1056–1062.

[Zh3] Zhang Cun Quan, Arc–disjoint circuits in digraphs, Dis. Math. 41(1982) 79–96.

[Zh4] Zhang Cun Quan, Hamilton paths in Multipartite Oriented Graphs, Ann. Dis. Math. 41(1989)

499–514.

[Zh5] Zhang Ke Min, Completely strong path–connected tournaments, J. Combin. Theory. B 33 (1982) 166–177.

[Zh6] Zhang Ke Min, Vertex even–pancyclicity in bipartite tournaments. J. of Nanjing University, Math. Biquarterly 1(1) (1984), 85–88.

[Zh7] Zhang Ke Min, Longest paths and Cycles in bipartite oriented graph. J. Graph Theory 11 (1987) 339–348.

[ZMS] Zhang Ke Min, Y. Manoussakis and Song Zeng Min, Complementary cycles containing a fixed arc in the bipartite tournaments. Dis. Math., in press.

[ZSW] Zhang Ke Min, Song Zeng Min and Wang Jian Zhong, On Hamiltonian bipartite tournaments, J. of Nanjing University, Math. Biquarterly 8(1) (1991).

[ZW1] Zhang Ke Min and Wang Jian Zhong, Complementary cycles containing a fiexed arc and a fixed vertex in bipartite tournaments, Ars Combinatorica, in press.

[ZW2] Zhang Ke Min and Wu Zheng Sheng, A necessary and sufficient condition for complete strong path connectivity of tourna– ments. Chinese Ann. Math. A(4) (1983), 385–392.

[ZWZ] Zhang Ke Min, Wu Zheng Sheng and Zou Yuan, On the existence of a class of tournaments with completely strong path connectivity. J. of Nanjing University (Natrual science edition) ? (1981), 303–307.

[Zhu] Zhu Xiao Fei, On arc–Hamiltonian tournaments. Advances in Math. 15(3) (1986), 321–324.

[ZhT] Zhu Yong Jing and Tian Feng, On the strong path connectivity of a tournament. Scientia Sinica, Special Issue (II) (1979), 18–28.

[ZTC] Zhu Yong Jing, Tian Feng, Chen Chuan Ping and Zhang Cun Quan, Are–pancyclic property of tournament under some degree conditions, J. Infor. Optim. Sci. 5 (1984), 1–16.

[ZhC] Zhu Y. J. and C. Chen, An extrence problem concerning k–arc–cyclic property for a class of tournaments. Dis. Math. 85(3) (1990), 301–311.

有向图中的回路综述

张克民 宋增民
(南京大学数学系) (东南大学数学系)

摘 要

本文的主要目的是综述有向图,定向图中的回路,最长回路,2–回路性质和围长;竞赛图中的各种泛回路,广义回路的最新结果,以及和它们相关联的文献和未解决的问题.

关键词: 有向图,竞赛图,回路,综述.