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ON EDGE-PANCYCLIC GRAPHS

BY

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Abstract. Let G have the property that for any non-adjacent vertices $u.v \in VG$, $d(u) + d(v) \geq 1 + |VG|$. We characterize the graphs of this type in which every edge lies on a cycle of every length from 3 to $|VG|$ inclusive.

1. Introduction

In this paper, all graphs G are simple. We let $n = |VG| \geq 5$. A graph is *pancyclic* if it contains a cycle of every length from 3 to n inclusive, while an edge (a vertex resp.) is *pancyclic* if it lies on a cycle of every length from 3 to n inclusive. A graph is *edge-pancyclic* (*vertex-pancyclic* resp.) if every edge (every vertex resp.) is pancyclic.

If $e \in EG$ does not lie on a k-cycle in G we say that e is *k^--cyclic.* Further $e \in EG$ is *k^--pancyclic* if it lies on a cycle of every length from 3 to n inclusive except k.

In this note we will consider graphs with the property $O(n+1)$: for any $u, v \in VG$ such that $uv \notin EG$, then $d(u) + d(v) \geq n + 1$.

Finally by $G \subseteq H$ we mean that G is a spanning subgraph of H.

Faudree and Schelp [3] proved the following result.

Theorem 1. *Let G have property $O(n+1)$. Then there is a path of*

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every length from 5 to n inclusive between any pair of vertices.

A shorter proof of this result is due to Cai [2].

The corollary follows immediately.

Corollary. *Let G have property $O(n+1)$. Then every edge in G lies on a cycle of length 5 to n inclusive.*

We will now construct some classes of graphs which will prove to be the only ones with property $O(n+1)$ that contain 3^--pancyclic or 4^--pancyclic edges.

Let $G_1 = (K_1 + K_{n-a-b-1}) \vee (K_a^c + K_b)$, where $a \geq 1$, $b = 0$ or $b \geq 2$ are positive integers. The graph operations join $\vee$, union $+$ and complement of H H^c are defined in [1]. Let sP_2 be a set of s independent edges and let $G_2 = (sP_2 + K_{n-b-2s}) \vee K_b^c$, where s and b are integers, and for $s = 1$ $3 \leq b \leq \frac{1}{2}(n-1)$ and for $2 \leq s \leq \lceil \frac{1}{2}b \rceil - 1$, and $n = 2b+1$.

2. Main Results

Theorem 2. *Let G satisfy the condition $O(n+1)$. Then*

(a) *G can not contain both 3^--pancyclic edges and 4^--pancyclic edges.*

(b) *G contains a 3^--pancyclic edge if and only if $G \subseteq G_1$.*

(c) *G contains a 4^--pancyclic edge if and only if $G \subseteq G_2$.*

Proof.

(i) We first prove that if G contains a 3^--cyclic edge then $G \subseteq G_1$.

Let u_1v_1, u_2v_2 be two independent 3^--cyclic edges in G. Let $B = N(u_1) \cap N(u_2)$, $A = N(u_1) \setminus (B \cup \{u_2\})$ and $C = N(u_2) \setminus (B \cup \{u_1\})$. Finally, let $|A| = a$, $|B| = b$, $|C| = c$. If $D = VG \setminus (\{u_1, u_2\} \cup A \cup B \cup C)$, then $|D| = n - a - b - c - 2$.

Since not both $u_2, v_2 \in N(u_1)$, we may assume without loss of generality that $u_1u_2 \notin EG$. We note that if $v_1 \in A$. then $d(v_1) \leq n - a - b - 1$. While if $v_1 \in B$, then $d(v_1) \leq n - a - b$. Hence $d(v_1) \leq n - d(u_1)$ and similarly $d(v_2) \leq n - d(u_2)$. Thus $d(v_1) + d(v_2) \leq 2n - (d(u_1) + d(u_2)) \leq n - 1$. Hence by the condition $O(n+1)$, $v_1v_2 \in EG$ and $v_1 \in A$, $v_2 \in C$. Without loss of generality we may assume that $a \geq c$. However $u_2v_1 \notin EG$, so $n+1 \leq d(u_2)+d(v_1) \leq (b+c)+(n-a-b-1) \leq n-1$.

This contradiction shows that G contains no two independent 3^--cyclic edges. Hence $G \subseteq G_1$.

(ii) We now prove that if G contains a 4^--cyclic edge then $G \subseteq G_2$.

Let $e = uv$ be a 4^--cyclic edge. Let $N(u) \setminus (N(v) \cup \{v\}) = A$, $N(u) \cap N(v) = B$ and $N(v) \setminus (N(u) \cup \{u\}) = C$, with $|A| = a$, $|B| = b$ and $|C| = c$. Let $VG \setminus (N(u) \cup N(v)) = D$. Then $|D| = n-a-b-c-2$.

Since e is 4^--cyclic, then $|EG[B]| = 0$.

Now $d(u) = a+b+1$. For $w \in C$, $d(w) \leq (c-1)+1+(n-a-b-c-2) = n-a-b-2$. Since $uw \notin EG$, we have $n+1 \leq d(u)+d(w) \leq n-1$, a contradiction. Hence $C = \phi$. Similarly $A = \phi$.

Let G contain s distinct 4^--cyclic edges. First consider $s = 1$. Suppose e is the only 4^--cyclic edge in G. Let $x \in D$. Then $d(x) \leq n-3$. However since $ux \notin EG$, $n+1 \leq d(u)+d(x) \leq (b+1)+(n-3)$. Hence $b \geq 3$.

Further, for $z_1, z_2 \in B$, $d(z_1) \leq n-b$ and $d(z_2) \leq n-b$. Since $z_1, z_2 \notin EG$, $n+1 \leq 2n-2b$. Hence $b \leq \frac{1}{2}(n-1)$. We have thus shown that $G \subseteq G_2$.

Suppose G contains the $s \geq 2$ distinct 4^--cyclic edges $e_1, e_2, \ldots, e_s$. Let $e_i = u_iv_i$. If $u_2 = v_1$, then we know from the argument above that $u_1v_2 \in EG$. Clearly there is no $v \in N(u_1) \cap N(v_1)$ and $v \neq v_2$, since e_2 is 4^--cyclic. Hence $d(u_1) = d(v_1) = 2$. Similarly $d(v_2) = 2$. Since G is connected by the condition $O(n+1)$, $G = K_3$, which contradicts $|VG| \geq 5$. It follows therefore that $\{e_1, e_2, \ldots, e_s\}$ is a set of independent edges.

Repeating the argument above which showed that $A = \phi = C$ on the neighbourhoods of e_1 and e_2, we see that $\{w : u_1w, v_1w \in EG \text{ and } u_2w, v_2w \notin EG\} = \phi$. Hence $N(u_i) \cap N(v_i) = N(u_1) \cap N(v_1) = B$ and $|D| = n-2s-b$.

Since G satisfies $O(n+1)$, $n+1 \leq d(u_1)+d(u_2) = 2b+2$. Hence $n \leq 2b+1$. Further if $z_1, z_2 \in B$ we have $n+1 \leq d(z_1)+d(z_2) \leq 2n-2b$. So $n \geq 2b+1$. Therefore $n = 2b+1$.

Finally, for $x \in D$ we have $n+1 \leq d(u_1)-d(x) \leq n-2s+b$.

Hence $s \leq \lceil\frac{1}{2}b\rceil - 1$. Therefore $2 \leq s \leq \lceil\frac{1}{2}b\rceil - 1$.

It follows that $G \subseteq G_2$.

(iii) We now show that if G contains a 4^--cyclic edge it contains no 3^--cyclic edge.

Suppose to the contrary, G contains 3^--cyclic edges. Let $e = uv$ be a 4^--cyclic edge in G. Let B, D and s be defined as in (ii). Let $z_1, z_2 \in B$ and $w_1, w_2 \in D$. Let $r = |N(w_1) \cap B|$ and $t = |N(w_1) \cap D|$. Consider two cases. First suppose $w_1w_2 \in EG$ is a 3^--cyclic edge. We have $N(w_1) \cap B \neq \phi$. Now for $z_1 \in N(w_1) \cap B$, $d(z_1) \leq n - b$. Further $d(w_2) \leq n - (r + t + 2s)$. Since $uw_1 \notin EG$, $n + 1 \leq d(u) + d(w_1) = r + t + b + 1$. Since $z_1w_2 \notin EG$, $n + 1 \leq d(z_1) + d(w_2) \leq 2n - b - t - r - 2s$. But this implies $n \geq b + t + r + 2s + 1 \geq n + 3$, a contradiction. Now Suppose z_1w_1 is a 3^--cyclic edge. Since $uw_1 \notin EG$, $n + 1 \leq d(u) + d(w_1) = r + t + b + 1$. Further, since $z_1z_2 \notin EG$, $n + 1 \leq d(z_1) + d(z_2) \leq 2n - 2b - t$. Hence $n \geq 2b + t + 1$. This implies that $r + t + b \geq 2b + t + 1$, or $r \geq b + 1$, a contradiction.

This completes the proof of (a).

(iv) We now show that if $G \subseteq G_1$, then G contains a 3^--pancyclic edge.

Suppose $G \subseteq G_1$ and there is no edge from K_1 to K_a^c in G. Let $u \in VK_1$ and $v \in K_a^c$. Then $d(u) + d(w) \leq n$, which contradicts $O(n + 1)$.

So there are edges from K_1 to K_a^c and clearly such edges lie on no 3-cycle. By (iii) they must lie on a 4-cycle and by the Corollary to Theorem 1, they lie on cycles of every higher length. Hence they are 3^--pancyclic.

Hence, combining (i) and (iv) we have completed the proof of (b).

(v) We now show that if $G \subseteq G_2$, then G contains a 4^--pancyclic edge.

If no edges of sP_2 remain in G, then let $u_1, u_2 \in V(sP_2) \cap VG$. We see that $d(u_1) + d(u_2) \leq 2b \leq n - 1$. This contradicts $O(n+1)$. Hence $u_1u_2 \in EG$.

Clearly u_1u_2 is a 4^--cyclic edge. Hence by (iii) and the Corollary to Theorem 1, u_1u_2 is a 4^--pancyclic edge.

Combining (ii) and (v) we have completed the proof of (c).

By Theorem 2 and the Corollary of Theorem 1, we have

Theorem 3. Let G satisfy the condition $O(n + 1)$. Then G is edge pancyclic iff $G \not\subseteq G_1$ and $G \not\subseteq G_2$.

Corollary. Let G satisfy the condition $O(n+1)$. Then G is vertex-pancyclic iff $G \not\subseteq G_1$ with $b = 0$.

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