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A NOTE ON 4-REGULAR 4-CHROMATIC GRAPHS WITH GIRTH 4†

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Abstract

A graph is a (k, k, g)-graph if it is k-regular, k-chromatic and all cycles have length at least g. In this note we prove that a (4, 4, 4)-graph of order n exists if and only if $n \geq 12$.

A graph is a (k, k, g)-graph if it is k-regular, k-chromatic and it has *girth* at least g (that is, all cycles have length at least g). Grünbaum [1] conjectured that (k, k, g)-graphs exist for all $k \geq 2$ and $g \geq 4$. For $k \geq 7$ and $g \geq 4$ the conjecture was disproved by Borodin and Kostochka [2], Catlin [3] and Lawrence [4]. For $k = 5$ and $g \geq 35$ the conjecture was disproved by Kostochka [5] and he also disproved it for $k = 6$ and g large enough. Moreover, we note that this is trivial for $g = 3$ and for $k = 2, 3$ the validity of the conjecture follows from the existence of the cages. Thus the major unsolved case is $k = 4$. In this case, only two such graphs are known (see [6]): a (4, 4, 4)-graph of order 12 and a (4, 4, 5)-graph of order 25. In this note we show that there are (4, 4, 4)-graphs of all orders greater than or equal to 12.

Lemma 1 [7]: Let G be a (4, 4, 4)-graph of order n. If $n \leq 12$, then G is isomorphic to the Chvátal graph (see Figure 1). ■

Lemma 2 [8]: There is a (4, 4, 4)-graph of order 15 (see Figure 2). ■

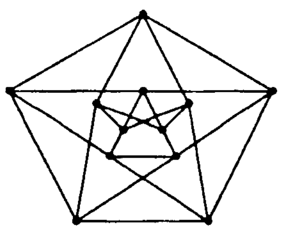

Figure 1: The Chvátal graph.

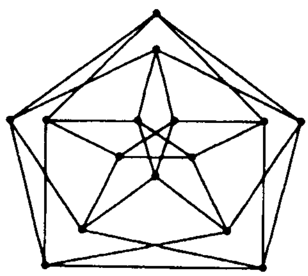

Figure 2: A (4, 4, 4)-graph of order 15.

Lemma 3 [9]: There are (4, 4, 4)-graphs of orders at least 20 (see Figure 3). ■

Lemma 4: There are (4, 4, 4)-graphs of orders 13, 14, 16, 17, 18 and 19.

Proof: Construct the (4, 4, 4)-graphs of orders 13, 14, 16, 17, 18 and 19 as shown in Figure 4. ■

Theorem 5: A (4, 4, 4)-graph of order n exists if and only if $n \geq 12$.

Proof: The result follows from Lemmas 1–4. ■

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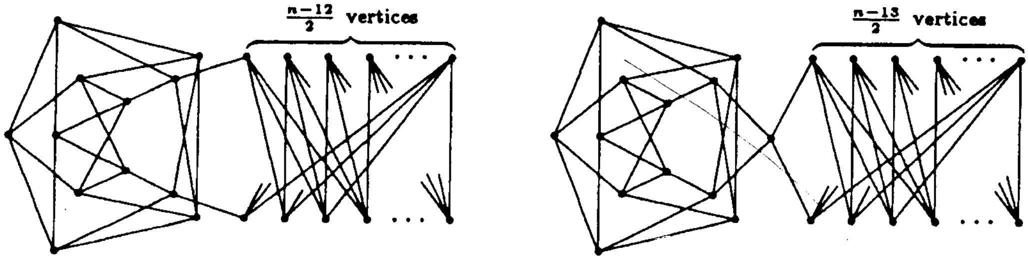
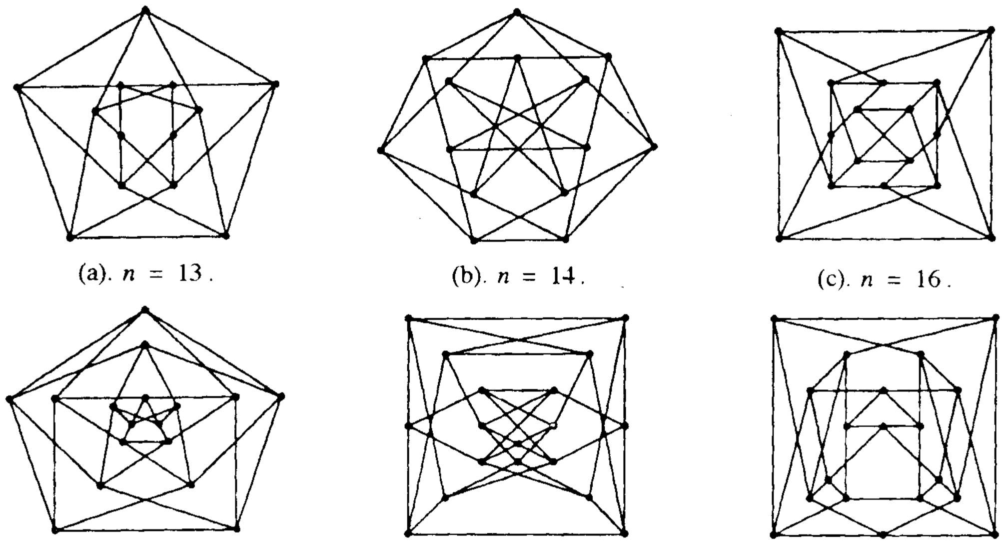


(a). n is odd. (b). n is even.

Figure 3: $(4,4,4)$-graphs of order $n \geq 20$.



(a). $n = 13$. (b). $n = 14$. (c). $n = 16$.

(d). $n = 17$. (e). $n = 18$. (f). $n = 19$.

Figure 4: Six $(4,4,4)$-graphs.

Remark: Let $f(4,4,g)$ denote the smallest number of vertices of a $(4,4,g)$-graph. Lemma 1 implies $f(4,4,g) = 12$. What is $f(4,4,g)$ for $g \geq 5$? We guess that $f(4,4,5) = 25$. Moreover, for any $n \geq f(4,4,g)$ with $g \geq 5$, does a $(4,4,g)$-graph of order n exist? All of these problems are still open.

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