Complementary Cycles Containing a Fixed Arc and a Fixed Vertex in Bipartite Tournaments

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Abstract. Suppose that $R = (V, A)$ is a diregular bipartite tournament of order $p \geq 8$. Denote a cycle of length k by C_k. Then for any $e \in A(R)$, $w \in V(R) \setminus V(e)$, there exists a pair of vertex-disjoint cycles C_4 and C_{p-4} in R with $e \in C_4$ and $w \in C_{p-4}$, except R is isomorphic to a special digraph $\widetilde{F}_{4k}$.

1. Introduction.

A bipartite tournament is an oriented complete bipartite graph. Just as ordinary tournaments may be used to represent a competition, so may bipartite tournaments. In the former case, each player competes against everyone else; while in the latter case, there are two teams and each player competes against everyone on the opposing team. Tournaments and bipartite tournaments are perhaps the most interesting two classes of oriented graphs. However, much less is known about the latter than the former. Properties of cycles in bipartite tournaments were investigated in [1, 3-10]. These include:

Theorem 1 [9]. *Suppose that R is a diregular bipartite tournament of order $p(p > 4)$, and u, v are two distinct vertices in R. Then there exists a pair of vertex-disjoint cycles C_4 and C_{p-4} in R with $u \in C_4$ and $v \in C_{p-4}$, except when R is a special 8-digraph.*

Theorem 2 [10]. *Suppose that R is a diregular bipartite tournament of order $p(p > 4)$ and e is any arc in R. Then there exists a pair of vertex-disjoint cycles C_4 and C_{p-4} in R with $e \in C_4$.*

In this paper, we shall prove a stronger result from which Theorem 1 and Theorem 2 follow as corollaries.

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2. Notation and some known results.

Let $R = (X, Y; A)$ be a bipartite tournament of order p, where (X, Y) is the bipartition of the vertex set $V = X \cup Y$ of R, and A is the set of arcs of R. For any $v \in V$ and any $S \subseteq V$, we define:

$$I(v) = \{u \in V \mid uv \in A\}, \qquad O(v) = \{u \in V \mid vu \in A\},$$
$$I(S) = \cup_{v \in S} I(v) \qquad \text{and} \qquad O(S) = \cup_{v \in S} O(v).$$

R is said to be *k-diregular* if $|I(v)| = |O(v)| = k$, for all $v \in V$. A *factor* of R is a spanning 1-diregular subgraph of R. Thus, a factor is a union of vertex disjoint cycles. For any subsets S, T of V, $S \Rightarrow T$ denotes that for all s in $S \cap X$ and t in $T \cap Y$, st is in A and for all s in $S \cap Y$ and t in $T \cap X$, st is in A. For any integer k, $F_{4k} = (V, A)$ is defined as follows: $V = \{v_1, v_2, \ldots, v_{4k}\}$, $v_i v_j \in A \Leftrightarrow j - i \equiv 1 \pmod 4$. F^*_{4k} is obtained from F_{4k} by reversing all arcs of a 4-cycle $uyxvu$ in F_{4k} (see Figure 1). And $\widetilde{F}_{4k}$ is obtained from F^*_{4k} by reversing all arcs of some 4-cycles: $v_1(i)\, v_2(i)\, v_3(i)\, x v_1(i)$, $i = 1, 2, \ldots, m$, where $v_1(i) \in V_1 \setminus \{v\}$, $v_2(i) \in V_2 \setminus \{u\}$, $v_3(i) \in V_3 \setminus \{y\}$ and when $i \neq j$, $v_1(i) \neq v_1(j)$, $v_3(i) \neq v_3(j)$. Clearly, when $m = 0$, $\widetilde{F}_{4k} = F^*_{4k}$. Denote a cycle of length k by C_k. We call C_k and C_{p-k} $(4 \leq k \leq p - 4)$ a pair of *complementary cycles* in R if C_k and C_{p-k} are two vertex-disjoint cycles in R. Other terms and symbols are found in [2].

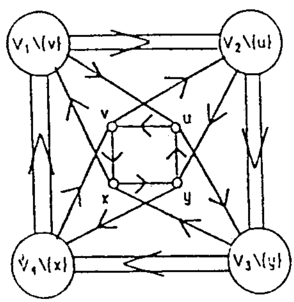

Figure 1. F^*_{4k}

In order to prove the main result, we will need the following Theorems.

Theorem 3 [5]. *A bipartite tournament R is hamiltonian if and only if both of the following conditions hold: (a) R is strong; (b) there is a factor in R.*

Theorem 4 [5]. *Let R be a bipartite tournament containing a factor. Then R is not strong if and only if there exists a factor F in R consisting of cycles $C^1, C^2, \ldots, C^m$, $m \geq 2$, such that $C^i \Rightarrow C^j$ if $i < j$.*

Let $R = (X, Y; A)$ be a bipartite tournament. And let $A_1 = \{ab \in A \mid a \in X, b \in Y\}$, $A_2 = A \setminus A_1$, $R(1) = R[A_1]$ and $R(2) = R[A_2]$. By the definition of factor, we have:

Theorem 5. *There exists a factor in R if and only if both of the following conditions hold: (a) $|X| = |Y|$, (b) there is a perfect matching in $R(1)$ and $R(2)$ respectively.*

3. Main result.

Theorem. *Suppose that $R = (X, Y; A)$ is a k-diregular bipartite tournament ($k \geq 2$). Then for any $e = uv \in A(R)$ and $w \in V(R) \setminus V(e)$, there exists a pair of vertex-disjoint cycles C_4 and C_{4k-4} in R such that $e \in C_4$, $w \in C_{4k-4}$, unless R is isomorphic to $\widetilde{F}_{4k}$.*

Proof: Clearly, $|X \cup Y| = |V| = 2|X| = 2|Y| = 4k$. Without loss of generality, suppose $u \in X$. We establish three claims.

Claim 1. R contains a cycle C_4 such that $uv \in C_4$ and $w \notin C_4$.

If $w \in X$, we may pick an $x \in O(v)$ and $x \neq w$ since $k \geq 2$. By k-diregularity of R, there exists $y \in I(u)$ with $xy \in A(R)$. Thus, there is a $C_4 = uvxyu$ in R. Similarly, for $w \in Y$.

Claim 2. If R is not isomorphic to $\widetilde{F}_{4k}$, then there exists a cycle C_4 through e such that $R_1 = R - C_4$ has a factor containing w.

Suppose there is a cycle C_4 containing e such that R_1 has no factor. By Theorem 5 and König-Hall's theorem on matching, it follows that there exists a subset S either of $X \setminus \{u, x\}$ or of $Y \setminus \{v, y\}$ such that $|S| > |O(S)|$. Without loss of generality, we assume that $S = X_1$ is in $X \setminus \{u, x\}$. Let $O(X_1) = Y_1$, $X_2 = X \setminus (X_1 \cup \{u, x\})$ and $Y_2 = Y \setminus (Y_1 \cup \{v, y\})$. Thus, $Y_2 \Rightarrow X_1$. Since R is k-diregular, $k \geq |X_1| > |O(X_1)| = |Y_1| \geq k - 2$. We will consider three subcases as follows:

(a) $|X_1| = k$ and $|Y_1| = k - 2$. By the k-diregularity of R, we have that $X_1 \Rightarrow Y_1 \cup \{v, y\}$. Hence, $|I(v)| = |X_1| + 1 = k + 1$, a contradiction.

(b) $|X_1| = k$ and $|Y_1| = k - 1$. By the k-diregularity of R, $X_2 \cup \{u, x\} \Rightarrow Y_2$, and since $|I(v)| = k$, there exists a vertex $x_1 \in X_1$ such that $vx_1, x_1y \in A$. Let $C'_4 = uvx_1yu$ and $R'_1 = R - C'_4$.

If $x_1 \neq w$, we have to prove that there is no subset S' of $X \setminus \{u, x_1\}$ (or of $Y \setminus \{v, y\}$ resp.) such that $|S'| > |O_{R'_1}(S')|$. In fact, if $|S'| = k$, then both $S' \cap (X_1 \setminus \{x_1\}) \neq \phi$ and $S' \cap (X_2 \cup \{x\}) \neq \phi$ ($S' \cap Y_1 \neq \phi$ and $S' \cap Y_2 \neq \phi$ resp.) hold. Hence, $|O_{R'_1}(S')| \geq |S'| = k$. If $|S'| = k - 1$, once more, we may easily verify that $|S'| \leq |0_{R_1'}(S')|$ unless $S' = Y_1$ or $S' = X_1 \setminus \{x_1\}$. When $S' = Y_1$, note that $x_1 \Rightarrow Y_1$. By k-diregularity of R, $0_{R_1'}(Y_1)| \geq k-1$. When $S' = X_1 \setminus \{x_1\}$ and $|S'| > |0_{R_1'}(S')|$, there exists a vertex $y_1 \in Y_1$ such that $y_1 \Rightarrow X_1 \setminus \{x_1\}$. Hence, $X_1 \setminus \{x_1\} \Rightarrow y$. Note that $x_1 y, xy \in A$, thus, we have $|I(y)| > k$, a contradiction.

If $x_1 = w$ and $X_1 \setminus \{x_1\} \Rightarrow v$, we will prove that $R \cong \widetilde{F}_{4k}$, with $V_1 \setminus \{v\}$ in Figure 1, corresponding to $X_1 \setminus \{x_1 = w\}$, $V_2 \setminus \{u\}$ to Y_1, $V_3 \setminus \{y\}$ to $X_2 \cup \{x\}$, $V_4 \setminus \{x\}$ to Y_2 and $uvxyu$ to $vx_1(=w)yuv$. Clearly, $X_2 \cup \{x\} \Rightarrow Y_2 \Rightarrow X_1 \setminus \{w\}$, $Y_1 \Rightarrow \{u, x\} \Rightarrow Y_2 \Rightarrow w \Rightarrow Y_1$, $X_1 \setminus \{w\} \Rightarrow v \Rightarrow X_2 \cup \{x\}$. Suppose $X_2' y_1' \in A$, where $x_2' \in X_2$, $y_1' \in Y_1$. By k-diregularity of R, then $yx_2' \in A$, and there exists a vertex $x_1' \in X_1 \setminus \{w\}$ such that $y_1' x_1' \in A$, thus, $x_1' y \in A$. Hence, $x_2' y_1'$ lies on a 4-cycle $x_2' y_1' \; x_1' y x_2'$ in R. Suppose $yx_2'' \in A$, where $x_2'' \in X_2$. Note that $vx_1'' \in A$. By k-diregularity of R, there exist vertices $y_1'' \in Y_1$ and $x_1'' \in X_1 \setminus \{w\}$ such that $x_2'' y_1'' \in A$ and $y_1'' x_1'' \in A$. And then $x_1'' y \in A$. Hence, yx_2'' lies on a 4-cycle $yx_2'' \; y_1'' \; x_1'' y$ in R. Using a similar argument, we can show that if $y_1 x_1' \in A$ or $x_1' y \in A$, where $y_1 \in Y_1$, $x_1' \in X_1 \setminus \{w\}$, then $y_1 x_1'$ or $x_1' y$ also lies on a 4-cycle, respectively, as above in R. Finally, if there are two 4-cycles $y_1(i) x_1(i) y x_2(i) \; y_1(i)$, $i = 1, 2$ in R where $y_1(i) \in Y_1$, $x_1(i) \in X_1 \setminus \{w\}$, $x_2(i) \in X_2$, then, once more, by the k-diregularity of R, we have $x_1(1) \neq x_1(2)$ and $x_2(1) \neq x_2(2)$. Therefore, $R \cong \widetilde{F}_{4k}$.

(c) $|X_1| = k-1$ and $|Y_1| = k-2$. We have $Y_2 \Rightarrow X_1 \Rightarrow Y_1 \cup \{v, y\}$, and $\{v, y\} \Rightarrow X_2$. Since $|0(u)| = k$, there exists a vertex y_2 in Y_2 such that $y_2 u$, $xy_2 \in A$. Let $C_4'' = uvxy_2 u$ and $R_1'' = R - C_4''$.

Suppose $y_2 \neq w$. If R_1'' has no factor, then, as above, there exists a subset S'' either of $X \setminus \{u, x\}$ or of $Y \setminus \{v, y_2\}$ such that $k \geq |S''| > |0_{R_1''}(S'')| \geq k-2$. If $|S''| = k$, then case (a) or (b) applies. So we assume that $|S''| = k-1$. Note that in this case, it is enough to consider that $S'' = X_2$ or $Y_1 \cup \{y\}$. Since $v \Rightarrow X_2$ and R is k-diregular, $|0_{R_1''}(X_2)| \geq k-1 = |X_2|$. Since $|0_{R_1''}(y)| \geq k-1$, $|0_{R_1''}(Y_1 \cup \{y\})| \geq k-1 = |Y_1 \cup \{y\}|$.

If $y_2 = w$, we may assume $u \Rightarrow Y_2 \setminus \{y_2\}$. Hence, $Y_1 \cup \{y\} \Rightarrow u$ and $X_2 \Rightarrow w$. Using a similar argument to that in (b), we can prove that $R \cong \widetilde{F}_{4k}$ with $V_1 \setminus \{v\}$ in Figure 1, corresponding to $Y_1 \cup \{y\}$, $V_2 \setminus \{u\}$ to X_2, $V_3 \setminus \{y\}$ to $Y_2 \setminus \{y = w\}$, $V_4 \setminus \{x\}$ to X_1 and $uvxyu$ to $uvxy_2 u$.

Claim 3. For $k \geq 4$ and R and C_4 as in Claim 2, $R_1 = R - C_4$ is strong.

Suppose R_1 is not strong. Let $C^1, C^2, \ldots, C^m$, $m \geq 2$, be cycles of R, as they are described in Theorem 4. If $|V(C^1)| \leq |V(C^2) \cup \ldots \cup V(C^m)|$, then there exists a vertex $z \in C^1$ such that $k = |0(z)| \geq |V(C^1)|/4 + |V(C^2) \cup \ldots \cup V(C^m)|/2 \geq (|V(C^1)| + |V(C^2)| + \ldots + |V(C^m)|)/4 + (|V(C^2)| + \ldots + |V(C^m)|)/4 \geq (p-4)/4 + (p-4)/8$. Since $p = 4k$, this is a contradiction for $k \geq 4$. On the other hand, if $|V(C^1)| \geq |V(C^2) \cup \ldots \cup V(C^m)|$, then using a similar argument, we obtain a contradiction by considering $|I(z)|$, where $z \in C^m$.

We can now settle the Theorem. Suppose R is not isomorphic to $\widetilde{F}_{4k}$ and R_1 has a factor containing w as in Claim 2. If $k \neq 3$, then R_1 is hamiltonian by Claim 3 and Theorem 3. Hence, Theorem is true. If $k = 3$, then either R_1 is hamiltonian or, by Theorem 4, R_1 consists of two 4-cycles $C' = 12341$ and $C'' = 56785$ such that $C' \Rightarrow C''$. By the 3-diregularity of R, we have $C'' \Rightarrow C_4 \Rightarrow C'$. Thus, there exist two pairs of complementary cycles as follows: $uv18\,u$ and $y3456\,x27\,y$ which satisfy the Theorem if $w \neq 1, 8$; or $uv36\,u$ and $y1278\,x45\,y$ which satisfy the Theorem if $w = 1$ or 8. This completes the proof of Theorem. ∎

Acknowledgement.

We wish to thank the referee for his comments.

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