SOME TREE-STARS RAMSEY NUMBERS

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In this paper, we consider the generalized Ramsey number $R(T_p, K_{1,q_1}, \cdots, K_{1,q_t})$ and its lower and upper bounds. Furthermore, we obtain some Ramsey numbers in special cases.

1. Introduction

Let $G_1, G_2, \cdots, G_k$ be simple graphs. The generalized Ramsey number $R = R(G_1, G_2, \cdots, G_k)$ is the smallest integer such that if the edges of a complete graph K_n with $n \geq R$ are painted arbitrarily with k colours, then the i-th coloured subgraph contains G_i as a subgraph for at least one i. Let T_p, $K_{1,p-1}$, K_p denote a tree, a star and a complete graph of order p, respectively.

Below are some known results about Ramsey numbers of stars or trees.

THEOREM A ([1]). *Let* $R(K_{1,q_1}, \cdots, K_{1,q_t}) = R$ *and* $\Sigma = \sum_{i=1}^{t} (q_i - 1)$. *Then,*

(i) $R = \Sigma + 2$ *if* Σ *is odd;*
(ii) $R = \Sigma + 2$ *if* Σ *is even and* q_i $(i = 1, 2, \cdots, t)$ *are odd;*
(iii) $R = \Sigma + 1$ *if* Σ *is even and there exists* $i \in \{1, 2, \cdots, t\}$ *such that* q_i *is even.*

THEOREM B ([3]). *Let* $p > 1$. *Then,*

(i) $R(T_p, K_{1,q}) = p + q - 1$ *if* $q \equiv 1 \pmod{p-1}$,
(ii) $R(T_p, K_{1,q}) \leq p + q - 1$.

THEOREM C ([2]). *Let* T_p *be a tree with a vertex of degree one adjacent to a vertex of degree two. Then*

$$R(T_p, K_{1,q}) = p + q - 2$$

provided that one of the following four conditions holds:

$$q \equiv 0,2 \pmod{p-1},$$
$$q \not\equiv 1 \pmod{p-1} \text{ and } q \geq (p-3)^2,$$
$$q \not\equiv 1 \pmod{p-1} \text{ and } q \equiv 1 \pmod{p-2},$$
$$\text{or } q \equiv p-2 \pmod{p-1} \text{ and } q > p-2.$$

Up to date ([5]), $R(T_p, K_{1,q})$ is still not known in general. In this paper we study the generalized Ramsey number $R(T_p, K_{1,q_1}, K_{1,q_2}, \cdots, K_{1,q_t})$.

2. The Lower Bounds

First, we need the following Lemmas :

LEMMA 1 ([4]). *Let $G = K_{n,n,\cdots,n}$ be a l-partite complete graph. If ln is odd and $l \geq 2$, then G is 2-factorable.*

LEMMA 2 ([6, Theorem 2.10]). *Let $G = K_{n,n}, \cdots, n$ be a l-partite complete graph. If ln is even and $l \geq 2$, then G is 1-factorable.*

THEOREM 1. *Let $p > 1$ and $\Sigma = \sum_{i=1}^{t}(q_i - 1)$. Then*

$$R(T_p, K_{1,q_1}, \cdots, K_{1,q_t}) \geq p + \Sigma - \theta_0,$$

where $\theta_0 = \min_{s,u}\{u + s - 1\}$ and $u, m \in \mathbb{N}^+, s \in \mathbb{N}^+ \cup \{0\}$ satisfy the following condition: $\Sigma - s = m(p-u)$, and the number of the even numbers in $\{q_1, q_2, \cdots, q_t\} \leq s$, if $p + \Sigma - s - u$ is odd.

PROOF. Let s and u satisfy the condition of Theorem such that $\theta_0 = s + u - 1$. Thus $p + \Sigma - \theta_0 - 1 = (\Sigma - s) + (p - u) = (m+1)(p-u)$. Let $G = K_{(m+1)(p-u)}$. We consider two cases.

Case I: $(m+1)(p-u)$ is even. Let $(V_1, V_2, \cdots, V_{m+1})$ be a partition of $V(G)$ with $|V_i| = p - u$ $(i = 1, 2, \cdots, m+1)$, and let $H = K_{|V_1|,|V_2|,\cdots,|V_{m+1}|}$. By Lemma 2, H is 1-factorable. Hence H is a union of $m(m-u)$ 1-factors. Thus H can be divided into internally-disjoint subgraphs H_i $(i - 1, 2, \cdots, t)$, where H_i is a union of $q'(\leq q_i - 1)$ 1-factors with $\sum_{i=1}^{t} q_i' = m(p-u) = \Sigma - s$. Thus there is an assignment of the i-th colour to H_i $(i = 1, 2, \cdots, t)$, and of the $(t+1)$-th colour to H^C. Clearly, there are no monochromatic T_p whose edges are in colour $t+1$ and $K_{1,q_i}(i = 1, 2, \cdots, t)$ whose edges are in colour i. Hence, the theorem is true in this case.

Case II: $(m+1)(p-u)$ is odd. Without loss of generality, we assume that $q_1, q_2, \cdots, q_r$ $(r \leq s)$ are even, and the other q_i $(i > r)$ are odd. Let H be as in case I. Since $(m+1)(p-u)$ is odd, m is even. Thus $m(p-u)$ is even. By Lemma 1, H is 2-factorable. Hence, H is a union of $\frac{1}{2}m(p-u)$ 2-factors. Note that $\Sigma - s = [(q_1-2)+(q_2-2)+\cdots+(q_r-2)]+[(q_{r+1}-1)+\cdots+(q_r-1)]-(s-r)$. So, there always exist nonnegative integers a_i satisfying $a_i \le \frac{1}{2}(q_i-2)$ if $i=1,2,\cdots,r$; $a_i \le \frac{1}{2}(q_i-1)$ if $i=r+1,r+2,\cdots,t$ and $a_1+a_2+\cdots+a_t = \frac{1}{2}(\Sigma - s) = \frac{1}{2}m(p-u)$. Hence H can be divided into internally-disjoint subgraphs H_i $(i=1,2,\cdots,t)$, where H_i is a union of a_i 2-factors. Thus there is an assignment of the i-th colour to H_i $(i=1,2,\cdots,t)$, and of the $(t+1)$-th colour to H^C. Clearly there are no monochromatic T_p whose edges are in colour $t+1$ and $K_{1,q_j}(j=1,2,\cdots,t)$ whose edges are in colour j too. Hence in this case, the theorem is true. □

When $t=1$, we have a stronger result.

THEOREM 2. *$R(T_p,K_{1,q}) \ge p+q-\theta$, where $\theta = \min\{I_1 \cup I_2\}$ with $I_1 = \{s_1 | q-s_1 = m_1(p-u_1),\ u_1 \le s_1,\ s_1,u_1,m_1 \in \mathbb{N}^+\}$, $I_2 = \{u_2 | q-s_2 = m_2(p-u_2),\ u_2 > s_2,\ s_2,u_2,m_2 \in \mathbb{N}^+\}$ and $p \ge 2$.*

PROOF. Clearly, there always exists θ satisfying the condition of the theorem.

Case I. $\theta = \min\{I_1\}$, i.e. there is $s_1 = \theta$ such that

$$p+q-\theta-1 = p+q-s_1-1 = m_1(p-u_1)+p-1;$$

$$(m_1-1)(p-u_1)+p-1 = m_1(p-u_1)+u_1-1 = q-(s_1-u_1)-1 \le q-1$$

and $m_1(p-u_1) = q-s_1 \le q-1$.

Let $G_1 = K_{m_1(p-u_1)+p-1}$. We divide $V(G_1)$ into m_1+1 parts V_i $(i=1,2,\ldots,m_1+1)$ such that $|V_i| = p-u_1$ $(i=1,2,\ldots,m_1)$ and $|V_{m_1+1}| = p-1$. Thus there is an assignment of the 1st colour to all $G_1[V_i]$ $(i=1,2,\ldots,m_1+1)$ and of the 2nd colour to the remaining edges of G_1. Clearly, there is no monochromatic T_p whose edges are in colour 1 and $K_{1,q}$ whose edges are in colour 2. Hence $R(T_p,K_{1,q}) \ge m_1(p-u_1)+p = p+q-\theta$.

Case II. $\theta = \min\{I_2\}$, i.e. there is $u_2 = \theta$ such that

$$p+q-\theta-1 = (q-s_2)+p-u_2+s_2-1 = m_2(p-u_2)+p-(u_2-s_2)-1;$$

$$p-(u_2-s_2)-1 < p-1;$$

$$(m_2-1)(p-u_2)+p-(u_2-s_2)-1 = m_2(p-u_2)+s_2-1 = q-1$$

and $m_2(p-u_2) = q-s_2 \le q-1$.

Let $G_2 = K_{m_2(p-u_2)+p-(u_2-s_2)-1}$. We divide $V(G_2)$ into m_2+1 parts V_i $(i=1,2,\ldots,m_2+1)$ such that $|V_i| = p-u_2$ $(i=1,2,\cdots,m_2)$ and $|V_{m_2+1}| = p-(u_2-s_2)-1$. A similar argument as in case I yields that $R(T_p,K_{1,q}) \ge m_2(p-u_2)+p-(u_2-s_2) = p+q-\theta$. □

3. The Upper Bounds

THEOREM 3. *If $p > 1$ and $\Sigma = \sum_{i=1}^{t}(q_i - 1)$, then $R(T_p, K_{1,q}, \cdots, K_{1,q_t}) \le p + \Sigma$.*

PROOF. Suppose that there is an assignment of $t+1$ colours, $1, 2, \cdots, t+1$, to the edges of $K_{p+\Sigma-1}$. If there is no monochromatic T_p whose edges are in colour $t+1$, then by Theorem B(ii) there must be a $K_{1,\Sigma+1}$ whose edges are in the former t colours. Hence there must be a monochromatic K_{1,q_i} for some $i \in \{1, 2, \cdots, t\}$ whose edges are in colour i. The proof is completed. □

THEOREM 4. *If $p(\ge 3)$ is odd and q is even, then $R(T_p, K_{1,q}) \le p + q - 2$.*

PROOF. Let $G = K_{p-q-2}$. Suppose that there is an assignment of 2 colours, 1, 2, to the edges of G. Let G' be the edge-deduced subgraph whose edge set is the set of all colour 1 edges in G. If there is no monochromatic $K_{1,q}$ whose edges are in colour 2, then the minimum degree $\delta(G') \ge p - 2$. The maximum degree is $\Delta(G') \ge p - 1$. Note that $\Delta(T_p) \le p - 1$. Using $\delta(G') \ge p - 2$ and $\Delta(G') \ge p - 1$, it is easy to check that T_p as a subgraph is contained in G'. Therefore the proof is completed. □

THEOREM 5. *If $p(\ge 3)$ and $\Sigma = \sum_{i=1}^{t}(q_i - 1)$ are odd, then*

$$R(T_p, K_{1,q_1}, \cdots, K_{1,q_t}) \le p + \Sigma - 1.$$

PROOF. Suppose that that there is an assignment of $t+1$ colours, $1, 2, \cdots, t+1$, to the edges of $K_{p+\Sigma-1}$. If there is no monochromatic T_p whose edges are in colour $t+1$, then by theorem 4, there must be a $K_{1,\Sigma+1}$ whose edges are in the former t colours. Hence there must exist a monochromatic K_{1,q_i} for some $i \in \{1, 2, \cdots, t\}$ whose edges are in colour i. The proof is completed. □

4. Some Ramsey Numbers in Special Cases

By Theorems 1, 3, and 5, we have:

THEOREM 6. *Let $\Sigma = \sum_{i=1}^{t}(q_i - 1)$.*

1. *If $p > 1$, $\Sigma \equiv 0 \pmod{p-1}$ and $p + \Sigma$ is odd, then $R(T_p, K_{1,q_1}, \cdots, K_{1,q_t}) = p + \Sigma$;*
2. *If $p > 1$, $\Sigma \equiv 1 \pmod{p-1}$ or $\Sigma = 0 \pmod{p-2}$ and $p + \Sigma$ is even, then $R(T_p, K_{1,q_t}) = p + \Sigma - 1$ or $p + \Sigma$;*
3. *If $p > 1$, $\Sigma \equiv 1 \pmod{p-2}$ and only one of $\{q_1, q_2, \cdots, q_t\}$ is even and p is odd, then $R(T_p, K_{1,q_1}, \cdots, K_{1,q_t}) = p + \Sigma - 2$ or $p + \Sigma - 1$.*

By Theorem B(ii) and Theorem 2, we have:

THEOREM 7. 1. *If* $p(>1)$ *and* q *satisfy one of the following conditions: (i)* $q \equiv 1$ *or 2* $\pmod{p-2}$, *(ii)* $q \equiv 2 \pmod{p-1}$, *then* $R(T_p, K_{1,q}) = p+q-2$ *or* $p+q-1$.

2. *If* $p(>1)$ *and* q *satisfy one of the following conditions: (i)* $q \equiv 1$ *or 2 or 3* $\pmod{p-3}$, *(ii)* $q \equiv 3 \pmod{p-2}$, *(iii)* $q \equiv 3 \pmod{p-1}$, *then* $R(T_p, K_{1,q}) = p+q-3$ *or* $p+q-2$ *or* $p+q-1$.

By Theorem 4 and 7, we have:

COROLLARY. *Let* $p(>1)$ *be odd and* q *even.*

1. *If* p *and* q *satisfy one of the following conditions: (i)* $q \equiv 1$ *or* 2 $\pmod{p-2}$, *(ii)* $q \equiv 2 \pmod{p-1}$, *then* $R(T_p, K_{1,q}) = p+q-2$;
2. *If* p *and* q *satisfy one of the following conditions: (i)* $q \equiv 2 \pmod{p-3}$, *(ii)* $q \equiv 3 \pmod{p-2}$, *then* $R(T_p, K_{1,q}) = p+q-3$ *or* $p+q-2$.

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