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Note

Complementary cycles containing a fixed arc in diregular bipartite tournaments

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Abstract

Let (x, y) be a specified arc in a k-regular bipartite tournament B. We prove that there exists a cycle C of length four through (x, y) in B such that $B-C$ is hamiltonian.

1. Introduction

A pair of vertex-disjoint cycles are called complementary if they span the vertex-set of graph. Complementary cycles in bipartite tournaments were discussed in [5] and [6]. In [5], nearly regular bipartite tournaments were studied, and in [6], two of the authors of this paper investigated complementary cycles containing a pair of specified vertices in regular bipartite tournaments. In this note, we prove that if B is a k-regular bipartite tournament and (x, y) any specified arc of B, then there exists a cycle C of length four through (x, y) in B such that $B-C$ is hamiltonian. Related conjectures are proposed at the end of Section 2.

We let in what follows $B(X, Y, E)$ denote a bipartite tournament with bipartition (X, Y), vertex set $V(B)=X \cup Y$ and arc set $E(B)$. If A and L are vertex-joint subsets of $V(B)$, we write $A \rightarrow L$ if every arc of B between A and L goes from A to L. Moreover, $\Gamma^+(A)$ (resp. $\Gamma^-(A)$) denotes the set of vertices of $B-A$ which are dominated by (resp. dominate) at least one vertex of A. If $A=\{x\}$, we write $\Gamma^+(x)$ (resp. $\Gamma^-(x)$) instead of

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$\Gamma^+(A)$ (resp. $\Gamma^-(A)$). A bipartite tournament is k-regular if for every vertex x of B we have $|\Gamma^+(x)| = |\Gamma^-(x)| = k$. A 1-factor of B is a spanning regular subgraph of B with indegree and outdegree one. It is well-known that B has a 1-factor, if and only if it contains a perfect matching from X to Y and from Y to X in B. We let F_{4k} denote the k-regular bipartite tournament consisting of four sets K, L, M, N each of cardinality k, and all possible arcs from K to L, from L to M, from M to N and from N to K.

The following results of [1–4] are used in Section 2.

Theorem 1.1 (Häggkvist and Manoussakis [2] and Manoussakis [3]). *Any bipartite tournament is hamiltonian if and only if it has a 1-factor and is strong.*

Lemma 1.2 (Häggkvist and Manoussakis [2] and Manoussakis [3]). *Let B be a bipartite tournament containing a 1-factor, B is not strong if and only if there exists a 1-factor consisting of cycles $C_1, C_2, \ldots, C_m$, $m \geqslant 2$ such that $C_1 \rightarrow C_j$ if $i<j$.*

Theorem 1.3 (Amar and Manoussakis [1], Manoussakis [3] and Wang Jian Zhong and He Shu Quang [4]). *Let B be a k-regular bipartite tournament and let (x, y) be any arc of B. There are cycles of all even length m, $4 \leqslant m \leqslant 4k$, through (x, y) unless B is isomorphic to F_{4k}.*

2. Main results

In this section we prove the following theorem.

Theorem 2.1. *Let B be a k-regular bipartite tournament and (x, y) any arc of B. There exists a cycle C of length four through (x, y) such that $B - C$ is hamiltonian.*

Proof. Let C: $x \rightarrow y \rightarrow w \rightarrow z \rightarrow x$ be any cycle of length four through the arc (x, y) in B. Such a cycle exists by Theorem 1.3, if B is not isomorphic to F_{4k}; Otherwise it is very easy to find such a cycle. Put $R = B - C$. Firstly we have to prove the following claim.

Claim. *There exists a cycle C of length four through (x, y) such that R has a 1-factor.*

Proof of the claim. Assume that for any cycle C of length four through (x, y), R has no 1-factor. It follows from a well-known theorem of König–Hall on matchings (see, for example, C. Berge, Graphs and Hypergraphs) that there exists a subset P either of $X - \{x, w\}$ or of $Y - \{y, z\}$ such that $|P| > |\Gamma^+(P)|$. Assume without loss of generality that $X - \{x, w\} \supseteq P$. Put $\Gamma^+(P) = Q$, $M = X - (P \cup \{x, w\})$ and $L = Y - (Q \cup \{y, z\})$. Since B is k-regular, $k \geqslant |P| > |Q| \geqslant k - 2$. We consider the following three possible cases:

(i) $|P| = k$ *and* $|Q| = k - 2$. By using regularity on degrees, we can see that $P - Q \cup \{y, z\}$. It follows that $|\Gamma^-(y)| = |P| + 1$, a contradiction.

(ii) $|P|=k$ *and* $|Q|=k-1$. As in (i), notice that $L\to P$ and $M\cup\{x,w\}\to L$. Consider now a vertex p in P such that both the arcs (y,p) and (p,z) are present in B. Such a vertex exists, since it follows from the regularity on degrees that $\Gamma^+(y)\cap\Gamma^-(z)\cap P\neq\emptyset$. Put C': $x\to y\to p\to z\to x$ and $R'=B-C'$. We have to prove that R' has a 1-factor. In particular, we have to prove that there is no subset P' of $X-\{x,p\}$ (the proof for $Y-\{y,z\}\supseteq P'$ is similar) such that $|P'|>|\Gamma^+(P')|$. Namely, if P' has k vertices, then both $P'\cap(P-p)\neq\emptyset$ and $P\cap(M\cup w)\neq\emptyset$ hold and therefore $|P'|\leqslant|\Gamma^+(P')|$. If on the other hand, the cardinality of P' is $k-1$, then, once more, we may easily verify that $|P'|\leqslant|\Gamma^+(P')|$.

(iii) $|P|=k-1$ *and* $|Q|=k-2$. We have $P\to Q\cup\{y,z\}$ and $L\to P$. Find, as above, a vertex p in L such that both the arcs (w,p) and (p,x) are present in B. Consider the cycle C': $x\to y\to w\to p\to x$. Put $R'=B-C'$. We have to prove that R' has a 1-factor. Let P' be defined as in case (ii). If P' has k vertices, we find cases (i) and (ii). Assume that the cardinality of P' is $k-1$. In this case, notice that $P'=M$, and therefore there exists a vertex g in L which is dominated by no vertex of P'. It follows that $|\Gamma^+(g)|=|P|+|M|=2k-2$, a contradiction for $k>2$. Assume $k=2$. In this particular case we have $Q=\emptyset$. Furthermore, g is dominated by both x and w. However this is another contradiction, since it follows that the outdegree of w is three. This completes the proof of the claim. □

Proof of Theorem 2.1 (*Conclusion*). Let now C and R be as they are described in the above claim. If R is strong, we have finished by Theorem 1.1, so assume that it is not the case. It follows that $k\geqslant 3$. If $k=3$, then R consists of two cycles C_1: $1\to 2\to 3\to 4$ and C_2: $5\to 6\to 7\to 8$, each of length four, such that $C_2\to C_1$, by Lemma 1.2. Now by studying conditions on degrees, we can see that $C_2\to C$ and $C\to C_1$ in B. Consequently, the cycles $x\to y\to 1\to 8\to x$ and $z\to 3\to 4\to 5\to 6\to w\to 2\to 7\to z$ are desired. Assume, therefore, that $k\geqslant 4$. Let $C_1, C_2, \ldots, C_m$, $m\geqslant 2$, be cycles of R, as given in Lemma 1.2. Let the length of C_i be n_i. Now, if $n_1\leqslant n_2+\cdots+n_m$, we can see that there exists a vertex r in C_1 such that $k=|\Gamma^+(r)|\geqslant n_1/4+(n_2+\cdots+n_m)/2\geqslant (n_1+n_2+\cdots+n_m)/4+(n_2+\cdots+n_m)/4\geqslant(n-4)/4+(n-4)/8$, a contradiction for $k\geqslant 4$, since $n=4k$. On the other hand, if $n_1\geqslant n_2+\cdots+n_m$, then using similar arguments, we obtain a contradiction by considerng $|\Gamma^-(r)|$, where r is now a vertex of C_m. This completes the proof of the theorem. □

We conclude this paper with some conjectures which could extend Theorem 2.1 and the theorem of [6].

Conjecture 2.2. Let B be a k-regular bipartite tournament, $k\geqslant 2$ on n vertices. If B is not isomorphic to F_{4k}, then there are complementary cycles of all possible lengths in B.

Conjecture 2.3. Let B be a k-regular bipartite tournament, $k\geqslant 2$, on n vertices and let (x,y) be any specified arc of B. If B is isomorphic neither to F_{4k} nor to some other specified families of digraphs, then there are complementary cycles C and C' of all possible lengths in B, and C goes through the arc (x, y).

Conjecture 2.4. Let B be a k-regular bipartite tournament, $k \geqslant 2$, on n vertices and let x, y be two specified vertices of B. If B is isomorphic neither to F_{4k} nor to some other specified families of digraphs, then there are complementary cycles C andc C' of all possible lengths in B such that C contains x (resp. C' contains y).

Obviously, Conjectures 2.3 and 2.4 are stronger than Conjecture 2.2. Furthermore, Conjectures 2.3 and 2.4 do not imply each other. Notice also that a support for these conjectures could be obtained from Theorem 2.1 and the theorem of [6].

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