A New Idea For Hamiltonian Problem*

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ABSTRACT. Let G be a 2-connected graph of order n with the connectivity k and the independence number α. In this paper, we show that if for each independent set S with $|S| = k+1$, there are $u, v \in S$ such that satisfying one of the following conditions:

(a) $d(u) + d(v) \geq n$; or $|N(u) \cap N(v)| \geq \alpha$; or $|N(u) \cup N(v)| \geq n - k$;

(b) for any $x \in \{u, v\}$, $y \in V(G)$ and $d(x, y) = 2$, it implies that $\max\{d(x), d(y)\} \geq n/2$,

then G is hamiltonian. This result reveals the internal relations among several well-known sufficient conditions: (1) it shows that it does not need to consider all pair of nonadjacent or distance two vertices in G; (2) it makes known that for the different pair of vertices in G, it permits to satisfy the different conditions.

1 Introduction

This paper uses terms and notations of [2]. Throughout this paper $G = (V, E)$ denotes an undirected connected simple graph of order n (≥ 3) with the connectivity $\kappa(G) = k$ and the independence number $\alpha(G) = \alpha$. Let $L \subseteq V(G)$, $F \subseteq G$ and v be any vertex in G. Define $N_L(V) = \{u | u \in L$ and $uv \in (G)\}$ and $N_L(F) = \cup_{v \in F} N_L(V)$. If no ambiguity can arise, we

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write F instead of $V(F)$ etc. . The addition of subscript is taken modulo t.

So far, there are many sufficient conditions for the existence of Hamilton graphs such as:

C1: For every pair of nonadjacent vertices u and v, $d(u) + d(v) \geq n$ (Ore 1960 [7]).

C2: For $k \geq 2$ and any u, $v \in V$ with $d(u,v) = 2$, $\max\{d(u), d(v)\} \geq n/2$ (Fan 1984 [4]).

C3: $\alpha \leq \kappa$ (Chvátal & Erdös 1972 [3]).

C4: For every pair of nonadjacent vertices u and v, $|N(u) \cap N(v)| \geq (2n-1)/3$ (Fandree et al. 1989 [5]).

C5: For every pair of nonadjacent vertices u and v, $|N(u) \cap N(v)| \geq \alpha$ (Song & Qin 1991 [8]).

C6: The condition of the Theorem of this paper.

The relation of these conditions is described in Figure 1.

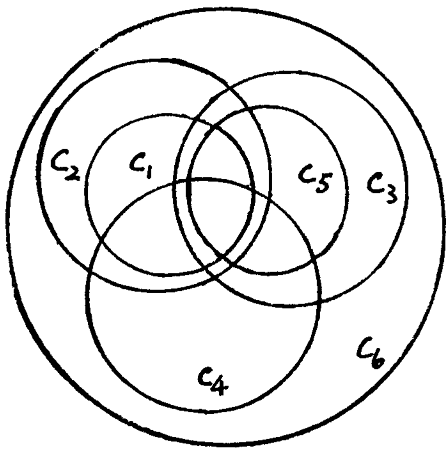


Figure 1

In this paper, the principal purpose is to reveal the internal relations among them. For this reason, we lead to an idea of "or", and turn to check nonadjacent vertices satisfying the condition as less as possible. In fact, in our result, it is enough to check a pair of vertices satisfying one of several conditions for each $k+1$ independent set in G. We think that this train of thought and the idea of "or" are very useful for the research of Hamilton problem.

2 Main Results

Theorem. *Let G be a 2-connected simple graph of order n with the connectivity k and the independence number α. If for any independent set S with $|S| = k+1$, we have*

(a) $\exists t \leq k$ such that $\forall X \subset S$ with $|X| = t$, $|N(X)| \geq \min\{\frac{t(n-1)+1}{t+1}, n-\delta\}$ or

(b) $\exists u, v$ in S with $d(u) + d(v) \geq n$ or

(c) $\exists u, v$ in S with $|N(u) \cap N(v)| \geq \alpha$ or

(d) $\exists u, v$ in S with $|N(u) \cup N(v)| \geq n - k$ or

(e) $\exists u, v$ in S such that $\forall x \in \{u, v\}$ and $\forall y \in V$, $d(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq n/2$,

then G is hamiltonian.

By the Theorem, it is easy to deduce a lot of new sufficient conditions of hamiltonian problem. And as corollaries, many well-known results, follow naturally.

Corollary 1. *Let G be a connected simple graph. If for any independent set S with $|S| = k+1$, there is a $t \leq k$ such that for each t subset X in S, $|N(X)| \geq \min\left\{\frac{t(n-1)+1}{t+1}, n-\delta\right\}$, then G is hamiltonian.*

Comparing Corollary 1 with the Theorem in [7], first it permits to take different t for different S in Corollary 1. And then note that $\frac{t(n-1)+1}{t+1} > n - \delta$, when $t \geq 2$ and for a sufficient large δ. Hence the latter is improved by the former.

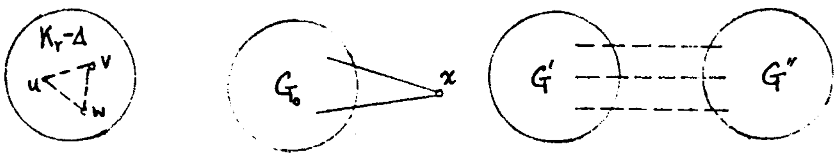
Corollary 2. *Let G be a 2-connected simple graph. If for any independent set S with $|S| = k+1$, there are $u, v \in S$ satisfying one of the following conditions:*

(a) $d(u) + d(v) \geq n$ or $|N(u) \cap N(v)| \geq \alpha$ or $|N(u) \cup N(v)| \geq n - k$;

(b) for any $x \in \{u, v\}$ and $y \in V(G)$, $d(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq n/2$, then G is hamiltonian.

This Corollary shows that it doesn't need to check every pair of nonadjacent or distance two vertices in G satisfying a same specific condition as traditional known results. It is enough to check only one pair of vertices for each $k+1$ independent set S, and it permits different pair of vertices in the different $k+1$ independent set satisfying a different specific condition. So those traditional known results are substantially generalized by Corollary 2.

Now, we use the following example to illustrate Corollary 2. Let G_1, denote $K_r - \{uv, vw, wu\}$, G_2 denote $G_0 \vee \{x\}$, and let G', G'' be graphs of order n and G_3 denote $G' \vee G'' \backslash M$, where M is a perfect matching between $V(G')$ and $V(G'')$ (see Figure 2).



G_1 G_2 G_3

Figure 2

We construct a graph G with order $n \geq 22$ (see Figure 3) as follows:

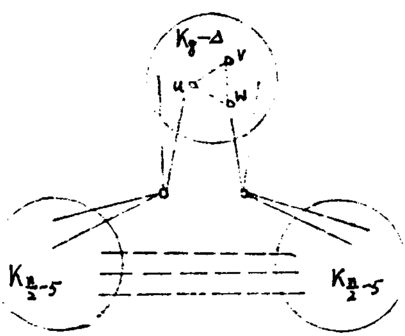


Figure 3

Clearly, any one of the conditions from C1 to C5 doesn't justify that G is hamiltonian, but G satisfies the condition of Corollary 2. In fact $\kappa(G) = k = 2$, $\alpha(G) = \alpha = 5$. Any S, which contains at least two vertices of $\{u, v, w\}$, contains two vertices x and y with $|N(x) \cap N(y)| \geq \alpha$. If S contains at most one of $\{u, v, w\}$, then it contains vertices l and m, where l and m are in the different sets $K_{n/2-5}$. Since $d(l) = d(m) = n - 11 \geq \frac{n}{2}$, the condition "for any $x \in \{l, m\}$ and $y \in V(G)$, it implies $\max\{d(x), d(y)\} \geq n/2$, if $d(x, y) = 2$" follows. So by Corollary 2, G is hamiltonian.

Corollary 2.1. *Let G be a 2-connected simple graph with connectivity k and independence number α. If for each independent set S with $|S| = k+1$, there are $u, v \in S$ satisfying the following conditions:*

$$d(u) + d(v) \geq n \text{ or } |N(u) \cap N(v)| \geq \alpha \text{ or } |N(u) \cup N(v)| \geq n - k$$

then G is hamiltonian.

Corollary 2.1.1. *Let G be a 2-connected simple graph with connectivity k. If for each independent set S with $|S| = k+1$, there are $u, v \in S$ such that $d(u) + d(v) \geq n$, then G is hamiltonian.*

Clearly, Dirac's Theorem and Ore's Theorem is a corollary of the Corollary.

Corollary 2.1.2. *Let G be a 2-connected simple graph with independence number α. If for each independent set S with $|S| = k+1$, there are $u, v \in S$ such that $|N(u) \cap N(v)| \geq \alpha$, then G is hamiltonian.*

Corollary 2.1.3. *Let G be a 2-connected simple graph with connectivity k. If for each independent set S with $|S| = k+1$, there are $u, v \in S$ such that $|N(u) \cup N(v)| \geq n - k$, then G is hamiltonian.*

Corollary 2.2. *Let G be a 2-connected simple graph with connectivity k. If for each independent set S with $|S| = k+1$, there are $u, v \in S$ such that for any $x \in \{u, v\}$ and $y \in V(G)$, $d(x, y) = 2$ implies that* $\max\{d(x), d(y)\} \geq n/2$, *then G is hamiltonian.*

Clearly, Fan's result [4] is generalized by Corollary 2.2.

Corollary 2.3. *Let G be a 2-connected simple graph with independence number α. If for each pair of u, v with $d(u, v) = 2$,* $\max\{d(u), d(v)\} \geq n/2$ *or $|N(u) \cap N(v)| \geq \alpha$, then G is hamiltonian.*

Corollary 3. *Let G be a 2-connected simple graph. If for each pair of nonadjacent vertices u, v, $d(u) + d(v) \geq n$ or $|N(u) \cap N(v)| \geq \alpha$ or $|N(u) \cup N(v)| \geq \min\{\frac{2n-1}{3}, n - \delta\}$, then G is hamiltonian.*

Corollary 3.1. *Let G be a 2-connected simple graph. If for each pair of nonadjacent vertices u, v, $|N(u) \cup N(v)| \geq \min\{\frac{2n-1}{3}, n - \delta\}$, then G is hamiltonian.*

Theorem 2 in [5] is generalized by this Corollary.

Corollary 4 [3]. *If $\alpha \leq k$, then G is hamiltonian.*

Proof: By the definition of α and $\alpha \leq k$, there is no independent set S with $|S| = k+1$ in G. Hence the condition of the Theorem is automatically satisfied. So G is hamiltonian. $\square$

3 The proof of the Theorem

First of all, we prove the following Lemma.

Lemma. *Let G be a nonhamiltonian graph, and let $C = v_1v_2 \ldots v_hv_1$ be a longest cycle containing a specific vertex set A in G. Let B be a component* *of* $G - V(C)$,

$$N_C(B) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}, N^- = \{v_{i_1-1}, v_{i_2-1}, \ldots, v_{i_m-1}\},$$
$$N^+ = \{v_{i_1+1}, v_{i_2+1}, \ldots, v_{i_m+1}\}$$

and let w *be any vertex in* B, *then*

(a) $N^- \cup \{w\}$ *and* $N^+ \cup \{w\}$ *are independent sets; and*

(b) *For any* $u, v \in N^- \cup \{w\}$ *or* $N^+ \cup \{w\}$,

$$d(u) + d(v) < n, |N(u) \cap N(v)| < \alpha \text{ and } |N(u) \cup N(v)| < n - k.$$

Proof:

(a) Clearly, w and any vertex in N^- are nonadjacent. If there are $v_{i_s-1}, v_{i_t-1} \in N^-$ and $v_{i_s-1}v_{i_t-1} \in E(G)$, then there is a cycle:

$$v_{i_s-1}v_{i_t-1}v_{i_t-2}\ldots v_{i_s}P_B v_{i_t}v_{i_t+1}\ldots v_{i_s-1}$$

containing A, which is longer than C, where P_B is a path in B and the end vertices of P_B are adjacent to v_{i_s}, v_{i_t} respectively, a contradiction. Hence $N^- \cup \{w\}$ is an independent set in G. Similarly, so is $N^+ \cup \{w\}$.

(b) If it is not true, there are $u, v \in N^- \cup \{w\}$ or $N^+ \cup \{w\}$, say $N^- \cup \{w\}$, such that $d(u) + d(v) \geq n$ or $|N(u) \cap N(v)| \geq \alpha$ or $|N(u) \cup N(v)| \geq n - k$. Clearly, $N(u) \cap N(v) \subseteq V(C)$, since C is a longest cycle in G. In the following we always assume that $N(u) \cap N(v) \subseteq V(C)$.

Case 1: $d(u) + d(v) \geq n$.

If $u = w$ or $v = w$, say $u = w$, then $v = v_j$, thus $v_{j+1} \in N_C(B)$. Let $P = v_{j+1}\ldots w$ be a path with $V(P) \cap V(C) = \{v_{j+1}\}$ in G. If there is $v_i \in V(C)$ such that $v_jv_i, wv_{i+1} \in E(G)$, then there is a cycle containing A in G as follows: $v_jv_iv_{i-1}\ldots v_{j+2}Pv_{i+1}v_{i+2}\ldots v_j$, which is longer than C, this is a contradiction. So $d_C(v_j) + d_C(w) \leq h$. Hence $d(u) + d(v) < n$, a contradiction.

If $u, v \in N^-$, let $u = v_i$, $v = v_j$; with $i < j$. Thus, by the definition of N^-, there is a path P' from v_{i+1} to v_{j+1} via B. If there is $v_p \in V(C)$ such that $v_iv_p, v_jv_{p+1} \in E(G)$ and $j < p < i$, thus there is a cycle containing A in G as follows:

$$v_iv_pv_{p-1}\ldots v_{j+2}P'v_{i+2}\ldots v_jv_{p+1}v_{p+2}\ldots v_i$$

which is longer than C. If there is $v_p \in V(C)$ such that $v_iv_p, v_jv_{p-1} \in E(G)$ and $i < p < j$, thus there is a cycle containing A in G as follows:

$$v_iv_pv_{p+1}\ldots v_jv_{p-1}v_{p-2}\ldots v_{i+2}P'v_{j+2}\ldots v_i$$

which is longer than C. So $dc(v_i) + dc(v_j) \leq h$. Hence $d(u) + d(v) < n$, a contradiction.

Case 2: $|N(u) \cap N(v)| \geq \alpha$

If $w \in \{u, v\}$, then $N(u) \cap N(v) \subseteq N_C(B)$. Hence

$$|N^- \cup \{w\}| = |N_C(B)| + 1 \geq |N(u) \cap N(v)| + 1 \geq \alpha + 1.$$

By (a), this is a contradiction.

If $u, v \in N^-$, let $u = v_i$, $v = v_j$ with $i < j$ and take paths P and P' as in case 1. Let

$$N(u) \cap N(v) = \{v_{r_1}, v_{r_2}, \ldots, v_{r_p}\}.$$

By the hypothesis of this case, $p \geq \alpha$. Let $S = \{v_{r_1+1}, v_{r_2+1}, \ldots, v_{r_p+1}\}$. Clearly w a the vertex v_{r_k+1} of S are nonadjacent, otherwise there is a cycle containing A as follows:

$$v_j v_{r_k} v_{r_k-1} \ldots v_{j+2} P v_{r_k+1} v_{r_k+2} \ldots v_j,$$

which is longer than C, a contradiction. If there are $v_s, v_q \in S$ with $s < q$ such that $v_s, v_q \in E(G)$, three subcases must be considered: (1) $i < s < q < j$; (2) $j < s < q < i$; (3) $i < s < j < q < i$. For all of subcases, there are cycles containing A as follows:

$$v_i v_{q-1} v_{q-2} \ldots v_s v_q v_{q+1} \ldots v_j v_{s-1} v_{s-2} \ldots v_{i+2} P' v_{j+2} v_{j+3} \ldots v_i;$$
$$v_i v_{s-1} v_{s-2} \ldots v_{j+2} P' v_{i+2} v_{i+3} \ldots v_j v_{q-1} v_{q-2} \ldots v_s v_q v_{q+1} \ldots v_i;$$

and

$$v_i v_{q-1} v_{q-2} \ldots v_{j+2} P' v_{i+2} v_{i+3} \ldots v_{s-1} v_j v_{j-1} \ldots v_s v_q v_{q+1} \ldots v_i,$$

respectively. All of these cycles are longer than C, a contradiction. Hence $S \cup \{w\}$ is an independent set in G. Note that $|S \cup \{w\}| = p + 1 \geq \alpha + 1$. This is a contradiction.

Case 3: $|N(u) \cup N(v)| \geq n - k$.

By (a), $N^- \cup \{w\}$ is an independent set. Note that $|N^- \cup \{w\}| = |N_C(B)| + 1 = m + 1$ and $m \geq k$. So it is impossible for this case.

Up to now, three subcases as above are not true. So the proof of (b) is completed. □

Now we turn on to prove the Theorem.

Let $A = \{u | u \in V(G)$ and $d_G(u) \geq \frac{n}{2}\}$ and $E' = \{xy | x, y \in A, xy \notin E(G)\}$. Let $H = G + E'$, then $H[A]$ is complete. By Lemma 4.4.2 in [2], H is Hamiltonian iff G is Hamiltonian. If H is not a Hamilton graph, let B be a component of $H - V(C)$, where C is a longest cycle containing A in H. let $C = v_1 v_2 \ldots v_t v_1$ and

$$N_C(B) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$$

where $i_1 < i_2 < \cdots < i_m$, $m \geq k$, since G is k-connected. Let w and v_{i_1} be adjacent in H, where $w \in B$. And let $N^- = \{v_{i_1-1}, v_{i_2-1}, \ldots, v_{i_k-1}\}$ and $S = N^- \cup \{w\}$. Hence by (a) of the Lemma, S is a $k+1$ independent set in H, thus S is also an independent set in G.

Case 1: There is $t \leq k$ such that for any t subset $X \subseteq S$,

$$|N_G(X)| \geq \min\left\{\frac{t(n-1)+1}{t+1}, n - \delta(G)\right\}.$$

Clearly, we also have $|N_H(X)| \geq \min\left\{\frac{t(n-1)+1}{t+1}, n - \delta(H)\right\}$. If for any t subset $X \subseteq S$, $|N_H(X)| \geq \frac{t(n-1)+1}{t+1}$, then using an analogous proof of the Theorem of [6], we can get a t subset X_1 in S such that $|N_H(X_1)| < \frac{t(n-1)+1}{t+1}$, a contradiction; If for any t subset $X \subseteq S$, $|N_H(X)| \geq n - \delta(H)$. Note that $m + |B| > \delta(H)$, consider subset $X_2 \subseteq N^-$ with $|X| = t$. By (a) of the Lemma, we have $|N_H(X_2)| \leq n - |N^-| - |B| \leq n - m - |B|$, a contradiction.

Case 2: There are $u, v \in S$ such that $d(u) + d(v) \geq n$ or $|N(u) \cap N(v)| \geq \alpha$ or $N(u) \cup N(v)| \geq n - k$.

It is clear by (b) of the Lemma.

Case 3: For any $x \in \{u, v\}$ and $y \in V(G)$, $d(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq n/2$.

Since $G \subseteq H$ and $V(G) = V(H)$, from the choice of C, $d_G(b) < n/2$ if $b \in B$. We consider $x \in \{u, v\}$. If $x \in A$, then $d_G(x) \geq n/2$. If $x \notin A$ and $x = v_{i_j-1} \in N^-$, thus there is a $b \in B$ such that $d_H(x, b) = 2$. Clearly $bv_{i_j} \in E(G)$. Since $x = v_{i_j-1} \notin, A$, $v_{i_j-1}v_{i_j} \in E(G)$. Thus $d_G(x, b) = 2$. Hence $d_G(x) \geq n/2$. This is a contradiction. So, if $u, v \in N^-$, then $d_G(u) + d_G(v) \geq n$. By (b) of the Lemma, this is a contradiction. Hence, in the following, we always assume $u \in N^-$ and $v = w \in B$.

If $v_{i_1-1} \notin A$, then $v_{i_1-1}v_{i_1} \in E(G)$ and $d_G(w, v_{i_1-1}) = 2$. By the condition (b) of the Theorem, we have $d_G(v_{i_1-1}) \geq \frac{n}{2}$, a contradiction. Hence $v_{i_1-1} \in A$, i.e. $d_G(v_{i_1-1}) \geq \frac{n}{2}$.

If $u \neq v_{i_1-1}$, then $d_G(u) + d_G(v_{i_1-1}) \geq n$. By (b) of the Lemma, this is a contradiction. So it must be $u = v_{i_1-1}$. Let w' and v_{i_2}; be adjacent in H, where $w' \in B$, and let $S' = N^- \cup \{w'\}$. Without loss of generality, we can assume that S and S' have same property. Using an analogous discussion as above mentioned, we have $d_G(v_{i_2-1}) \geq n/2$. Hence

$$d_G(v_{i_1-1}) + d_G(v_{i_2-1}) \geq n.$$

Again by (b) of the Lemma, this is a contradiction.

Combining cases 1, 2 and 3, we have that H is hamiltonian. Hence G is hamiltonian. The proof is completed. $\square$

Note: Using an analogous condition and method of proof of the Theorem, we can get an analogous theorem about Hamilton-connected graphs.

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