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Note

Neighborhood unions and Hamiltonian properties*

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Abstract

Let $G$ be a simple graph of order $n$ with connectivity $k \geq 2$, independence number $\alpha$. We prove that if for each independent set $S$ of cardinality $k+1$, one of the following conditions holds:
1. there exist $u \neq v$ in $S$ such that $d(u) + d(v) > n$ or $|N(u) \cap N(v)| > \alpha$;
2. for any distinct pair $u$ and $v$ in $S$, $|N(u) \cup N(v)| > n - \max \{d(x) | x \in S\}$, then $G$ is Hamiltonian. Many known results on Hamiltonian graphs are corollaries of this result.

1. Introduction

This paper uses terms and notation of [1]. Throughout, $G$ denotes an undirected connected simple graph of order $n \geq 3$ with connectivity $k$ and independence number $\alpha$. Let $L$ be a subset of $V(G)$, $F$ a subgraph of $G$ and $u$ a vertex in $G$. Define $N_L(v) = \{u | u \in L, uv \in E(G)\}$, $N_L(F) = \bigcup_{v \in V(F)} N_L(v)$. For the special case when $L = V(G)$, we simply write $N(v)$ and $N(F)$. If no ambiguity can arise, we sometimes write $F$ instead of $V(F)$. Let $S \subseteq V$, define $\Delta(S) = \max \{d(u) | u \in S\}$.

It is well known that there are many sufficient conditions of Hamiltonian graphs, which are divided into various types. Degree conditions are a fundamental type. The inspiration for this development was the classical result of Ore [7].

Theorem 1.1 ([7]). Let $G$ be a graph of order $n \geq 3$. If for every pair of nonadjacent vertices $u$ and $v$, $d(u) + d(v) \geq n$, then $G$ is Hamiltonian.

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Neighborhood conditions are also a type, which begin with the result obtained by Faudree et al. [5].

**Theorem 1.2 ([5]).** Let \( G \) be a 2-connected simple graph of order \( n(\geq 3) \). If for every pair of nonadjacent vertices \( u \) and \( v \), \( |N(u)\cap N(v)| \geq (2n-1)/3 \), then \( G \) is Hamiltonian.

Chvátal and Erdős's Theorem [4] is another type.

**Theorem 1.3.** Let \( G \) be a simple graph of order \( n \geq 3 \) with connectivity \( k \) and independence number \( \alpha \). If \( \alpha < k \), then \( G \) is Hamiltonian.

Using an idea of 'or', we combine these conditions to obtain the following result.

**Theorem 1.4.** Let \( G \) be a 2-connected simple graph of order \( n(\geq 3) \) with connectivity \( k \) and independence number of \( \alpha \). If for every independent set \( S \) of cardinality \( k+1 \), one of the following conditions holds:

1. there exist \( u \neq v \) in \( S \) such that \( d(u)+d(v) \geq n \) or \( |N(u)\cap N(v)| \geq \alpha \);
2. for any distinct pair \( u \) and \( v \) in \( S \), \( |N(u)\cap N(v)| \geq n - k(S) \),

then \( G \) is Hamiltonian.

It is easy to prove that Theorem 1.4 is stronger than the first three theorems listed above, and Theorem C in [2] is a corollary of this theorem. On the other hand, there are a lot of Hamilton graphs which do not satisfy conditions of the first theorems listed above, but satisfy the condition of Theorem 1.4. An example is depicted in Fig. 1, where \( G_1 \) and \( G_2 \) are complete graphs with \( |V(G_1)| = r_1, |V(G_2)| = r_2 \) and \( r_1 > r_2 \geq 4 \); \( x_1 \) is not adjacent to \( x_2 \); \( d_{G_1}(y_1) = r_1, d_{G_1}(y_2) = r_2 \). In addition, for each vertex, \( x \in V(G_1), d_{G_2}(x) \geq 3 \), and for each vertex \( x \in V(G_2), d_{G_1}(x) \geq 3 \). We denote the graph by \( G_n \). Clearly \( \kappa(G_n) = k = 2 \), \( \chi(G_n) = \alpha = 5 \), \( d(u_1) + d(u_2) = 10 < n \) and \( |N(u_1)\cap N(u_2)| = 5 \). Hence \( G_n \) does not satisfy conditions of Theorems 1.1-1.3. But for any independent set \( S \) of cardinality 3 in \( G_n \), if \( |S\cap \{u_1, u_2, u_3\}| \geq 2 \), then \( S \) satisfies condition (1) of

![Fig. 1.](image-url)
Theorem 1.4. For any other $S$, by $A(S) = \max \{d(u) | u \in S\}$ and a simple calculation, it is easy to obtain that $S$ satisfies condition (2) of Theorem 1.4. For example, for $S = \{w_2, y_1, z\}$, where $w_2 \in V(G_2)$ and $z \notin V(G_1 \cup G_2)$, it is clear that $d(w_2) = r_2 + d_{G_2}(w_2)$ and $d(y_1) = r_1 + 3$. By $A(S) = d(w_2)$ or $d(y_1)$, respectively, we obtain that $S$ satisfies condition (2) of Theorem 1.4. Thus, $G_n$ satisfies the condition of Theorem 1.4.

Remark 1.5. In [6], Flandrin et al. have proved Theorem 1.6.

Theorem 1.6. Let $G$ be a 2-connected graph of order $n$ such that
\[ d(u) + d(v) + d(w) > n + |N(u) \cap N(v) \cap N(w)|, \]
for any independent set $\{u, v, w\}$, then $G$ is Hamiltonian.

Consider an independent set $\{u_1, u_2, u_3\}$. We obtain that $G_n$ does not satisfy the condition of Theorem 1.6. On the other hand, Fig. 1 in [6] satisfies the condition of Theorem 1.4.

Remark 1.7. In [3], Chen and Schelp obtain the following result.

Theorem 1.8. Let $G$ be a simple graph with connectivity $k = 2$. If for each independent set $S$ of order 3, $s_1 + 2s_2 + 2s_3 > n - 1$ holds, then $G$ is Hamiltonian, where $s_i = |\{v \in V(G) | N(v) \cap S = i\}|$.

Consider an independent set $\{u_1, u_2, u_3\}$. $s_1 + 2s_2 + 2s_3 = 10 < n - 1$, this implies that $G_1$ does not satisfy the conditions of Theorem 1.8.

In the following section we prove Theorem 1.4 and in the last section we discuss a corollary.

2. Proof of Theorem 1.4

Let $G$ satisfy the conditions of the theorem. If $G$ is not Hamiltonian, let $C$ be a cycle of maximum length in $G$, then $|V(C)| < n$. Let $B$ be any component of $G \setminus V(C)$. By $C$ we denote the cycle with a given orientation. Let $u, v \in V(C)$. By $uCv$ we denote the consecutive vertices on $C$ from $u$ to $v$ in the direction specified by $C$. The same vertices, in reverse order are given by $vCu$. We will consider $uCv$ and $vCu$ both as paths and as vertex sets. We use $uBu$ to denote a path from $u$ via $B$ to $v$. We use $u^+$ to denote the successor of $u$ on $C$ and $u^-$ to denote its predecessor. We write $u^{++}$ instead of $(u^+)^+$ and $u^{--}$ instead of $(u^-)^-$. Let $S \subseteq V(C)$, define $S^+ = \{x^+ | x \in S\}$ and $S^- = \{x^- | x \in S\}$. Put $N_+(B) = \{v_1, v_2, \ldots, v_m\}$, where $v_i$ occurs on $C$ in the order of their indices. Clearly, $m \geq k \geq 2$. We write $N^+(B)$ instead of $(N_+(B))^+$ and $N^-(B)$ instead of $(N_+(B))^-$, for any $j (1 \leq j \leq m)$. Let $x_j$ be any vertex in $B$ which is adjacent to $v_j$. It is possible that $x_i = x_j$ for $i \neq j$. 


Claim 2.1. For any \( x \in V(B) \), \( N^+_x(B) \cup \{x\} \) and \( N^-_x(B) \cup \{x\} \) are independent, and for any \( u, v \in N^+_x(B) \cup \{x\} \) or \( N^-_x(B) \cup \{x\} \), \( N(u) \backslash V(C) = \emptyset \) or \( N(v) \backslash V(C) = \emptyset \) or \( N(u) \) and \( N(v) \) are not connected in \( G - V(C) \).

For any \( j (1 \leq j \leq m) \), Claim 1.2 implies \( v^+_j \not\in N(v^+_j) \). Thus since \( C \) is a longest cycle and \( v^+_j \in N(v^+_j) \), there exists a vertex \( u_j, u_j \in v^+_j \) such that \( u_j \not\in N(v^+_j) \), and \( v \in N(v^+_j) \) for all \( v \in u_j \). Put \( N = \{u_1, u_2, \ldots, u_m\} \). Let \( x \) be a vertex in \( B \).

Claim 2.2. For any \( x \in B \), \( N \cup \{x\} \) is independent.

In fact, if there exist \( u, v \in N \cup \{x\} \) with \( uv \in E(G) \), then \( x \not\in \{u, v\} \) by the definition of \( N \). Let \( u = u_i, v = u_j \) with \( i < j \). The cycle
\[
\text{in longer than } C. \text{ This is a contradiction.}
\]

Claim 2.3. For any \( u, v \in N \cup \{x\} \), \( d(u) + d(v) < n \) and \( |N(u) \cap N(v)| < x \).

Proof of Claim 2.3. It is clear that \( N(u) \cap N(v) \subseteq V(C) \), since \( C \) is a longest cycle in \( G \). In the following we always assume that \( N(u) \cap N(v) \subseteq V(C) \). If there are \( u, v \in N(x) \), \( d(u) + d(v) \geq n \) or \( |N(u) \cap N(v)| \geq x \), then by the proof of Claim 2.2 and Lemma 4.4.1 in \([1]\) (if \( d(u) + d(v) \geq n \), then \( G \) is Hamiltonian if and only if \( G + uv \) is Hamiltonian), we can assume \( |N(u) \cap N(v)| \geq x \).

If \( x \in \{u, v\} \), then \( N(u) \cap N(v) \subseteq N(B) \), Hence
\[
|N \cup \{x\}| = |N(B)| + 1 \geq |N(u) \cap N(v)| + 1 \geq x + 1.
\]

By Claim 2.2, this is a contradiction.

If \( x \not\in \{u, v\} \), let \( u = u_i, v = v_j \) with \( i < j \). Let \( N(u) \cap N(v) = \{y_1, y_2, \ldots, y_p\} = Y \). Then \( p \geq x \). We prove that \( Y^+ \cup \{x\} \) is an independent set. Claim 2.2 implies \( N \cap Y = \emptyset \). In fact, if there exists \( y^+ \in Y^+ \) such that \( xy^+ \in E(G) \). Without loss of generality, we assume \( y_i \in u \tilde{C}v \), the cycle
\[
y_i v \tilde{C} y_i^+ x B y_j v_j \tilde{C} y_j^+ v_j \tilde{C} y_i
\]
is longer than \( C \). This is a contradiction. If there are \( y^+_s, y^+_t \in Y^+ \) with \( s < t \) such that \( y^+_s v^+_s, y^+_t v^+_t \in E(G) \), by the symmetry, two subcases must be considered: (1) \( y^+_s \in u \tilde{C}v \); (2) \( y^+_t \in u \tilde{C}v \). For each subcase, the cycle
\[
uy_i \tilde{C} y_i^+ y^+_t \tilde{C} y_y y^+_s \tilde{C} y_s^+ y^+_t \tilde{C} y_y \tilde{C} y_s^+ v_j \tilde{C} u
\]
and the cycle
\[
u \tilde{C} y_i^+ y^+_s \tilde{C} y_y y^+_t \tilde{C} y_s^+ y^+_t \tilde{C} y_y \tilde{C} y_s^+ v_j \tilde{C} u
\]
are longer than C. There are contradictions. Hence, $Y^+ \cup \{x\}$ is an independent set in $G$. Note that $|Y^+ \cup \{x\}| > p + 1 > \pi + 1$ there is a contradiction. Claim 2.3 holds.

Let $S = \{u_1, u_2, \ldots, u_k\}$ and $S_j = S \cup \{x_j\}$ for $1 \leq j \leq m$. Then $S_j \subseteq N \cup \{x_j\}$ and $S_j$ is an independent set of cardinality $k + 1$. By the conditions of Theorem 1.4 and Claim 2.3, for any $u, v \in S_j$, $|N(u) \cap N(v)| \geq n - \Delta(S_j)$. Put $t_j = \Delta(S_j)$.

**Claim 2.4.** $d(x_j) < \Delta(S_j)$ for any $j (1 \leq j \leq m)$.

If there exists $x_j \in V(B)$ with $d(x_j) = \Delta(S_j)$. Consider $u_1, u_2$, we have $(N(u_1) \cup N(u_2)) \cap N(N(u_2)V(B)) = \emptyset$ by the definition of $N$. This implies that $|N(u_1) \cap N(u_2)| \leq n - (|B| + m) \leq n - t_j - 1$, a contradiction.

We know that $\Delta(S) \geq t_j$ for any $j$, $1 \leq j \leq k$, and by Claim 2.4, we can assume $d(u_1) = d(u_2) = t_1$. Then $d(u_1) \geq t_2$. Consider $N(u_2) \cup N(x_2)$. By the conditions of Theorem 1.4, $|N(u_2) \cup N(x_2)| \geq n - t_2$.

**Claim 2.5.** Let $v \in N(u_2) \cup N(x_2)$. Then $v \notin N(u_2) \cup N(x_2)$ and
1. if $v \notin v_1 \tilde{C}u_2$, then $u_1v^+ \notin E(G)$;
2. if $v \notin u_1 \tilde{C}u_2$, then $u_1v^- \notin E(G)$.

**Proof of Claim 2.5.** If there exists $v \in N(u_2) \cup N(x_2)$ such that $u_1v^+ \in E(G)$, then since $u_1v^+ \notin E(G)$, we have $v \in v_1 \tilde{C}u_2$. Hence by the definition of $N_\epsilon(B)$, $v \in N(u_2)$, and the cycle

$$u_1v^+ \tilde{C}u_2vCv_1u_1 \tilde{C}v_1x_1Bx_2v_2 \tilde{C}u_2v_2 \tilde{C}u_1$$

is longer than $C$. If there exists $v \in u_2 \tilde{C}u_1$ such that $u_1v^- \in E(G)$, and then when $v \in N(x_2)$, the cycle

$$u_1v^- \tilde{C}v_1u_1 \tilde{C}v_1x_1Bx_2vCv_1u_1$$

is longer than $C$; when $v \in N(u_2)$, $v \in v_2 \tilde{C}u_1$ and the cycle

$$u_1v^- \tilde{C}v_2u_2 \tilde{C}v_2x_2Bx_1v_1 \tilde{C}u_2v_1 \tilde{C}u_2v \tilde{C}u_4$$

is longer than $C$. These are contradictions. Claim 2.5 holds.

We define a bijection $f$ on $N(u_2) \cup \{x_2\}$ as follows: Let $u \in N(u_2) \cup N(x_2)$,

$$f(u) = \begin{cases} 
  u & \text{for } u \notin V(C), \\
  u^+ & \text{for } u \in u_1 \tilde{C}u_2^- \\
  u_1 & \text{for } u = u_2^- \\
  u^- & \text{for } u = u_2^+. 
\end{cases}$$

From the previous arguments and Claim 2.5, for any $u \in f(N(u_2) \cup N(x_2))$, we have $uu_1 \notin E(G)$. Note that $x_2 \notin f(N(u_2) \cup N(x_2))$ and $x_2u_1 \notin E(G)$, and we obtain

$$t_2 \leq d(u_1) \leq n - |N(u_2) \cup N(x_2)| - 1 \leq t_2 - 1,$$

a contradiction. Therefore, Theorem 1.4 holds.
3. A corollary

Theorem 1.4 can deduce many sufficient conditions for Hamiltonian graphs, one of them is the following corollary.

**Corollary 3.1.** Let $G$ be a simple graph of order $n(\geq 3)$ with connectivity $k \geq 2$. If for each independent set $S$ of cardinality $k+1$, and any distinct pair $u$ and $v$ in $S$, $|N(u)\cup N(v)| \geq n - \Delta(S)$, then $G$ is Hamiltonian.

Take a complete graph $K_n$ and attach $k$ independent vertices $X = \{x_1, x_2, \ldots, x_k\}$ of $\min \{d(x) : x \in X\} = k$ and for any $u, v \in X$, $|N(u)\cup N(v)| = k+1$. and there exists a vertex $x \in V(K_n - k)$, $x$ is nonadjacent to all vertices in $X$. We denote the graph by $G(k)$. Let $x \in V \setminus N(X)$, then $S = X \cup \{x\}$ is an independent set with $\Delta(S) = n - (k+1)$. For any $u, v \in X$, $|N(u)\cup N(v)| = k+1 = n - \Delta(S)$. This implies that $G(k)$ satisfies the condition of Corollary 3.1, thus, $G(k)$ is Hamiltonian. However, $|N(u)\cup N(v)| = k+1 < n - \max \{d(u), d(v)\}$, this implies that $G(k)$ does not satisfy the condition of Corollary 2 in [6]. Moreover, for $k \geq 3$, $n \geq 3k+4$ and any $u, v, w \in X$, $d(u)+d(v)+d(w) \leq 3k+3 < n + |N(u)\cup N(v)\cup N(w)|$. This implies that $G(k)$ does not satisfy the condition of Theorem 1.6.

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