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ON GENERALIZED VERTEX–PANCYCLIC GRAPHS

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ABSTRACT. Let G be a graph G of order $n \geq 4$ such that $d(u) + d(v) \geq n$ for all non–adjacent vertices u, v. Thus each vertex of G lies on a cycle of every length from 4 to n inclusive except if n is even, $n \neq 4$ and $G \cong K_{\frac{1}{2}n, \frac{1}{2}n}$. A similar result, without the exceptional case, holds if $d(u) + d(v) \geq n + 1$ for each pair of vertices u and v a distance two apart in G. We show that upper bounds are given for the number of vertices which do not lie on 3–cycles in the above two types of graphs.

1. INTRODUCTION

In this paper we consider only simple graphs. Throughout, we essentially use the terminology and notation of Bondy and Murty [2]. Hence we use $N(v)$ for the neighbourhood of a vertex v, $d(v) = |N(v)|$ and $d(u, v)$ for the distance between u and v. In addition we will let $\bar{N}(v) = N(v) \cup \{v\}$. A graph of order n is said to be pancyclic if it contains a cycle of length l for all l such that $3 \leq l \leq n$. In this paper we consider the concept of pancyclicity from the point of view of a vertex. So we say that a vertex is pancyclic if that vertex lies on a cycle of every length from 3 to n inclusive. We will be particularly interested in vertices which are not quite pancyclic. Hence we say that a vertex is 3^-–pancyclic if it lies on a cycle of

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every length from 4 to n inclusive and it does not lie on a 3–cycle. We say that G is vertex–pancyclic if every vertex is pancyclic; and vertex 3^-–pancyclic if every vertex is at least 3^--pancyclic. It is convenient to say that a vertex is l–cyclic if it lies on a cycle of length l. A 3^-–pancyclic vertex is not 3–cyclic.

We consider the 3^--pancyclic property of vertices in two classes of graphs. The first class is that for which if $u, v \in V(G)$ and $uv \notin E(G)$, then $d(u) + d(v) \geq n$. This is Ore's condition on a graph.

Fan [5] introduced a different condition to imply the existence of a hamiltonian cycle.

The second class is that we combine Fan's and Ore's condition to give the distance two condition, for all $u, v \in V(G)$ with $d(u, v) = 2$, $d(u) + d(v) \geq n + 1$.

The following theorem follows immediately by [5].

Theorem 1. *Let G be a 2-connected graph which satisfies the distance two condition. Then G is hamiltonian.*

To enable us to prove our results on 3^-–pancyclicity we will need the following.

Theorem 2. *If G is a graph of order $n \geq 5$ with $d(u) + d(v) \geq n + 1$ for distinct non–adjacent vertices u, v, then G contains a path of every length from 4 to $n - 1$ inclusive, between any pair of distinct vertices in G.*

Theorem 2 is due to Faudree and Schelp [6]. A shorter proof was given by Cai in [4]. In Cai's proof the full power of the hypothesis of Theorem 2 is not used. Instead only the distance two condition is used.

Corollary 3. *Let G be a grapph of order $n \geq 5$ which satisfies the distance two condition. Then G contains a path of every length from 4 to $n - 1$ inclusive, between any pair of distinct vertices in G.*

Finally we also need a result of Schmeichel and Hakimi (see [7] Lemma 1). Again, the proof of their result contains more than its original statement. We give a fuller result below.

Theorem 4. *Let u, v be adjacent vertices on a hamiltonian cycle in a graph G. If $d(u) + d(v) \geq n + 1$, then u and v are pancyclic.*

2. ORE'S CONDITION

Theorem 5. [3, corollary 4] *Let G be a graph of order $n \geq 4$ where, for all non–adjacent vertices u, v in G, $d(u) + d(v) \geq n$. Then G is vertex 3^-–pancyclic unless n is even $n \neq 4$ and $G \cong K_{\frac{1}{2}n,\frac{1}{2}n}$.*

Consider the class of graphs $G \ncong K_{\frac{1}{2}n,\frac{1}{2}n}$ such that for $u, v \in V(G)$ where if $uv \notin E(G)$, we have $d(u) + d(v) \geq n$. Let M_1 be the maximum number of 3^-–pancyclic vertices in a graph with the above property.

Theorem 6. *If $n \geq 6$, then* $M_1 = \begin{cases} 2 & for\ n\ odd \\ \frac{1}{2}n - 2 & for\ n\ even \end{cases}$

Proof. We first show that there are graphs with the stated number of 3^-–pancyclic vertices.

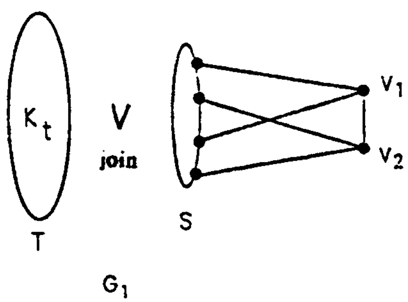

Figure 1.

The join $G \vee H$ of disjoint graphs G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H. Let G_1 be the graph of order n as shown in Figure 1 where $n = |T| + 6 = t + 6$. We see that $d(v_1) = d(v_2) = 3$, and $d(w) = t + 3$ for $w \in S \cup T$. Clearly, if $w \in T$, then $wv_i \notin E(G_1)$, and $d(w) + d(v_i) = t + 6 = |V(G_1)|$, for $i = 1, 2$. If $w \in S$ and $wv_i \notin E(G_1)$, again $d(w) + d(v_i) = t + 6$. Finally, if $w, w' \in S$ and $ww' \notin E(G)$, then $d(w) + d(w') = 2t + 6 \geq |V(G_1)|$. Hence G_1 satisfies the Ore's condition and contains only two 3^-–pancyclic vertices. If t is odd, then G_1 shows that $M_1 \geq 2$.

Now let $G_2 = K_{\frac{1}{2}n,\frac{1}{2}n} + e$. Let the two partite sets of $K_{\frac{1}{2}n,\frac{1}{2}n}$ be X and Y. Let $e = x_1x_2$ and $X = X_1 \cup X_2$, such that $X_1 = \{x_1, x_2\}$ and $X_1 \cap X_2 = \Phi$. We see that for $x \in X_1$, $d(x) = \frac{1}{2}n + 1$, for $x \in X_2$, $d(x) = \frac{1}{2}n$. Clearly $d(x) + d(x') \geq n$ for $xx' \notin E(G)$. It is easily seen that every vertex of X_2 is 3^-–pancyclic. So, for n even, $M_1 \geq |X_2| = \frac{1}{2}n - 2$.

Note that both G_1 and G_2 have at least 6 vertices. We now produce the reverse inequalities for M_1. In what follows, the graph G satisfies the Ore's condition. Let $R = \{v_i : 1, 2, \cdots, r\}$ be the set of 3^-–pancyclic vertices in G. Since $M_1 \geq 2$ we may assume $r = |R| \geq 2$. Let $C = u_1u_2\cdots u_nu_1$ be a hamiltonian cycle in G. We have $d(v_1) \leq \frac{1}{2}n$ for every i. Otherwise $d(v_1) > \frac{1}{2}n$ and there is a 3–cycle containing v_1, a contradiction.

Case 1. $d(v_i) = \frac{n}{2}$, for some $i \in \{1, 2, \cdots, r\}$.Clearly, n is even. Without loss of generality, suppose $d(v_1) = \frac{1}{2}n$. For any $u_1, u_2 \in N(v_1)$. Since v_1 is not 3–cyclic, we have $u_1u_2 \notin E(G)$. Hence $N(u_1) = N(u_2) \subseteq V(G)\backslash N(v_1)$ and, by Ore's condition, $d(u_1) + d(u_2) \geq n$. So $d(u_1) = d(u_2) = \frac{n}{2}$. Now $G \not\cong K_{\frac{n}{2},\frac{n}{2}}$. So $K_{\frac{n}{2},\frac{n}{2}} + e$ is a spanning subgraph of G, where e is an edge. Hence $M_1 \leq \frac{1}{2}n - 2$.

Case 2. $d(v_i) < \frac{n}{2}$ for any $i \in \{1, 2, \cdots, r\}$. Then Ore's condition gives $v_kv_j \in E(G)$ for any $k, j \in \{1, 2, \cdots, r\}$ with $k \neq j$. Furthermore, $G[R] \cong K_r$. Hence $r \leq 2$.

Combining the two cases with G_1 and G_2, the theorem is true.

3. THE DISTANCE TWO CONDITION

In this section we consider the problem of vertex 3^-–pancyclicity and the number of 3^-–pancylic vertices in graphs satisfying the distence two condition.

Theorem 7. *Let G be a graph of order $n \geq 5$. If $d(u) + d(v) \geq n + 1$ for any $u, v \in V(G)$ with $d(u, v) = 2$, then G is vertex 3^--pancyclic.*

Proof. By Corollary 3, every pair of distinct vertices are joined by paths of length 4 to $n - 1$ inclusive. Hence adjacent vertices must lie on t–cycles for $5 \leq t \leq n$. We now only have to show that any vertex of G lies on a 4–cycle. Since G

is hamiltonian, let $v_1 v_2 \cdots v_n v_1$ be a hamiltonian cycle. Suppose that v_n is on no 4-cycle. Then for all k such that $3 \le k \le n-1$, at most one of $v_2 v_k$, $v_k v_n$ is in $E(G)$. Hence $d(v_2) + d(v_n) \le (n-3)+1+1 = n-1$. By the conditions on G, this implies $v_2 v_n \in E(G)$. Replacing v_2 and v_n by v_1 and v_{n-1} similarly $v_1 v_{n-1} \in E(G)$. So the 4–cycle $v_1 v_2 v_n v_{n-1} v_1$ esists in G. This contradiction shows that v_n, and hence any vertex, lies on a 4–cycle. ◇

Unfortunately the conditions of the last theorem are not sufficient to imply vertex pancyclicity. The graph of Figure 2 satisfies the conditions of Theorem 7 but the vertex v_7 does not lie on a 3–cycle. In this graph vertices v_1, v_2, v_3, v_5, v_6 are adjacent to all vertices of K_{n-6} where $n \ge 7$.

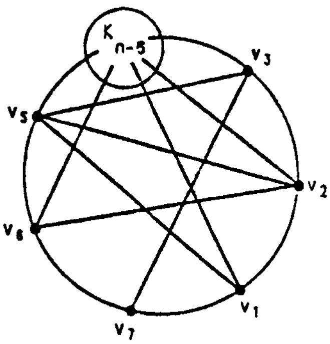

Figure 2.

Consider all graphs G with the property that for $u, v \in V(G)$ and $d(u,v) = 2$, $d(u)+d(v) \ge n+1$. Let M_2 denote the maximum number of 3^-–pancyclic vertices in a graph with the above property.

Theorem 8. $M_2 = \left[\frac{1}{2}(-3+\sqrt{11+2n}\,)\right]$.

Proof. Consider the graph F with $V(F) = A \cup \left(\bigcup_{i=1}^{r} B_i\right) \cup D$, where $F[A] = K_r^C$, $F[B_i] = K_{r+2}^C$ for all i and $F[D] = K_{r^2+3r-1}$. Further $N(v_i) = B_i$ for all $v_i \in F[A]$ and each vertex of $\bigcup_{i=1}^{r} B_i$ is joined to every vertex in D.

We note that $d(v_i) = r+2$ for all $v_i \in A$, $d(w_j) = r^2+3r$ for all $w_j \in \bigcup_{i=1}^{r} B_i$ and $d(z) = 2r^2+5r-2$ for all $z \in D$. Clearly $n = |V(F)| = 2r^2+6r-1$.

It is easily checked that F satisfies the distance two condition and that the vertices of degree $r+2$ are 3^--pancyclic. Now $r = [\frac{1}{2}(-3+\sqrt{11+2n})]$. So $M_2 = [\frac{1}{2}(-3+\sqrt{11+2n})]$.

Let G satisfy the distance two condition. Let $R = \{v_i : i = 1, 2, \cdots, r\}$ be the set of 3^--pancyclic vertices in G. Assume without loss of generality, that $d(v_i) \geq d(v_j)$ for $i \leq j$. The remainder of the proof now proceeds in four steps.

Step 1. $d(v_i) < \frac{1}{2}n$. Suppose $d(v_i) \geq \frac{1}{2}n$ and let $w_i \in N(v_i)$. By Theorem 2, there is a hamiltonian cycle containing $v_i w_i$. If $d(w_i) > \frac{1}{2}n$, then $d(v_i) + d(w_i) \geq n+1$. By Theorem 4, v_i is pancyclic. Hence $d(v_i) \leq \frac{1}{2}n$. Indeed, for all $w_i \in N(v_i)$, we have $d(w_i) \leq \frac{1}{2}n$. Since G is hamiltonian, $d(v_i) \geq 2$. Take $u, v \in N(v_i)$. Since v_i is not on a 3–cycle, $d(u,v) = 2$. From above we see that $d(u) + d(v) \leq n$ and so $uv \in E(G)$. However, this gives the 3–cycle vv_iuv in G. The contradiction shows that $d(w_i) < \frac{1}{2}n$.

Step 2. $v_iv_j \notin E(G)$. Assume that $v_iv_j \in E(G)$ and let $w_i \in N(v_i)$. Clearly $w_i \notin N(v_j)$ since v_i and v_j are not 3–cyclic. Hence $d(w_i, v_j) = 2$ and $d(v_i)+d(w_i) \geq d(v_j) + d(w_i) \geq n+1$. But this implies that $N(v_i) \cap N(w_i) \neq \Phi$ and so v_i lies on a 3–cycle. Hence $v_iv_j \notin E(G)$.

Step 3. $N(v_i) \cap N(v_j) = \Phi$ and if $w_i \in N(v_i)$, then $N(w_i) \cap N(v_i) = \Phi$. Let $w_i \in N(v_i)$. If $w_i \in N(v_j)$, then $d(v_i, v_j) = 2$ so $d(v_i) + d(v_j) \geq n+1$. However, by Step 1 we know that $d(v_i) + d(v_j) < n$. Hence $N(v_i) \cap N(v_j) = \Phi$. Now since v_i is 3^--pancyclic, $N(v_i) \cap N(w_i) = \Phi$, so $d(v_i)+d(w_i) \leq n$. If $N(w_i) \cap N(v_j) \neq \Phi$, then $d(w_i, v_j) = 2$, so $d(w_i)+d(v_j) \geq n+1$. However, $d(w_i)+d(v_j) \leq d(w_i)+d(v_i) \leq n$. Consequently $N(w_i) \cap N(v_j) = \Phi$.

Step 4. $r \leq [\frac{1}{2}(-3+\sqrt{11+2n})]$. Let $z \in V(G) \backslash \left[\bigcup_{i=1}^{r} \bar{N}(v_i)\right]$. Such *a* z exists since G is connected. Let $|N(v_i)| = s_i$, $\left|\bigcup_{i=1}^{r} N(v_i)\right| = s$ and $t = \left|V(G) \backslash \left[\bigcup_{i=1}^{r} \bar{N}(v_i)\right]\right|$. Then for some v_i, $r+s+t+1 = n+1 \leq d(z) + d(v_i) \leq [s + (t-1)] + s_1$. Hence $s_1 \geq r+2$ and $s \geq r(r+2)$. Further, for w_i, $w_{i'} \in N(v_i)$, $r+s+t+1 = n+1 \leq d(w_i)+d(w_{i'}) \leq 2t+2$. Hence $t \geq s+r-1$. So $n = r+s+t \geq 2s+2r-1 \geq 2r^2+6r-1$. This implies that $r \leq [\frac{1}{2}(-3+\sqrt{11+2n})]$. Hence $M_2 \leq [\frac{1}{2}(-3+\sqrt{11+2n})]$.

Combining the two inequalities for M_2 we complete the proof of the theorem. ◇

Note that $M_2 = 0$ for $n \leq 6$. However, if $r \geq 1$ then, from Step 4 of the proof, $s_1 \geq 3$ and $t \geq 3$. Hence $n \geq 7$.

We note that combining, in some sense Ore's condition and the distance two condition reduces the maximum number of 3^-–pancyclic vertices.

Theorem 9. *Consider the set of graphs of order $n \geq 5$ for which $d(u)+d(v) \geq n+1$ for all u, v such uv is not an edge. Then the largest number of 3^--pancyclic vertices, M_3, in a graph of this set is one.*

Proof. Let v_1, v_2 be 3^-–pancyclic vertices. Steps 1 and 2 of Theorem 8 still hold. By Step 2, $d(v_1) + d(v_2) \geq n + 1$ but by Step 1, $d(v_1) + d(v_2) < n$. This contradiction shows that $M_3 \leq 1$.

The graph F of Theorem 8 with $r = 1$, shows that $M_3 \geq 1$. ◇

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