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Note

Neighborhood unions and Hamiltonian properties☆

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Abstract

Let G be a simple graph of order n with connectivity $k \geqslant 2$, independence number α. We prove that if for each independent set S of cardinality $k+1$, one of the following condition holds: (1) there exist $u \neq v$ in S such that $d(u)+d(v) \geqslant n$ or $|N(u) \cap N(v)| \geqslant \alpha$; (2) for any distinct pair u and v in S, $|N(u) \cup N(v)| \geqslant n - \max\{d(x) \mid x \in S\}$, then G is Hamiltonian. Many known results on Hamiltonian graphs are corollaries of this result.

1. Introduction

This paper uses terms and notation of [1]. Throughout, G denotes an undirected connected simple graph of order $n(\geqslant 3)$ with connectivity k and independence number α. Let L be a subset of $V(G)$, F a subgraph of G and v a vertex in G. Define $N_L(v)=\{u \mid u \in L,\ uv \in E(G)\}$, $N_L(F)=\bigcup_{v \in V(F)} N_L(v)$. For the special case when $L=V(G)$, we simply write $N(v)$ and $N(F)$. If no ambiguity can arise, we sometimes write F instead of $V(F)$. Let $S \subseteq V$, define $\Delta(S)=\max\{d(u) \mid u \in S\}$.

It is well known that there are many sufficient conditions of Hamiltonian graphs, which are divided into various types. Degree conditions are a fundamental type. The inspiration for this development was the classical result of Ore [7]

Theorem 1.1 ([7]). *Let G be a graph of order $n \geqslant 3$. If for every pair of nonadjacent vertices u and v, $d(u)+d(v) \geqslant n$, then G is Hamiltonian.*

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Neighborhood conditions are also a type, which begin with the result obtained by Faudree et al. [5].

Theorem 1.2 ([5]). *Let G be a 2-connected simple graph of order $n(\geqslant 3)$. If for every pair of nonadjacent vertices u and v, $|N(u)\cup N(v)|\geqslant(2n-1)/3$, then G is Hamiltonian.*

Chvátal and Erdős's Theorem [4] is another type.

Theorem 1.3. *Let G be a simple graph of order $n\geqslant 3$ with connectivity k and independence number α. If $\alpha\leqslant k$, then G is Hamiltonian.*

Using an idea of 'or', we combine these conditions to obtain the following result.

Theorem 1.4. *Let G be a 2-connected simple graph of order $n(\geqslant 3)$ with connectivity k and independence number of α. If for every independent set S of cardinality $k+1$, one of the following conditions holds:*
(1) there exist $u\neq v$ in S such that $d(u)+d(v)\geqslant n$ or $|N(u)\cap N(v)|\geqslant\alpha$;
(2) for any distinct pair u and v in S, $|N(u)\cup N(v)|\geqslant n-\Delta(S)$,
then G is Hamiltonian.

It is easy to prove that Theorem 1.4 is stronger than the first three theorems listed above, and Theorem C in [2] is a corollary of this theorem. On the other hand, there are a lot of Hamilton graphs which do not satisfy conditions of the first theorems listed above, but satisfy the condition of Theorem 1.4. An example is depicted in Fig. 1, where G_1 and G_2 are complete graphs with $|V(G_1)|=r_1$, $|V(G_2)|=r_2$ and $r_1\geqslant r_2\geqslant 4$; x_1 is not adjacent to x_2; $d_{G_1}(y_1)=r_1$, $d_{G_2}(y_2)=r_2$. In addition, for each vertex, $x\in V(G_1)$, $d_{G_2}(x)\geqslant 3$, and for each vertex $x\in V(G_2)$, $d_{G_1}(x)\geqslant 3$. We denote the graph by G_n. Clearly $\kappa(G_n)=k=2$, $\alpha(G_n)=\alpha=5$, $d(u_1)+d(u_2)=10<n$ and $|N(u_1)\cup N(u_2)|=5$. Hence G_n does not satisfy conditions of Theorems 1.1–1.3. But for any independent set S of cardinality 3 in G_n, if $|S\cap\{u_1,u_2,u_3\}|\geqslant 2$, then S satisfies condition (1) of

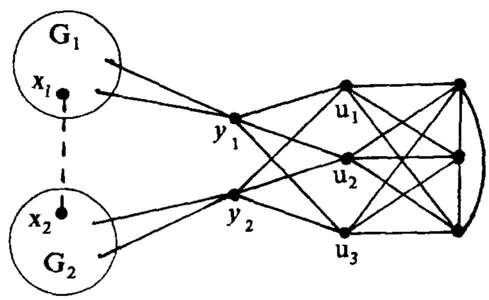

Fig. 1.

Theorem 1.4. For any other S, by $\Delta(S)=\max\{d(u)\,|\,u\in S\}$ and a simple calculation, it is easy to obtain that S satisfies condition (2) of Theorem 1.4. For example, for $S=\{w_2, y_1, z\}$, where $w_2\in V(G_2)$ and $z\notin V(G_1\cup G_2)\cup\{y_1, y_2, u_1, u_2, u_3\}$, it is clear that $d(w_2)=r_2+d_{G_1}(w_2)$ and $d(y_1)=r_1+3$. By $\Delta(S)=d(w_2)$ or $d(y_1)$, respectively, we obtain that S satisfies condition (2) of Theorem 1.4. Thus, G_n satisfies the condition of Theorem 1.4

Remark 1.5. In [6], Flandrin et al. have proved Theorem 1.6.

Theorem 1.6. *Let G be a 2-connected graph of order n such that*

$$d(u)+d(v)+d(w)\geqslant n+|N(u)\cap N(v)\cap N(w)|,$$

for any independent set $\{u, v, w\}$, *then G is Hamiltonian.*

Consider an independent set $\{u_1, u_2, u_3\}$. We obtain that G_n does not satisfy the condition of Theorem 1.6. On the other hand, Fig. 1 in [6] satisfies the condition of Theorem 1.4.

Remark 1.7. In [3], Chen and Schelp obtain the following result.

Theorem 1.8. *Let G be a simple graph with connectivity* $k=2$. *If for each independent set S of order* 3, $s_1+2s_2+2s_3>n-1$ *holds, then G is Hamiltonian, where* $s_i=|\{v\in V(G)\,|\,N(v)\cap S|=i\}|$.

Consider an independent set $\{u_1, u_2, u_3\}$. $s_1+2s_2+2s_3=10<n-1$, this implies that G_1 does not satisfy the conditions of Theorem 1.8.

In the following section we prove Theorem 1.4 and in the last section we discuss a corollary.

2. Proof of Theorem 1.4

Let G satisfy the conditions of the theorem. If G is not Hamiltonian, let C be a cycle of maximum length in G, then $|V(C)|<n$. Let B be any component of $G\setminus V(C)$, By $\vec{C}$ we denote the cycle with a given orientation. Let u, $v\in V(C)$. By $u\vec{C}v$ we denote the consecutive vertices on C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use uBv to denote a path from u via B to v. We use u^+ to denote the successor of on $\vec{C}$ and u^- to denote its predecessor. We write u^{++} instead of $(u^+)^+$ and u^{--} instead of $(u^-)^-$. Let $S\subseteq V(C)$, define $S^+=\{x^+\,|\,x\in S\}$ and $S^-\{x^-\,|\,x\in S\}$. Put $N_C(B)=\{v_1, v_2, \ldots, v_m\}$, where v_i occurs on $\vec{C}$ in the order of their indices. Clearly, $m\geqslant k\geqslant 2$. We write $N_C^+(B)$ instead of $(N_C(B))^+$ and $N_C^-(B)$ instead of $(N_C(B))^-$. For any $j\,(1\leqslant j\leqslant m)$, Let x_j be any vertex in B which is adjacent to v_j. it is possible that $x_i=x_j$ for $i\neq j$.

Claim 2.1. *For any $x \in V(B)$, $N_C^+(B) \cup \{x\}$ and $N_C^-(B) \cup \{x\}$ are independent, and for any $u, v \in N_C^+(B) \cup \{x\}$ or $N_C^-(B) \cup \{x\}$, $N(u) \setminus V(C) = \phi$ or $N(v) \setminus V(C) = \phi$ or $N(u)$ and $N(v)$ are not connected in $G - V(C)$.*

For any $j\,(1 \leqslant j \leqslant m)$, Claim 1.2 implies $v_{j-1}^+ \notin N(v_j^+)$. Thus since C is a longest cycle and $v_j \in N(v_j^+)$, there exists a vertex u_j, $u_j \in v_{j-1}^+ \vec{C} v_j^-$ such that $u_j \notin N(v_j^+)$, and $v \in N(v_j^+)$ for all $v \in u_j^+ \vec{C} v_j$. Put $N = \{u_1, u_2, \ldots, u_m\}$. Let x be a vertex in B.

Claim 2.2. *For any $x \in B$, $N \cup \{x\}$ is independent.*

In fact, if there exist $u, v \in N \cup \{x\}$ with $uv \in E(G)$, then $x \notin \{u, v\}$ by the definition of N. Let $u = u_i$, $v = u_j$ with $i < j$. The cycle

$$u_i u_j \overleftarrow{C} v_i^+ u_i^+ \vec{C} v_i x_i B x_j v_j \overleftarrow{C} u_j^+ v_j^+ \vec{C} u_i$$

is longer than C. This is a contradiction.

Claim 2.3. *For any $u, v \in N \cup \{x\}$, $d(u) + d(v) < n$ and $|N(u) \cap N(v)| < \alpha$.*

Proof of Claim 2.3. It is clear that $N(u) \cap N(v) \subseteq V(C)$, since C is a longest cycle in G. In the following we always assume that $N(u) \cap N(v) \subseteq V(C)$. If there are $u, v \in N \cap \{x\}$, $d(u) + d(v) \geqslant n$ or $|N(u) \cap N(v)| \geqslant \alpha$, then by the proof of Claim 2.2 and Lemma 4.4.1 in [1] (if $d(u) + d(v) \geqslant n$, then G is Hamiltonian if and only if $G + uv$ is Hamiltonian), we can assume $|N(u) \cap N(v)| \geqslant \alpha$.

If $x \in \{u, v\}$, then $N(u) \cap N(v) \subseteq N_C(B)$, Hence

$$|N \cup \{x\}| = |N_C(B)| + 1 \geqslant |N(u) \cap N(v)| + 1 \geqslant \alpha + 1.$$

By Claim 2.2, this is a contradiction.

If $x \notin \{u, v\}$, let $u = u_i$, $v = v_j$ with $i < j$. Let $N(u) \cap N(v) = \{y_1, y_2, \ldots, y_p\} = Y$. Then $p \geqslant \alpha$. We prove that $Y^+ \cup \{x\}$ is an independent set. Claim 2.2 implies $N \cap Y = \phi$. In fact, if there exsts $y_t^+ \in Y^+$ such that $xy_t^+ \in E(G)$. Without loss of generality, we assume $y_t \in u\vec{C}v^-$, the cycle

$$y_t v \overleftarrow{C} y_t^+ x B x_j v_j \overleftarrow{C} v^+ v_j^+ \vec{C} y_t$$

is longer than C. This is a contradiction. If there are $y_s^+, y_t^+ \in Y^+$ with $s < t$ such that $y_s^+ y_t^+ \in E(G)$, by the symmetry, two subcases must be considered: (1) $y_s^+, y_t^+ \in u\vec{C}v$; (2) $y_s^+ \in u\vec{C}v$, $y_t^+ \in v\vec{C}u$. For each subcase, the cycle

$$u y_t \overleftarrow{C} y_s^+ y_t^+ \vec{C} v y_s \overleftarrow{C} v_i^+ u^+ \vec{C} v_i x_i B x_j v_j \overleftarrow{C} v^+ v_j^+ \vec{C} u$$

and the cycle

$$u \overleftarrow{C} y_t^+ y_s^+ \vec{C} v y_s^+ \overleftarrow{C} v_i^+ u^+ \vec{C} v_i x_i B x_j v_j \overleftarrow{C} v^+ v_j^+ \vec{C} u$$

are longer than C. There are contradictions. Hence, $Y^+ \cup \{x\}$ is an independent set in G. Note that $|Y^+ \cup \{x\}| \geqslant p+1 \geqslant \alpha+1$ there is a contradiction. Claim 2.3 holds. □

Let $S=\{u_1, u_2, \ldots, u_k\}$ and $S_j = S \cup \{x_j\}$ for $1 \leqslant j \leqslant m$. Then $S_j \subseteq N \cup \{x_j\}$ and S_j is an independent set of cardinality $k+1$. By the conditions of Theorem 1.4 and Claim 2.3, for any $u, v \in S_j$, $|N(u) \cup N(v)| \geqslant n - \Delta(S_j)$. Put $t_j = \Delta(S_j)$.

Claim 2.4. *$d(x_j) < \Delta(S_j)$ for any j $(1 \leqslant j \leqslant m)$.*

If there exists $x_j \in V(B)$ with $d(x_j) = \Delta(S_j)$, Consider u_1, u_2, we have $(N(u_1) \cup N(u_2)) \cap N(N \cup V(B)) = \emptyset$ by the definition of N. This implies that $|N(u_1) \cup N(u_2)| \leqslant n - (|B| + m) \leqslant n - t_j - 1$, a contradiction.

We know that $\Delta(S) \geqslant t_j$ for any j, $1 \leqslant j \leqslant k$, and by Claim 2.4, we can assume $d(u_1) = \Delta(S) \geqslant t_1$. Then $d(u_2) \geqslant t_2$. Consider $N(u_2) \cup N(x_2)$. By the conditions of Theorem 1.4, $|N(u_2) \cup N(x_2)| \geqslant n - t_2$.

Claim 2.5. *Let $v \in N(u_2) \cup N(x_2)$. Then $v \notin u_1 \vec{C} v_1^-$ and*
(1) *if $v \in v_1 \vec{C} u_2^-$, then $u_1 v^+ \notin E(G)$;*
(2) *if $v \in u_1^+ \vec{C} u_1^-$, then $u_1 v^- \notin E(G)$.*

Proof of Claim 2.5. If there exists $v \in v_1 \vec{C} u_2^-$ such that $u_1 v^+ \in E(G)$, then since $u_1 v_1^+ \notin E(G)$, we have $v \in v_1^+ \vec{C} u_2^-$. Hence by the definition of $N_C(B)$, $v \in N(u_2)$, and the cycle

$$u_1 v^+ \vec{C} u_2 v \overleftarrow{C} v_1^+ u_1^+ \vec{C} v_1 x_1 B x_2 v_2 \vec{C} u_2^+ v_2^+ \vec{C} u_1$$

is longer than C, If there exists $v \in u_2^+ \vec{C} u_1^-$ such that $u_1 v^- \in E(G)$, and then when $v \in N(x_2)$, the cycle

$$u_1 v^- \overleftarrow{C} v_1^+ u_1^+ \vec{C} v_1 x_1 B x_2 v \vec{C} u_1$$

is longer than C; when $v \in N(u_2)$, $v \in v_2^+ \vec{C} u_1$ and the cycle

$$u_1 v^- \overleftarrow{C} v_2^+ u_2^+ \vec{C} v_2 x_2 B x_1 v_1 \overleftarrow{C} u_1^+ v_1^+ \vec{C} u_2 v \vec{C} u_1$$

is longer than C. These are contradictions. Claim 2.5 holds. □

We define a bijection f on $N(u_2) \cup \{x_2\}$ as follows: Let $u \in N(u_2) \cup N(x_2)$,

$$f(u) = \begin{cases} u & \text{for } u \notin V(C), \\ u^+ & \text{for } u \in u_1 \vec{C} u_2^{--}, \\ u_1 & \text{for } u = u_2^-, \\ u^- & \text{for } u \in u_2^+ \vec{C} u^-. \end{cases}$$

From the previous arguments and Claim 2.5, for any $u \in f(N(u_2) \cup N(x_2))$, we have $u u_1 \notin E(G)$. Note that $x_2 \notin f(N(u_2) \cup N(x_2))$ and $x_2 u_1 \notin E(G)$, and we obtain

$$t_2 \leqslant d(u_1) \leqslant n - |N(u_2) \cup N(x_2)| - 1 \leqslant t_2 - 1,$$

a contradiction. Therefore, Theorem 1.4 holds. □

3. A corollary

Theorem 1.4 can deduce many sufficient conditions for Hamiltonian graphs, one of them is the following corollary.

Corollary 3.1. *Let G be a simple graph of order* $n(\geqslant 3)$ *with connectivity* $k\geqslant 2$. *If for each independent set S of cardinality* $k+1$, *and any distinct pair u and v in S,* $|N(u)\cup N(v)|\geqslant n-\Delta(S)$, *then G is Hamiltonian.*

Take a complete graph K_{n-k} and attach k independent vertices $X=\{x_1, x_2, \ldots, x_k\}$ of $\min\{d(x)|x\in X\}=k$ and for any $u,v\in X$, $|N(u)\cup N(v)|=k+1$, and there exists a vertex $x\in V(K_{n-k})$, x is nonadjacent to all vertices in X. We denote the graph by $G(k)$. Let $x\in V\setminus N(X)$, then $S=X\cup\{x\}$ is an independent set with $\Delta(S)=n-(k+1)$. For any $u,v\in X$, $|N(u)\cup N(v)|=k+1=n-\Delta(S)$. This implies that $G(k)$ satisfies the condition of Corollary 3.1, thus, $G(k)$ is Hamiltonian. However, $|N(u)\cup N(v)|=k+1<n-\max\{d(u), d(v)\}$, this implies that $G(k)$ does not satisfy the condition of Corollary 2 in [6]. Moreover, for $k\geqslant 3$, $n\geqslant 3k+4$ and any $u,v,w\in X$, $d(u)+d(v)+d(w)\leqslant 3k+3<n+|N(u)\cup N(v)\cup N(w)|$. This implies that $G(k)$ does not satisfy the condition of Theorem 1.6.

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