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The Characterization of Symmetric Primitive Matrices with Exponent $2n - 2r(\geq n)$*

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In 1986, Shao [3] proved that the exponent set of symmetric primitive matrices is $\{1, 2, \ldots, 2n-2\}\backslash S$, where S consists of all odd numbers among $\{n, n+1, \ldots, 2n-2\}$. This paper gives the complete description of symmetric primitive matrices with exponent $2n - 2r\,(\geq n)$.

1. INTRODUCTION

An $n \times n$ nonnegative matrix $A = (a_{ij})$ is *primitive if* $A^k > 0$ for some positive integer k, the least such k being called the *exponent* of A and denoted by $\gamma(A)$. The *associated graph* of symmetric matrix A, denoted by $G(A)$, is the graph with a vertex set $V(G(A)) = \{1, 2, \ldots, n\}$ such that there is an edge from i to j in graph $G(A)$ iff $a_{ij} > 0$. Hence $G(A)$ contains loops if $a_{ii} > 0$ for some i. A graph G is *primitive* if there exists an integer $k > 0$ such that for all ordered pairs of vertices $i, j \in V(G)$ (not necessarily distinct), there is a walk from i to j with length k. The least such k is called the *exponent* of G, denoted by $\gamma(G)$. Clearly, A symmetric matrix A is primitive iff its associated graph $G(A)$ is primitive. And in this case, we have $\gamma(A) = \gamma(G(A))$. By this reason as above, we will use graph theory as a major tool and consider $\gamma(G(A))$ to prove our result.

Let $SE_n = \{k \in Z^+ \mid k = \gamma(A)$ for some $n \times n$ symmetric primitive matrix $A\}$ be the exponent set of $n \times n$ symmetric primitive matrices. In 1986, Shao Jiayu proved that SE_n is $\{1, 2, \ldots, 2n-2\}\backslash S$, where S consists of all odd numbers among $\{n, n+1, \ldots, 2n-2\}$. In this paper, we give the complete description of $n \times n$ symmetric primitive matrices with exponent $2n - 2r\,(\geq n)$. This result is a generalisation of the result in [2], [3], where $r = 1, 2$.

Other notation and terminology not defined in this paper can be found in [1].

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2. SOME LEMMAS ABOUT $\gamma(G)$

The *exponent from vertex u to vertex v*, denoted by $\gamma(u.v)$, is the least integer k such that there exists a walk of length m from u to v for all $m \geq k$. We denote $\gamma(u,u)$ by $\gamma(u)$.

LEMMA 1[3] *If G is a primitive graph, then* $\gamma(G) = \mathrm{Max}_{u,v \in V(G)} \gamma(u,v)$.

LEMMA 2[3] *G is a primitive graph iff G is connected and has odd cycles.*

LEMMA 3[2] *Let G be a primitive graph, and let u and* $v \in V(G)$. *If there are two walks from u to v with lengths* k_1 *and* k_2 *respectively, where* $k_1 + k_2 \equiv 1 (\mathrm{mod}\ 2)$, *then* $\gamma(u,v) \leq \mathrm{Max}\{k_1, k_2\} - 1$.

LEMMA 4 *Let G be a primitive graph with order n* $(\geq 2r)$. *If there are* $u, v \in V(G)$ *such that* $\gamma(u,v) = \gamma(G) = 2n - 2r$, *and C is any order cycle in G, then* $V[P_{\min}(u,v)] \cap V(C) = \Phi$, *where* $P_{\min}(u,v)$ *is a shortest path from u to v in G.*

Proof Let $P = P_{\min}(u,v) = v_0 v_1 \cdots v_m$ (where $v_0 = u, v_m = v$) with $V(P) \cap V(C) \neq \Phi$, and let $i = \mathrm{Min}\{s \mid v_s \in V(P) \cap V(C)\}, j = \mathrm{Max}\{s \mid v_s \in V(P) \cap V(C)\}$, $(0 \leq i \leq j \leq m)$. Then v_i, v_j divide C into internally disjoint (v_i, v_j)-paths L_1 and L_2. Clearly, $Q_k = P(v_0, v_i) \cup L_k \cup P(v_j, v_m), k = 1, 2$, are two disjoint (u,v)-walks in G with $|Q_1| + |Q_2| \equiv 1 (\mathrm{mod}\, 2)$ and $|Q_k| \leq n$. Thus by Lemma 3, $n \leq 2n - 2r = \gamma(u,v) \leq \mathrm{Max}\{|Q_1|, |Q_2|\} - 1 < n$, a contradiction. ■

Let G be a primitive graph with $u, v \in V(G)$ and $k_0 = d(u,v)$, where $d(u,v)$ is the distance between u and v in G. Thus by Lemma 2, there is an (u,v)-walk with length k' such that $k_0 + k' \equiv 1 (\mathrm{mod}\ 2)$. Let $k'_0 = k'_0(u,v) = \mathrm{Min}\{k' \mid k_0 + k' \equiv 1 (\mathrm{mod}\ 2)$ and there is a walk of length k' from u to v in $G\}$. Then $k'_0 > k_0$ and $k_0 + k'_0 \equiv (\mathrm{mod}\ 2)$. By Lemma 3, we have $\gamma(u,v) = k_0^1(u,v) - 1$.

Let G_1, G_2 be two subgraphs of G. $P_{\min}(G_1, G_2)$ denotes a shortest path between G_1 and G_2. Its length $d(G_1, G_2) = \mathrm{Min}_{(v_i \in V(G_i)/i=1,2)} d(v_1, v_2)$. If for $u, v \in V(G)$ and an odd cycle C in G, there is an $x \in V(C)$ such that $d(u,C) = d(u,x)$ and $d(v,C) = d(v,x)$, then we define $\gamma(u,v,C) = d(u,x) + d(v,x) + |C| - 1$. For convenience, we denote $\gamma(u,u,C)$ by $\gamma(u,C)$. And let $\gamma(u) = \mathrm{Min}_{|c| \equiv 1 (\mathrm{mod}\ 2)} \gamma(u,C)$.

LEMMA 5 *Let G be a primitive graph with order* $n (\geq 2r)$. *Suppose there are* $u, v \in V(G)$ *such that* $\gamma(u,v) = \gamma(G) = 2n - 2r$. *Then for any odd cycle C in* $G, \gamma(u,v,C)$ *is well defined. Furthermore, there is an odd cycle* C_0 *in G such that* $k'_0 = k'_0(u,v) = \gamma(u,v,C_0) + 1$. *i.e.* $\gamma(u,v) = \gamma(u,v,C_0)$.

Proof Let C be an odd cycle in G. Let $P_{\min}(u,C)$ with $d(u,i) = d(u,C), i \in C$ and $P_{\min}(v,C)$ with $d(v,j) = d(v,C), j \in C$. If $I = V(P_{\min}(u,C)) \cap V(P_{\min}(v,C)) = \phi$, then using an analogous proof of Lemma 4, produces a contradiction. So $I \neq \Phi$. Let $w \in I$ such that the section $P(w,i)$ of $P_{\min}(u,C)$ satisfying $V(P(w,i)) \cap I = \{w\}$. Since $P_{\min}(u,C)(P_{\min}(v,C)$, resp.) is the shortest path from $u(v$, resp.) to $C, |P(w,i)|$ must be equal to $|P(w,j)|$, where $P(w,j)$ is the section of $P_{\min}(v,C)$. Thus $|P(v,w) \cup P(w,i)| = d(v,C)$, where $P(v,w)$ is section of $P_{\min}(v,C)$. Now, we change $P_{\min}(v,C)$ to $P(v,w) \cup P(w,i)$, and let $x = i$. Hence $\gamma(u,v,C)$ is well defined.

Furthermore, by the definition of k_0', there is an (u,v)-walk $P_{k_0'}(u,v)$ with length k_0'. Since $k_0+k_0'\equiv 1 \pmod 2$, the symmetric difference $P_{\min}(u,v)\triangle P_{k_0'}(u,v)$ must be contained an odd cycle C_0. By Lemma 4, we have $C_0\subseteq P_{k_0'}(u,v)$. Thus $\gamma(u,v,C_0)+1\leq |P_{k_0'}(u,v)|=k_0'$. Hence by the definition of k_0', $k_0'=k_0'(u,v)=\gamma(u,v,C_0)+1$.

LEMMA 6 *Let G be a primitive graph of order $n\,(\geq 2r)$ with $\gamma(G)=2n-2r$. Then there exists a vertex $u\in V(G)$ such that $\gamma(u)=\gamma(G)$.*

Proof Suppose the lemma is false. Let $m=\min_{x,y\in V(G)}\{d(x,y)|\gamma(x,y)=\gamma(G)\}$, thus there exist $u,v\in V(G)$ such that $\gamma(u,v)=\gamma(G)$ and $m=d(u,v)>0$. Let $P_0=P_{\min}(u,v)$ and let C_0 be an odd cycle in G, thus $d(P_0,C_0)=t\geq 1$ by Lemma 4. Now, we choose P_0 and C_0 such that $2t+d$ as small as possible, where $d=|C_0|$. Let $P_1=P_{\min}(P_0,C_0)=u_0u_1\cdots u_t$, where $u_0\in V(P_0)$ and $u_t\in V(C_0)$. Then $2n-2r=\gamma(u,v)\leq\gamma(u,v,C_0)\leq m+2t+d-1$. Hence we have $n_2=|V(G)\backslash V(P_0\cup P_1\cup C_0)|=n-(m+t+d)\leq n-(2n-2r-t+1)\leq t-1$.

If $\gamma(u,C_0)\geq\gamma(G)$. By $m>0$, there is an odd cycle C_1 in G such that $\gamma(u)=\gamma(u,C_1)<\gamma(G)\leq\gamma(u,C_0)$. Let $P_1'=P_{\min}(u,C_1)=uv_1v_2\cdots v_s$, where $v_s\in V(C_1)$.

CASE 1 $[V(P_1')\cup V(C_1)]\cap[V(P_1)\cup V(C_0)]=\Phi$. We have $2d(P_0,C_1)+|C_1|\leq 2|P_1'|+|C_1|\leq 2n_2+1\leq 2(t-1)+1<2t+d.$. This contradicts the choice of P_0 and C_0;

CASE 2 $V(P_1')\cap V(P_1)\neq\Phi$. Let $w\in V(P_1')\cap V(P_1)$. Thus $\gamma(u)=\gamma(u,C_1)=2|uP_1'w|+2|wP_1'v_s|+|C_1|-1<\gamma(u,C_0)\leq 2|uP_1'w|+2|wP_1u_t|+|C_0|-1$. Hence we have $2|wP_1'v_s|+|C_1|<2|wP_1u_t|+|C_0|$. Thus $2|P_{\min}(P_0,C_1)|+|C_1|\leq 2|u_0P_1w|+2|wP_1'v_s|+|C_1|<2|u_0P_1w|+2|wP_1u_t|+|C_0|=2t+d$. This contradicts the choice of P_0 and C_0;

CASE 3 $V(P_1')\cap V(P_1)=\Phi, V(C_1)\cap V(C_0)\neq\Phi$. In this case, it is easy to see that there is an odd cycle in $C_0\cup C_1\cup P_0\cup P_1\cup P_1'$, which intersects the path P_0 This contradicts Lemma 4.

Hence we can always assume that $\gamma(u,C_0)<\gamma(G)$. Using an analogous proof, we have $\gamma(v,C_0)<\gamma(G)$. Thus $\gamma(G)=\gamma(u,v)\leq\gamma(u,v,C_0)\leq d(u,C_0)+d(v,C_0)+d-1=\frac{1}{2}(\gamma(u,C_0)+\gamma(v,C_0))<\gamma(G)$, a contradiction. Therefore $m=0$, i.e. there exists a vertex $u\in V(G)$ such that $\gamma(u)=\gamma(G)$. ■

3. THE MAIN RESULTS

Firstly, we define a class of graphs $N_{n.r}(G)(n\geq 2r>2)$ as follows:

Let $G^*=(V^*,E^*)$ be a graph, where the vertex set $V^*=\cup_{0\leq i\leq n-r}V_i^*$ with $V_i^*\cap V_j^*=\Phi(0\leq i\leq j\leq n-r)$ and $|V_i^*|=r\ (0\leq i\leq n-r)$, the edge set $E^*=E_1^*\cup E_2^*$ with $E_1^*=\{uv\,|\,u\in V_i^*, v\in V_{i+1}^*, 0\leq i\leq n-r-1\}$ and $E_2^*=\{uv\,|\,u,v\in V_{n-r}^*$ and u,v are not necessarily distinct$\}$.

For any odd number d with $1\leq d\leq 2r-1$, let $t=n-\frac{1}{2}(2r+d-1)=n-r-\frac{1}{2}(d-1)$. When $r\geq 2$, we take two distinct vertices $u_i,u_{2n-2r+1-i}$ in $V_i^*(0\leq i\leq n-r)$, and put the path $P_t=u_0u_1\cdots u_t$ and the cycle $C_d=u_tu_{t+1}\cdots u_{t+d-1}u_t$. Let

$V(d) = V_1(d) \cup V_2(d)$, where $V_1(d) = \{u_0, u_1, \ldots, u_t, \ldots, u_{t+d-1}\}$ and $V_2(d) \subseteq V^* \backslash V_1(d)$ with $|V_2(d)| = r - \frac{1}{2}(d+1)$. Thus $|V(d)| = n$. The induced subgraph $G^*[V(d)]$ of G^* is denoted by G^*_d.

The edge-induced spanning subgraph $G^*_d[E(d)]$ of G^*_d is denoted by $\tilde{G}_d$, where $E(d) = E_1(d) \cup E_2(d), E_1(d) = E(P_t) \cup E(C_d)$,and $E_2(d) \subseteq E(G^*_d) \backslash E_1(d)$ such that $\tilde{G}_d$ satisfies the following two conditions: (a) $\tilde{G}_d$ is connected, (b) There always be $d(x, C_d) \leq t$ in $\tilde{G}_d$ for any vertex $x \in K$, where K is any component of $\tilde{G}_d[V_2(d)]$ with $V(K) \cap V(C) = \Phi$ and C is any odd cycle in $\tilde{G}_d$.

Now, we define $N_{n,r}(G) = \bigcup_{\substack{1 \leq d \leq 2r-1 \\ d \equiv 1 \pmod 2}} N_{n,r}(d)$, where $N_{n,r}(d) = \bigcup_{G^*_d \in \{G^*_d\}} \{\tilde{G}_d\}$.

THEOREM 1 *Let G be a primitive graph with order n $(\geq 2r)$, then $\gamma(G) = 2n - 2r$ iff $G \in N_{n,r}(G)$.*

Proof Necessity. Let G be a primitive graph of order $n \geq 2r$ with $\gamma(G) = 2n - 2r$. By Lemma 6, there is an $u_0 \in V(G)$ such that $\gamma(u_0) = \gamma(G)$. By Lemma 5, there is an odd cycle C_d with the length d in G such that $\gamma(u_0) = \gamma(u_0, C_d)$. And let $d(u_0, C_d) = t$. By Lemma 4, we have $t \geq 1$. Thus $\gamma(u_0) = 2d(u_0, C_d) + |C_d| - 1 = 2t + d - 1 = 2n - 2r$. Hence we have $t = n - r - \frac{1}{2}(d-1)$. Let $P_t = P_{\min}(u_0, C_d) = u_0 u_1 \cdots u_t, C_d = u_t u_{t+1} \cdots u_{t+d-1} u_t$, $V(d) = V(P_t) \cup V(C_d) = \{u_0, u_1, \ldots, u_t, \ldots, u_{t+d-1}\}$ and $V_2(d) = V(G) \backslash V_1(d)$, thus $n_1 = |V_1(d)| = t + d$ and $n_2 = |V_2(d)| = n - n_1 = n - t - d = r - \frac{1}{2}(d+1)$. Let $E_1(d) = E(P_t) \cup E(C_d)$ and $E_2(d) = E(G) \backslash E_1(d)$. Thus $E(G) = E_1(d) \cup E_2(d)$.

Let $X = \{u \mid d(u_0, u) < n - r, u \in V(G)\}$. If there is an odd cycle C in $G[X]$, let $d(u_0, C) = s$, thus $s + \frac{1}{2}(|C| - 1) < n - r$. We have $\gamma(u_0) \leq \gamma(u_0, C) \leq 2s + |C| - 1 = 2(s + \frac{1}{2}(|C| - 1)) < 2n - 2r = \gamma(G) = \gamma(u_0)$, a contradiction. Hence $G[X]$ is a bipartite graph.

G is connected by Lemma 2. Let K be a component of $G[V_2(d)]$ with $V(K) \cap V(C) = \Phi$, where C is any odd cycle in G. If there is an $x \in V(K)$ such that $d(x, C_d) > t$. Since $|V(K)| \leq n_2 = t - 1 + 2r - n \leq t - 1, N(K) \cap V(C_d) = \Phi$ and $\gamma(x, C_d) > \gamma(u_0, C_d) = \gamma(u_0)$. On the other hand, by Lemma 1, $\gamma(u_0) = \gamma(G) \geq \gamma(x)$. Hence by Lemma 5, there always exist an odd cycle $C' \neq C_d$ such that $\gamma(x) = \gamma(x, C') \leq \gamma(u_0)$. By the hypothesis $V(K) \cap V(C') = \Phi$ and $N(K) \cap V(C_d) = \Phi$, we have $P_{\min}(x, C') \cap P_t \neq \Phi$. Let $y \in P_{\min}(x, C') \cap P_t$. Note that $d(x, C_d) > t$ implies $d(x, y) > d(u_0, y)$. We then have $\gamma(u_0) \geq \gamma(x) = \gamma(x, C') = 2(dx, C') + |C'| - 1 = 2d(x, y) + 2d(y, C') + |C'| - 1 > 2d(u_0, y) + 2d(y, C')$ $|C'| - 1 \geq 2d(u_0, C') + |C'| - 1 = \gamma(u_0, C') \geq \gamma(u_0)$, a contradiction. Therefore for any $x \in V(K)$, we have $d(x, C_d) \leq t$.

Let $X_i = \{u \mid u \in X, d_G(u_0, u) = i\}, 0 \leq i < n - r$. If there is an edge $xy \in E(G[X_i])$, let $P_x = P_{\min}(u_0, x)$ and $P_y = P_{\min}(u_o, y)$ be two paths, then there is an odd cycle in $u_0 P_x xy P_y u_0 \subseteq G[X]$. This contradicts $G[X]$ being a bipartite graph. Hence $G[X_i]$ is an empty graph for any $i \in \{0, 1, \ldots, n - r - 1\}$. Note that $|X_i| \leq n_2 + 1 = r - \frac{1}{2}(d+1) + 1$. So $|X_i| \leq r$, where $i \in \{0, 1, \ldots, n - r - 1\}$. Hence $G[X]$ is a subgraph of $G^*[X]$. On the other hand, $\{u_0, u_1, \ldots, u_{n-r-1}\} \subseteq X$, hence $|X| \geq n - r$. Thus $|X_{n-r}| = |G \backslash X| = n - |X| \leq n - (n - r) = r$, and $G^*[V^*_{n-r}]$ is a "complete" graph with a loop at every vertex. Therefore G is a subgraph of G^*_d. i.e. $G \in N_{n,r}(d) \subseteq N_{n,r}(G)$.

Sufficiency. Let $G = \tilde{G}_d \in N_{n,r}(d) \subseteq N_{n,r}(G)$. Since G is connected and contains odd cycles, G is a primitive graph by Lemma 2. Also $|V(G)| = |V_1(d)| + |V_2(d)| = t + d + (r - \frac{1}{2}(d+1)) = (n - r - \frac{1}{2}(d-1)) + d + (r - \frac{1}{2}(d+1)) = n$. In the following, we divide into four steps the proof that $\gamma(G) = 2n - 2r$.

1. $\gamma(u_0) = 2n - 2r$.
   Clearly, $\gamma(u_0, C_d) = 2t + d - 1 = 2n - 2r$. Let $X_0 = \{u | d_G(u_0, u) < n - r, u \in V(G)\}$. It is easy to check that $G[X_0]$ does not contain any odd cycle since $G = \tilde{G}_d \in N_{n,r}(d)$. If there is an odd cycle C in G such that $\gamma(u_0, C) < 2n - 2r$. i.e. $2d(u_0, C) + |C| - 1 < 2n - 2r$. Thus $d(u_0, C) + \frac{1}{2}(|C| - 1) < n - r$. This implies that $V(C) \subseteq X_0$. Hence C is an odd cycle in $G[X_0]$, a contradiction. So $\gamma(u_0) = 2n - 2r$.
   Let $B = \{K | K$ is a component in $G[V_2(d)]$ and there is an odd cycle C in G such that $V(K) \cap V(C) \neq \Phi\}$, $X_2' = \cup_{K \in B} V(K)$ and $X_1' = V_2(d) \backslash X_2'$.
2. If $u, v \in V_1(d) \cup X_1'$, then $\gamma(u, v) \leq 2n - 2r$.
   By the condition (b) of the construction of $\tilde{G}_d$, we have that $\gamma(u, v) \leq \gamma(u, v, C_d) \leq d(u, C_d) + d(v, C_d) + |C_d| - 1 \leq 2t + d - 1 = \gamma(u_0) = 2n - 2r$.

In the following, without loss of generality, we always assume that: $v \in X_2'$. Thus there is a component $K \in B$ so that $v \in V(K)$.

3. If $N(K) \cap V(C_d) \neq \Phi$, then $\gamma(u, v) < 2n - 2r$.
   If $u \in V_1(d) \cup X_1'$, by the construction of $\tilde{G}_d$, we have $d(u, C_d) \leq t$, and note that $d(v, C_d) \leq |X_2'| \leq r - \frac{1}{2}(d+1) \leq t - 1$; If $u \in X_2'$, we have $d(u, C_d) + d(v, C_d) \leq \text{Max}\{2|X_2'|, |X_2'| + |V(P_t)|\} = |X_2'| + |V(P_t)|$. Thus we always have $d(u, C_d) + d(v, C_d) \leq 2t - 1$. Hence $\gamma(u, v) \leq \gamma(u, v, C_d) \leq d(u, C_d) + d(v, C_d) + d - 1 \leq 2t + d - 2 < 2n - 2r$.
4. If $N(K) \cap V(C_d) = \Phi$, then $\gamma(u, v) \leq 2n - 2r$.
   By the definition of K, there is an odd cycle C in G such that $V(K) \cap V(C) \neq \Phi$. Let $P_1 = P_{\min}(u, C)$ be a shortest path from u to C in G and let $P_2 = P_{\min}(v, C)$ be a shortest path from v to C in K. If P_1 and P_2 are internally disjoint paths in G, then using analogous proof of Lemma 4, we have $\gamma(u, v) \leq 2n - 2r$. In the following, we assume that P_1 and P_2 internally intersect each other.
   If $u \in K$, thus since P_t is a shortest path in G and C contains at most $d - 1$ edges in C_d, $|X_2' \cap V(C)| \geq \frac{1}{2}(|C| - d)$. Thus we have: $|V(K) \backslash V(C)| \leq |X_2' \backslash V(C)| \leq t - 1 - \frac{1}{2}(|C| - d)$. Hence we have $\gamma(u, v) \leq \gamma(u, v, C) \leq d(u, C) + d(v, C) + |C| - 1 \leq 2|V(K) \backslash V(C)| + |C| - 1 \leq 2(t - 1 - \frac{1}{2}(|C| - d)) + |C| - 1 = 2t + d - 3 < 2n - 2r$.
   If $u \notin K$, thus P_1 must intersect P_t. Along P_1 from u to C, u_i and u_j are denoted the first and the last intersections of P_t respectively.

   4.1. $V(C) \cap V(P_t) \neq \Phi$. Let $P_1' = P_1'(u, C) = uP_1u_iP_tC$. Clearly $V(P_1') \cup V(K) = \Phi$. Hence P_1' and P_2 are internally disjoint paths in G. Thus by the proof of above, we have $\gamma(u, v) < 2n - 2r$.
   4.2. $V(C) \cap V(P_t) = \Phi$. I.e. $V(C) \subseteq V(K)$ since $N(K) \cap V(C_d) = \Phi$.
   Since P_1 and P_t are the shortest paths in G, $|u_iP_1u_j| = |u_iP_tu_j| = |i - j|$. Hence, in the following, we always assume that $u_iP_1u_j \equiv u_iP_tu_j$. And along P_2 from v to C, v_0 is denoted the first intersection on P_1. Note that at this

case v_0 must be on $u_jP_1C \subseteq K \cup \{u_j\}$. Since v_0P_1C and v_0P_2C are the shortest paths from v_0 to C in K, $|v_0p_1C| = |v_0P_2C|$. Hence, in the following, we always assume that $v_0P_1C \equiv v_0P_1C$ too. If $d(v,v_0)+d(v_0,u_j)+d(u,u_i) \leq i+j$, thus we have $\gamma(u,v) \leq \gamma(u,v,C_d) \leq d(u,C_d)+d(v,C_d)+d-1 \leq d(u,u_i)+d(u_i,C_d)+d(v,v_0)+d(v_0,u_j)+d(u_j,C_d)+d-1 \leq i+d(u_i,C_d)+j+d(u_j,C_d)+d-1 = \gamma(u_0) = 2n-2r$. Hence in the following we assume that $d(v,v_0)+d(v_0,u_j)+d(u,u_i) \geq i+j+1$. Note that $d(v,C)+|C|+d(u_j,v_0)-1+d(u,u_i) \leq |V_2(d)| \leq t-1$ and $d(v,C) = d(v,v_0)+d(v_0,C)$. Thus we have $d(v_0,C) \leq t-1-(i+j)-|C|$. Therefore $\gamma(u,v) \leq \gamma(u,v,C) \leq d(u,C)+d(v,C)+|C|-1 \leq d(u,u_i)+d(u_i,u_j)+d(u_j,v_0)+d(v_0,C)+d(v,C)+|C|-1 \leq t-1+|i-j|+(t-1-i-j-|C|) = 2t-|C|-2-(i+j-|i-j|) < 2n-2r$.

Up to now, we have exhausted all possible cases and get that $\gamma(G) = 2n-2r$. Therefore the proof of the Theorem is completed. ∎

Using the connection between the exponent of a matrix and the exponent of a graph stated above, by Theorem 1, we have:

THEOREM 2 *Let A be a symmetric primitive matrix with order $n(\geq 2r)$, then $\gamma(A) = 2n-2r$ iff $G(A) \in N_{n,r}(G)$.*

References

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