NORTH-HOLLAND

The Upper Bounds of the Generalized Maximum Density Index of Irreducible Boolean Matrices*

Chen Xi and Zhang Ke Min
Department of Mathematics
Nanjing University
Nanjing, 210093, P.R. of China

Submitted by Richard A. Brualdi

ABSTRACT

Let IBM(n, p) denote the set of all $n \times n$ irreducible Boolean matrices with period p. This paper generalizes the concept of the index of maximum density of A, where $A \in$ IBM(n, p) with $p > 1$, and obtains upper bounds on the generalized maximum density index of IBM(n, p). © Elsevier Science Inc., 1997

1. INTRODUCTION

The maximum density index of a power sequence of $n \times n$ irreducible Boolean matrices with period p is an important combinatorial parameter. We consider such a memoryless communication system associated with a network D. Every vertex of D can have several different bits of information simultaneously. Let t denote the time. When $t = 0$, there are k bits of information distributing on k vertices of D respectively. When $t = 1$, every vertex transfers its information to its heads and loses its original information for the same time. The system operates in that way. If D is strong, the problems are: how much information can it store, and when will it attain this maximum?

Based on the above mathematical model, we generalize the concept of maximum density.

*This project was supported by NSFC.

LINEAR ALGEBRA AND ITS APPLICATIONS 256:75–93 (1997)

0024-3795/97/$17.00
PII S0024-3795(96)00169-3

If A is an $n \times n$ Boolean matrix, in the power sequence $A, A^2, A^3, \ldots,$ let $\mu_A^j(i)$ denote the number of 1's in the ith row of A^j. If $X \subset \{1, 2, \ldots, n\}$, we define:

DEFINITION 1. $\mu_A^j(X) := \sum_{i \in X} \mu_A^j(i)$.

DEFINITION 2. The generalized maximum density on X of A:

$$\mu_A(X) := \max_{j \in Z^+} \left\{ \mu_A^j(X) \right\}.$$

DEFINITION 3. The generalized index of maximum density on X of A:

$$h_A(X) := \min\left\{ m : m \in Z^+,\ \mu_A^m(X) = \mu_A(X) \right\}.$$

DEFINITION 4. The k-generalized maximum density of A:

$$\mu_A(k) := \max_{|X|=k} \left\{ \mu_A(X) \right\}.$$

DEFINITION 5. The k-generalized index of maximum density of A:

$$h_A(k) := \min\left\{ m : m \in Z^+, \text{ and there exists } X \subset \{1, 2, \ldots, n\} \text{ with } |X| = k \text{ such that } \mu_A^m(X) = \mu_A(k) \right\}.$$

DEFINITION 6. $h_A(n, p, k) := \max\{h_A(k) : A \in \mathrm{IBM}(n, p)\}$.

For undefined terminology, the reader is referred to [2].

If a Boolean matrix A is primitive, we have $\mu_A(X) = kn$, where $n = |X|$. So the problems are solved if $p = 1$.

Let $h(A) := h_A(n)$ denote the maximum density index of A, and let $h(n, p) := \max\{h(A) : A \in \mathrm{IBM}(n, p)\}$, where $n = rp + s$, $r = [n/p]$. In 1988, Shao Jiayu and Li Qiao [4] obtained the following results:

$$h(n, p) = \begin{cases} p(r^2 - 2r + 2), & r > 1, \quad s = 0, \\ p(r^2 - 2r + 3), & r > 1, \quad 0 < s < p, \\ p, & r = 1, \quad 0 < s < p, \\ 1, & r = 1, \quad s = 0. \end{cases}$$

THEOREM 1.1. *If $A \in \mathrm{IBM}(n, p)$, then there exists a permutation matrix Q such that A has the normal form*

$$A_Q = Q^T A Q = \begin{bmatrix} 0 & A_1 & & & \\ & 0 & A_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & A_{p-1} \\ A_p & 0 & & & 0 \end{bmatrix},$$

where A_k $(k = 1, 2, \ldots, p-1)$ is an $n_k \times n_{k+1}$ matrix, so that the diagonal blocks are square. Furthermore, we have $n_1 + n_2 + \cdots + n_p = n$.

THEOREM 1.2 [3]. *If $A \in \mathrm{IBM}(n, p)$, then for any integer $\zeta \geq 0$, $A_Q^{\zeta p}$ has the form*

$$A_Q^{\zeta p} = \begin{bmatrix} C_1^{\zeta} & & & 0 \\ & C_2^{\zeta} & & \\ & & \ddots & \\ 0 & & & C_p^{\zeta} \end{bmatrix},$$

where C_k $(k = 1, 2, \ldots, p)$ is an $n_k \times n_k$ primitive matrix, and the exponents of the C_k differ by at most unity.

Let $D(A)$ denote the digraph associated with a Boolean matrix A. By these two theorems, we can see that for every $A \in \mathrm{IBM}(n, p)$, the digraph $D(A)$ is a p-partite digraph with the partition $(V_1, V_2, \ldots, V_p)$ and $|V_i| = n_i$. And it is easy to see that from each vertex of V_i one can reach every vertex of V_j of $D(A)$ by walks of the same and sufficient length. We have

$$\mu_A(1) = \max_{i=1,\ldots,p} \{n_i\}.$$

If $A \in \mathrm{IBM}(n, p)$, let A_Q be the normal form of A in Theorem 1.1. Of the n_i $(i = 1, 2, \ldots, p)$, some may be equal. We denote by $\{\eta_1, \eta_2, \ldots, \eta_m\}$ with $\eta_1 > \eta_2 > \cdots > \eta_m$ the set of the multiset $\{n_1, n_2, \ldots, n_p\}$. In the partition of $D(A)$, there are x_i subsets with η_i vertices $(i = 1, 2, \ldots, p)$, $x_1 + x_2 + \cdots + x_m = p$. Thus we have:

THEOREM 1.3. *If* $A \in \mathrm{IBM}(n, p)$, *then*

$$\mu_A(k) = \begin{cases} k\eta_1, & 1 \leqslant k \leqslant x_1\eta_1, \\ x_1\eta_1^2 + (k - x_1\eta_1)\eta_2, & x_1\eta_1 + 1 \leqslant k \leqslant x_1\eta_1 + x_2\eta_2, \\ \sum_{j=1}^{i} x_j\eta_j^2 + \left(k - \sum_{j=1}^{i} x_j\eta_j\right)\eta_{j+1}, & \sum_{j=1}^{i} x_j\eta_j + 1 \leqslant k \leqslant \sum_{j=1}^{i+1} x_j\eta_j, \\ \sum_{j=1}^{m-1} x_j\eta_j^2 + \left(k - \sum_{j=1}^{m-1} x_j\eta_j\right)\eta_m, & \sum_{j=1}^{m-1} x_j\eta_j^2 + 1 \leqslant k \leqslant n. \end{cases}$$

In particular, $\mu_A(n) = \sum_{j=1}^{m} x_j\eta_j^2 = \sum_{j=1}^{p} \eta_j^2$.

Therefore, the problem of the k-generalized maximum density of A is completely solved. Now, we state the main theorem as follows:

THEOREM. *For any* $p > 1$, *we have that*

for $n = p$,

$$h(n, p, k) = 1 \qquad 1 \leqslant k \leqslant n;$$

for $n = 2p$,

$$h(n, p, k) = \begin{cases} p - 1, & k = 1, \\ p, & 2 \leqslant k \leqslant p, \\ k, & p + 1 \leqslant k \leqslant n; \end{cases}$$

for $n = 3p$,

$$h(n, p, k) = \begin{cases} 3p, & 3 \leqslant k \leqslant p, \\ 2p + k, & \text{otherwise}; \end{cases}$$

for $n = rp$ *and* $r \geqslant 4$,

$$h(n, p, k) = (r^2 - 3r + 2)p + k;$$

for $n = p + s$ *and* $1 \leqslant s \leqslant p - 1$,

$$h(n, p, k) = \begin{cases} \max\{1, s - 1\}, & k = 1, \\ p, & 2 \leqslant k \leqslant n; \end{cases}$$

for $n = rp + s$ *and* $r \geqslant 2$, $1 \leqslant s \leqslant p - 1$,

$$h(n, p, k) = \begin{cases} (r^2 - 3r + 2 + k)p + \max\{1, s - 1\}, & 1 \leqslant k \leqslant r - 1, \\ (r^2 - 2r + 2)p, & k = r, \\ (r^2 - 2r + 3)p, & r + 1 \leqslant k \leqslant n. \end{cases}$$

2. SOME PRELIMINARY LEMMAS

Let $D(A)$ be a strong digraph, and let $R_t(i)$ denote the set of vertices which can be reached from vertex i through a walk with length t in $D(A)$. In order to prove the main theorem, we need the following lemmas.

LEMMA 2.1 [1]. *Let* A *be an* $n \times n$ *primitive matrix. Then* $\gamma(A) \leqslant n^2 - 2n + 2$ *and* $\gamma(A) = n^2 - 2n + 2$ *if and only if* A *and* W_n *are isomorphic, where*

$$W_n = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & \mathbf{0} & \\ & & \ddots & \ddots & \\ & \mathbf{0} & & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \quad (n \geqslant 3) \quad \text{and} \quad W_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

LEMMA 2.2. *For* $D(W_n)$ *we have*

(1) $R_{n^2-3n+2}(n) = \{2, 3, \dots, n\}$,
(2) $R_{n^2-2n+1}(i) = \{1, 2, \dots, n\}$ *if* $i \in \{2, 3, \dots, n\}$.

Proof. Both results are easy to prove. ∎

LEMMA 2.3 [2]. $h_{W_n}(k) = n^2 - 3n + k + 2$, *where* $1 \leqslant k \leqslant n$.

LEMMA 2.4 [2]. *If A is an $n \times n$ primitive matrix, and s is the shortest cycle length of D(A), then*

$$h_A(k) \leqslant \begin{cases} s(n-1) & (k \leqslant s), \\ s(n-1+k-s) & (k > s). \end{cases}$$

LEMMA 2.5 [2]. *If k is an integer with $1 \leqslant k \leqslant n$, then*

$$h(n,1,k) = n^2 - 3n + k + 2.$$

LEMMA 2.6. *If*

$$B_n = \begin{bmatrix} 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n} \qquad (n \geqslant 3),$$

then B_n is primitive, and when $t = n^2 - 3n + 2$, B_n^t has an all 1 row.

Proof. We can see that the shortest cycle length is $n-1$ and another cycle length is n (see Figure 1). So B_n is primitive.

We have $R_1(1) = \{2, 3\}; \dots; R_{(n-3)n}(1) = \{1, 2, \dots, n-1\}$; $R_{n^2-3n+2}(1) = \{1, 2, \dots, n-1, n\}$. So, when $t = n^2 - 3n + 2$, the first row of B_n^t is all 1. ■

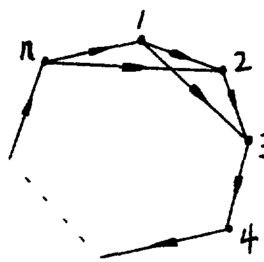

FIG. 1. $D(B_n)$.

The following two lemmas are obvious by Theorems 1.1 and 1.2.

LEMMA 2.7. *If* $A \in \mathrm{IBM}(n, p)$, *and a subset of the partition of* $D(A)$ *has only one vertex, then* $\gamma_i = 1$ $(i = 1, 2, \dots, p)$.

LEMMA 2.8. *If* $A \in \mathrm{IBM}(n, p)$, *then for every* $1 \leqslant k \leqslant n$ *we have* $h_A(k) \leqslant (\max_{1 \leqslant i \leqslant p}\{\gamma_i\})p$.

LEMMA 2.9. *Set* $A \in \mathrm{IBM}(n, p)$ *with* $n = rp$, $r \geqslant 2$. *If the normal form of* A *has* $n_1 = n_2 = \cdots = n_p = r$ *and* $\gamma_t = r^2 - 2r + 2$ *for every* $i \in \{1, 2, \dots, p\}$, *then* $D(A)$ *is isomorphic to the digraph* $W(r, p)$ (*see Figure* 2).

Proof. It is obvious that the adjacency matrix of $W(r, p)$ belongs to $\mathrm{IBM}(n, p)$. Since $\gamma_i = r^2 - 2r + 2$, C_i is isomorphic to W_r. Without loss of generality, suppose $C_1 = W_r$. We will show that for $1 \leqslant i \leqslant r - 1$, $1 \leqslant t \leqslant p$, we have $|R_{t-1}(i)| = 1$. First, any two distinct $R_{t-1}(i)$ have no common vertex. If t is different, the result is obvious. If $v_t \in R_{t-1}(i_1) \cap R_{t-1}(i_2)$, so that v_t can attain a vertex v_{j_0} in V_1 through a path with length $p - t + 1$, then the entries (i_1, j_0) and (i_2, j_0) of C_1 are both 1, which contradicts $C_1 = W_r$. Since $C_1 = W_r$, there exists an $u_t \in R_{t-1}(r)$ $(1 \leqslant t \leqslant p)$ which can attain vertex 1 of V_1 through a path with length $p - t + 1$. And we have $u_t \notin R_{t-1}(i)$ $(i = 1, 2, \dots, r - 1)$; otherwise there would exist i_0 with $1 \leqslant i_0 \leqslant r - 1$ and $u_t \in R_{t-1}(i_0)$, and thus i_0 could attain vertex 1 of V_1 through a path with length p, which contradicts $C_1 = W_r$. Since there are only r vertices in V_t, we have $|R_{t-1}(i)| = 1$ for $1 \leqslant i \leqslant r - 1$, $2 \leqslant t \leqslant p$. Define $v(t, r)$ as the vertex of $R_{t-1}(i)$, and $v(t, r)$ as u_t. There must exist an arc

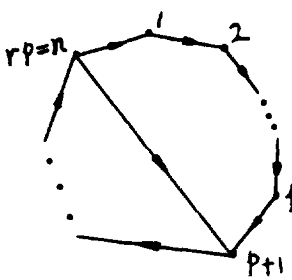

FIG. 2. $W(r, p)$.

from $v(t, r)$ to $v(t+1, 1)$ for some t $(1 \leqslant t \leqslant p)$; otherwise $C_1 \neq W_r$. We will show that there is only one such arc. Suppose there are arcs from $v(t_1, r)$ to $v(t_1+1, 1)$ and $v(t_2, r)$ to $v(t_2+1, 1)$ with $1 \leqslant t_1 < t_2 \leqslant p$. Note that $v(p+1, 1) = 2$, $v(p+1, 2) = 3, \ldots,$ $v(p+1, r-1) = r$, $v(p+1, r) = 1$, and $R_p(r) = \{1, 2\}$. Consider the paths

$$v(t_2, r-1) \to v(t_2+1, r-1) \to \cdots \to v(p, r-1)$$

$$\to v(p+1, r-1)\ [= r = v(1, r)] \to v(2, r) \to \cdots$$

$$\to v(t_1, r) \begin{cases} \to v(t_1+1, r) \to \cdots \to v(t_2-1, r) \to v(t_2, r) \\ \to v(t_1+1, 1) \to \cdots \to v(t_2-1, r) \to v(t_2, 1) \end{cases}$$

Therefore, the values of entries $(r-1, 1)$ and $(r-1, r)$ of C_{t_2} are both 1. Consider also the following paths:

$$v(t_2, r)$$

$$\to \begin{cases} \to v(t_2+1, r) \to \cdots \to v(p, r) \to v(p+1, r)\ [= 1 = v(1, 1)] \\ \quad \to v(2, 1) \to \cdots \to v(t_2, 1) \\ \to v(t_2+1, r) \to \cdots \to v(p, 1) \to v(p+1, 1)\ [= 2 = v(1, 2)] \\ \quad \to v(2, 2) \to \cdots \to v(t_2, 2). \end{cases}$$

So the values of entries $(r, 1)$ and $(r, 2)$ of C_{t_2} are both 1. The above conclusions imply that C_{t_2} is not similar to W_r, a contradiction. Hence, without loss of generality, we can assume that $t = p$; thus $D(A)$ is isomorphic to $W(r, p)$. ■

LEMMA 2.10. *Suppose* $A \in \mathrm{IBM}(n, p)$ *with* $n = rp$, *and* $r \geqslant 2$. *If* $n_1 = n_2 = \cdots = n_p = r$ *and* $\gamma_1 = r^2 - 2r + 2$, *where* $i = 1, 2, \ldots, p$, *then* $h_A(k) = (r^2 - 3r + 2)p + k$.

Proof. By Lemma 2.9, $D(A)$ is isomorphic to $W(r, p)$. Thus we have $R_t(n) = R_{n-i+1}(i)$ if $1 \leqslant i \leqslant n$. Now $R_0(n) = \{n\} = \{rp\}$, $R_1(n) = \{p+1, 1\}, \ldots,$ $R_p(n) = \{2p, p\}, \ldots,$ $R_{rp}(n) = \{rp, 2p, p\}; \ldots;$ $R_{(r^2-3r+2)p}(n)$

$= \{rp, (r-1)p, \ldots, 2p\}$; $R_{(r^2-3r+2)p+1}(n) = \{(r-1)p+1, (r-2)p+1, \ldots, 2p+1, p+1, 1\}$; and so on. Thus we have that $|R_t(n)| \leqslant r-1$ if $t < (r^2-3r+2)$; $|R_t(n)| = r$ if $t = (r^2-3r+2)p+1$; and $|R_{t_1}(i)| \leqslant |R_{t_2}(i)|$ if $t_1 \leqslant t_2$ and $1 \leqslant i \leqslant n$. By Theorem 1.3, we have $h_A(k) = (r^2-3r+2)p+k$. ■

LEMMA 2.11. *Suppose* $A \in \mathrm{IBM}(n, p)$ *with* $n = rp$ *and in the normal form of* A *there are* $n_1 = n_2 = \cdots = n_p = r$. *Then for any* $2 \leqslant k \leqslant n$, *we have* $h_A(k) \leqslant h_A(k-1)+1$.

Proof. If the number of nonzero elements of the jth row of A^t is r, then there are walks with length t from $v_j \in V_i$ to all vertices of V_{i+t}, where $(V_1, V_2, \ldots, V_p)$ is the partition of $D(A)$. Since $D(A)$ is strong, there are walks with length $t+1$ from V_j to all vertices of V_{i+t+1}. Hence the number of nonzero elements of the jth row of A^{l+1} is r too.

For any $2 \leqslant k \leqslant n$, let $\beta = h_A(k-1)$. There are at least $k-1$ rows of A^β whose number of nonzero elements is r. Let U denote the set of vertices which correspond to previous rows in the digraph $D(A)$. Thus $|U| \geqslant k-1$.

If $|U| = n$, we have $h_A(k-1) = h_A(k) = \cdots = h_A(n) = \beta$ and $h_A(k) \leqslant h_A(k-1)+1$.

If $|U| < n$, since $D(A)$ is strong, there must be a vertex v_0 of $D(A)$ with $v_0 \notin U$ such that there is an arc from v_0 to a vertex of U. So $|U \cup \{v_0\}| \geqslant k$ in $A^{\beta+1}$. Hence $h_A(k) \leqslant \beta+1 = h_A(k-1)+1$. ■

LEMMA 2.12. *Suppose* $A \in \mathrm{IBM}(n, p)$ *with* $n = rp+s$, $1 \leqslant s \leqslant p-1$. *If* $\min_{1 \leqslant i \leqslant p}\{n_i\} = r$ *in the normal form of* A, *then* $h_A(r) \leqslant (r^2-2r+2)p$.

Proof. Without loss of generality, we assume $n_1 = \max_{1 \leqslant i \leqslant p}\{n_i\} \geqslant r+1$ and $n_1 > n_p \geqslant r$. If $\max_{1 \leqslant i \leqslant p}\{n_i\} = r+1$, then $n_p = r$. If $\max_{1 \leqslant i \leqslant p}\{n_i\} \geqslant r+2$, then $s \geqslant 2$ and $n_q = r$, where $p-s+2 \leqslant q \leqslant p$. We consider the following two cases:

Case 1. $\gamma_q \leqslant r^2-2r+1$. Since $\gamma_q \leqslant r^2-2r+1$, $\max_{1 \leqslant i \leqslant p}\{\gamma_i\} \leqslant r^2-2r+2$. So $h_A(r) \leqslant (r^2-2r+2)p$.

Case 2. $\gamma_q \leqslant r^2-2r+2$. Label the vertices of V_q with $\{1, 2, \ldots, r\}$ so that $C_q = W_r$. Thus there is a path with length p from i to $i+1$ where $1 \leqslant i \leqslant r-1$, and there is a path with length p from r to 1. Obviously, each pair of these paths are disjoint unless they are in V_q, for otherwise $C_q \neq W_r$. Hence each of these r paths reaches only one distinct vertex in each V_i $(i = 1, 2, \ldots, p)$. Denote by $v(x, i)$ the vertex that the path departing from the ith vertex in V_q reaches in V_x. All these r paths together form a cycle C with length rp. There is another path with length p from vertex r to vertex 2 in V_q, and this path must intersect with the path from vertex 1 to vertex 2 in a subset of the partition of $D(A)$, say V_t, where t is the smallest such integer in $1 \leqslant t \leqslant p$. We consider two subcases:

(1) $q + 1 \leqslant t \leqslant p$ or $t = 1$. There is a path with length $p - q + 1$ from vertex r to $v(1, 1)$. Denote by X the set of vertices in V_1 not belonging to the cycle C. Suppose there exists a vertex $v(1, 0)$ of X from which there is a path with length $q - 1$ reaching a vertex in $\{2, 3, \ldots, r\}$ of V_q. By Lemma 2.2, for any i of $\{2, 3, \ldots, r\}$ of V_q and any j of V_1, there exists a path with length $(r^2 - 2r + 2)p - q + 1$ from i to j. Hence from any vertex of $\{v(1, 0), v(1, 1), \ldots, v(1, r - 1)\}$ one can reach any vertex of V_1 through a path with length $(r^2 - 2r + 2)p$. That, is $h_A(r) \leqslant (r^2 - 2r + 2)p$. Suppose, on the other hand, that from any vertex of X one can only reach vertex 1 in V_q through paths with length $q - 1$. Then from vertex r one can reach any vertex of $X \cup \{v(1, 1), v(1, r)\}$ through a path with length $p - q + 1$. Hence, by Lemma 2.2, from vertex r one can reach any vertex of V_1 through a path with length $(r^2 - 3r + 2)p + p - q + 1$. So from vertex i of V_q one can reach any vertex of V_1 through a path with length $(r^2 - 3r + 2)p + (r - i + 1)p - q + 1 \leqslant (r^2 - 2r + 2)p - q - 1$. Hence we have

$$h_A(r) \leqslant (r^2 - 2r + 2)p - q + 1 < (r^2 - 2r + 2)p.$$

(2) $2 \leqslant t \leqslant q$. Let v_{t-1} be the vertex in V_{t-1} which locates the path with length p from r to 2. Then $v_{t-1} \neq v(t - 1, i)$, where $i = 1, 2, \ldots, r - 1$; otherwise $C_q \neq W_r$. If $v_{t-1} = v(t - 1, r)$, then from it one can reach any vertex in V_q through paths with length $(r^2 - 3r + 2)p + q - t + 1$. Hence from any vertex of $\{v(t - 1, 1), v(t - 1, 2), \ldots, v(t - 1, r)\}$, one can reach any vertex of V_1 through paths with length $(r^2 - 2r + 2)p - t + 2$. So we have $h_A(r) \leqslant (r^2 - 2r + 2)p - t + 2$. Since $2 \leqslant t \leqslant q$, we have $h_A(r) \leqslant (r^2 - 2r + 2)p$. If $v_{t-1} \neq v(t - 1, r)$, from any vertex of $\{v(t - 1, 1), v(t - 1, 2), \ldots, v(t - 1, r - 1), v_{t-1}\}$, one can reach some vertices of $\{2, 3, \ldots, r\}$ in V_q through a path with length $q - t + 1$. And then, by Lemma 2.2, one can reach any vertex of V_1 through paths with length $(r^2 - 2r + 2)p - t + 2$. So we have $h_A(r) \leqslant (r^2 - 2r + 2)p + 2 - t \leqslant (r^2 - 2r + 2)p$. The proof is completed. ■

4. THE PROOF OF THE MAIN THEOREM

In order to prove the Theorem, we divide the proof into six lemmas.

LEMMA 4.1. *If $n = p$ then*

$$h(n, p, k) = 1, \qquad \textit{where} \quad 1 \leqslant k \leqslant n.$$

The proof is obvious.

LEMMA 4.2. *If $n = 2p$, then*

$$h(n, p, k) = \begin{cases} p - 1, & k = 1, \\ p, & 2 \leqslant k \leqslant p, \\ k, & p + 1 \leqslant k \leqslant n. \end{cases}$$

Proof. For any $A \in \mathrm{IBM}(n, p)$, we consider the following two cases:

Case 1. In the normal form of A, we have $n_1 = n_2 = \cdots = n_p = r = 2$.
(1) For any $i = 1, 2, \ldots, p$, we have $\gamma_i = r^2 - 2r + 2$. By Lemma 2.10, we have

$$h_A(k) = (r^2 - 3r + 2)p + k = k \qquad (1 \leqslant k \leqslant n).$$

(2) For some t, $\gamma_t = 1$. That is,

$$C_t = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

so there are two nonzero entries in a row of A. By Theorem 1.3 we can see that $h_A(1) = 1$. And by Lemma 2.11, we have

$$h_A(k) \leqslant h_A(1) + k - 1 = k.$$

Case 2. In the normal form of A, there are i, j, with $n_i \neq n_j$. In this case, there must exist $n_i = 1$. By Lemma 2.7, we have $\gamma_i = 1$ for any $i = 1, 2, \ldots, p$. Hence $h_A(k) \leqslant p$, where $1 \leqslant k \leqslant n$. Without loss of general-

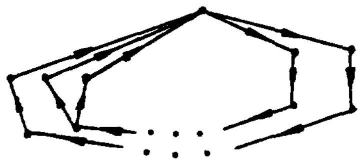

FIG. 3. D_3.

ity, we assume $n_1 = 1$; then the number of nonzero entries in the first row of A^t is n_{t+1}. So $h_A(1) \leqslant p - 1$. On the other hand, if $D(A)$ is isomorphic to D_3 (see Figure 3), then $h_A(1) = p - 1$. If $D(A)$ is isomorphic to D_4 (see Figure 4), then we have $h_A(1) = 1$ and $h_A(k) = p$, where $2 \leqslant k \leqslant p$.

Combining cases 1 and 2, Lemma 4.2 follows. ■

LEMMA 4.3. *If $n = rp$ and $r \geqslant 4$, we have*

$$h(n, p, k) = (r^2 - 3r + 2)p + k.$$

Proof. For any $A \in \mathrm{IBM}(n, p)$ there are two cases:

Case 1. In the normal form of A, we have $n_1 = n_2 = \cdots = n_p = r$. If $\gamma_i = r^2 - 2r + 2$ for any $i = 1, 2, \ldots, p$, by Lemma 2.10, we have

$$h_A(k) = (r^2 - 3r + 2)p + k.$$

If $\gamma_i < r^2 - 2r + 2$ for some t, let s be the length of a minimum cycle of $D(C_t)$.

(1) If $s \leqslant r - 2$, then by Lemma 2.4,

$$h_A(1) \leqslant (r^2 - 3r + 2)p < (r^2 - 3r + 2)p + 1.$$

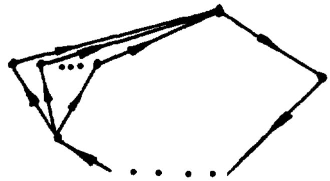

FIG. 4. D_4.

(2) If $s = r - 1$, then there is an r-cycle in $D(C_t)$. Since C_t is not similar to W_r, $D(C_t)$ must be isomorphic to $D(B_r)$ in Lemma 2.6. Hence

$$h_A(1) \leqslant (r^2 - 3r + 2)p < (r^2 - 3r + 2)p + 1.$$

By Lemma 2.11 we have

$$h_A(k) \leqslant h_A(k-1) + 1 \leqslant \cdots \leqslant h_A(1) + k - 1,$$

i.e., $h_A(k) < (r^2 - 3r + 2)p + k$.

Case 2. In the normal form of A, if there is some t with $n_t \leqslant r - 1$, then by Lemma 2.1 we have $\gamma_t \leqslant (r-1)^2 - 2(r-1) + 2 = r^2 - 4r + 5$, and by Theorem 1.2, $\gamma_i \leqslant \gamma_t + 1 \leqslant r^2 - 4r + 6 \leqslant r^2 - 3r + 2$ $(i = 1, 2, \dots, p)$ when $r \geqslant 4$. By Lemma 2.8, we have

$$h_A(k) \leqslant (r^2 - 3r + 2)p < (r^2 - 3r + 2)p + k \qquad (1 \leqslant k \leqslant n).$$

Combining cases 1 and 2, we can see that when $r \geqslant 4$ for any $A \in \mathrm{IBM}(n, p)$,

$$h_A(k) \leqslant (r^2 - 3r + 2)p + k.$$

Recalling the result of case 1, we have

$$h(n, p, k) = (r^2 - 3r + 2)p + k, \quad \text{where} \quad n = rp \text{ and } r \geqslant 4. \quad \blacksquare$$

LEMMA 4.4. *If $n = 3p$, then*

$$h(n, p, k) = \begin{cases} 3p, & 3 \leqslant k \leqslant p, \\ 2p + k & \text{otherwise.} \end{cases}$$

Proof. For any $A \in \mathrm{IBM}(n, p)$, there are two cases:

Case 1. In the normal form of A, suppose $n_1 = n_2 = \cdots = n_p = 3$. Thus a similar argument to the proof of Lemma 4.3 yields

$$h_A(k) \leqslant (r^2 - 3r + 2)p + k = 2p + k,$$

and these upper bounds can be attained.

Case 2. In the normal form of A, there exist $1 \leqslant i, j \leqslant p$ with $n_i \neq n_j$. If $\min_{1 \leqslant i \leqslant p}\{n_i\} = 1$, by Lemma 2.7, $\gamma_i = 1$ $(i = 1, 2, \ldots, p)$. Hence $h_A(k) \leqslant p$ $(k = 1, 2, \ldots, n)$. If $\min_{1 \leqslant i \leqslant p}\{n_i\} = 2$, without loss of generality, we assume that $n_1 = 2$.

(1) If $\gamma_1 = 1$, then for any $1 \leqslant i \leqslant p$ we have $\gamma_i \leqslant 2$; thus

$$h_A(k) \leqslant 2p \qquad (1 \leqslant k \leqslant n).$$

(2) If $\gamma_1 = 2$, i.e. $C_1 = W_2$, then for any $1 \leqslant i \leqslant p$ we have $\gamma_i \leqslant 3$. By Lemma 2.8, we have

$$h_A(k) \leqslant 3p \qquad (1 \leqslant k \leqslant n).$$

Next, let $n_t = \max_{1 \leqslant i \leqslant p}\{n_i\}$. If there exists i_0, $2 \leqslant i_0 \leqslant t$, such that $R_{i_0 - 1}(2) = V_{i_0}$, then $h_A(1) \leqslant t - 1$. Since $C_1 = W_2$, we have $R_{p+(t-1)}(1) = R_{t-1}(2) = V_t$ and $R_{p+(t-1)}(2) = R_{t-1}(2) = V_t$. Hence $h_A(2) \leqslant p + (t - 1)$. Since $2 \leqslant t \leqslant p$, we have $h_A(2) \leqslant 2p - 1 < 2p$. Now for any $2 \leqslant i \leqslant t$, we have $R_{i-1}(2) \neq V_i$. Hence there exists $v_i \in V_i$ such that $v_i \in R_{i-1}(2)$. Since $D(A)$ is strong, $v_i \in R_{i-1}(1)$. In particular, there exists $v_t \in V_t$ such that $v_t \in R_{t-1}(1)$, $v_t \notin R_{t-1}(2)$. Thus from v_t we can reach vertex 2, but not 1, in V_1 through a path with length $p - t + 1$. Since $C_1 = W_2$, there exists a vertex $v_0 \in R_{t-1}(2)$ such that from it we can reach vertex 2 through a path with length $p - t + 1$. On the other hand, from vertex 2 we can reach all vertices of V_t through paths with length $p + t - 1$, i.e., $R_{p+t-1}(2) = V_t$. Thus from v_t and v_0 we can reach any vertex of V_t through a path with length $(p + t - 1) + (p - t + 1) = 2p$, i.e., $h_A(2) \leqslant 2p$. Furthermore, we have $h_A(1) \leqslant h_A(2) \leqslant 2p$. So in case 2, we have

$$h_A(k) \leqslant \begin{cases} 2p, & k = 1, 2, \\ 3p, & 3 \leqslant k \leqslant n. \end{cases}$$

Combining cases 1 and 2, we have

$$h_A(k) \leqslant \begin{cases} 3p, & 3 \leqslant k \leqslant p, \\ 2p + k & \text{otherwise.} \end{cases}$$

Consider the digraph D_5 (see Figure 5). Its adjacency matrix $A \in \mathrm{IBM}(n, p)$ with $h_A(3) = h_A(4) = \cdots = h_A(3p) = 3p$. Hence we have $h(n, p, k) = 3p$ $(3 \leqslant k \leqslant p)$. The proof is completed. ■

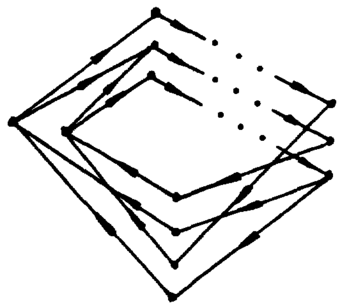

FIG. 5. D_5.

LEMMA 4.5. *If* $n = rp + s$ *with* $1 \leqslant s \leqslant p - 1$, *and* $r \geqslant 2$, *we have*

$$h(n,p,k) = \begin{cases} (r^2 - 3r + 2 + k)p + \max\{1, s-1\}, & 1 \leqslant k \leqslant r-1, \\ (r^2 - 2r + 2)p, & k = r, \\ (r^2 - 2r + 3)p, & r+1 \leqslant k \leqslant n. \end{cases}$$

Proof. We consider two cases:

Case 1. $s = 1$. For any $A \in \mathrm{IBM}(n,p)$ with $n = rp + 1$ and $r \geqslant 2$, we have:

(1) In the normal form of A, it has $\min_{1 \leqslant i \leqslant p}\{n_i\} < r$; let $n_t = \min_{1 \leqslant i \leqslant p}\{n_i\} \leqslant r - 1$. So $\gamma_t \leqslant (r-1)^2 - 2(r-1) + 2 = r^2 - 4r + 5$. Thus $\gamma_i \leqslant \gamma_t + 1 \leqslant r^2 - 4r + 6$ for any $1 \leqslant i \leqslant p$. Since $r^2 - 4r + 6 \leqslant r^2 - 3r + 3$ where $r \geqslant 3$,

$$h_A(k) \leqslant (r^2 - 4r + 6)p < (r^2 - 3r + 3)p + 1 \qquad (1 \leqslant k \leqslant n).$$

If $r = 2$, then $\min_{1 \leqslant i \leqslant p}\{n_i\} = 1$. By Lemmas 2.7 and 2.8 we have $\gamma_i = 1$ $(i = 1, 2, \ldots, p)$ and $h_A(k) \leqslant p$ $(1 \leqslant k \leqslant n)$.

(2) In the normal form of A, it has $\min_{1 \leqslant i \leqslant p}\{n_i\} = r$. When $s = 1$, without loss of generality, we assume $n_1 = r + 1$ and $n_2 = n_3 = \cdots = n_p = r$. Thus $\gamma_i \leqslant r^2 - 2r + 2$ $(i = 2, 3, \ldots, p)$ and $\gamma_1 \leqslant r^2 - 2r + 3$. By Lemmas 2.5 and 2.12, for any $A \in \mathrm{IBM}(n,p)$ with $n = rp + 1$ and $r \geqslant 2$, we have

$$h_A(k) \leqslant \begin{cases} (r^2 - 3r + 2 + k)p + 1, & 1 \leqslant k \leqslant r-1, \\ (r^2 - 2r + 2)p, & k = r, \\ (r^2 - 3r + 3)p, & r+1 \leqslant k \leqslant n. \end{cases} \tag{A}$$

Consider the p-partite digraph D_6 (see Figure 6) with $|V_1| = r + 1$ and $|V_i| = r$, $i = 2, 3, \ldots, p$. Its adjacency matrix $A \in \mathrm{IBM}(n, p)$ has

$$h_A(k) = \begin{cases} (r^2 - 3r + 2 + k)p + 1, & 1 \leqslant k \leqslant r - 1, \\ (r^2 - 2r + 2)p, & k = r, \\ (r^2 - 2r + 3)p, & r + 1 \leqslant k \leqslant n. \end{cases}$$

So the upper bound of (A) can be attained.

Case 2. $2 \leqslant s \leqslant p - 1$. For any $A \in \mathrm{IBM}(n, p)$ with $n = rp + s$, $2 \leqslant s \leqslant p - 1$, and $r \geqslant 2$, we consider two cases:

(1) In the normal form of A, it has $\min_{1 \leqslant i \leqslant p}\{n_i\} < r$. A similar argument to case 1(1) yields

$$h_A(k) < \begin{cases} (r^2 - 3r + 2 + k)p + s - 1, & 1 \leqslant k \leqslant r - 1, \\ (r^2 - 2r + 2)p, & k = r, \\ (r^2 - 2r + 3)p, & r + 1 \leqslant k \leqslant n. \end{cases}$$

(2) In the normal form of A, it has $\min_{1 \leqslant i \leqslant p}\{n_i\} = r$. Without loss of generality, we assume that $n_1 = \max_{1 \leqslant i \leqslant p}\{n_i\} > n_p$. It's obvious that there exists t where $p - s + 2 \leqslant t \leqslant p$ such that $n_t = r$. If $1 \leqslant k \leqslant r - 1$, by Lemma 2.5, we have $h_A(k) \leqslant (r^2 - 3r + 2 + k)p + (p - t + 1) \leqslant (r^2 - 3r + 2 + k)p + s - 1$. If $k = r$, we have $h_A(r) \leqslant (r^2 - 2r + 2)p$ by Lemma 2.12; if $r + 1 \leqslant k \leqslant n$, noting that $\gamma_i \leqslant r^2 - 2r + 3$ $(i = 1, 2, \ldots, p)$, we have $h_A(k) \leqslant (r^2 - 2r + 3)p$.

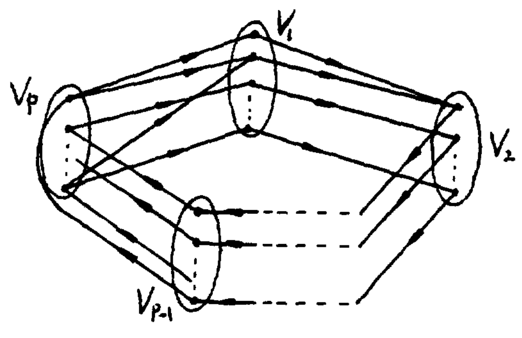

FIG. 6. D_6.

Combining cases 1 and 2, for any $A \in \mathrm{IBM}(n, p)$ with $n = rp + s$, $r \geqslant 2$, and $s \geqslant 2$, we have

$$h_A(k) \leqslant \begin{cases} (r^2 - 3r + 2 + k)p + s - 1, & 1 \leqslant k \leqslant r - 1, \\ (r^2 - 2r + 2)p, & k = r, \\ (r^2 - 2r + 3)p, & r + 1 \leqslant k \leqslant n. \end{cases} \tag{B}$$

For $r \geqslant 2$ and $2 \leqslant s \leqslant p - 1$, consider the p-partite digraph D_7 (see Figure 7) with $|V_1| = |V_2| = \cdots = |V_{s-2}| = r + 1$, $|V_{s-1}| = r + 2$, and $|V_s| = |V_{s+1}| = \cdots = |V_p| = r$. It is easy to check that its adjacency matrix $A \in \mathrm{IBM}(n, p)$ has

$$h_A(k) = \begin{cases} (r^2 - 3r + 2 + k)p + s - 1, & 1 \leqslant k \leqslant r - 1, \\ (r^2 - 2r + 2)p, & k = r, \\ (r^2 - 2r + 3)p, & r + 1 \leqslant k \leqslant n. \end{cases}$$

So the upper bounds of (B) can be attained.

Hence Lemma 4.5 follows. ■

LEMMA 4.6. *If $n = p + s$ with $1 \leqslant s \leqslant p - 1$, we have*

$$h(n, p, k) = \begin{cases} \max\{1, s - 1\}, & k = 1, \\ p, & 2 \leqslant k \leqslant n. \end{cases}$$

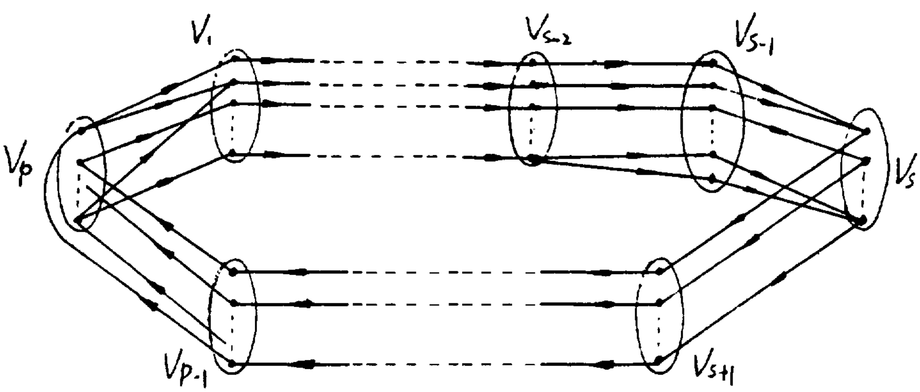

FIG. 7. D_7.

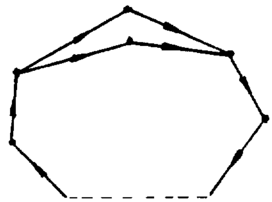

FIG. 8. D_8.

Proof. We consider the following two cases:

Case 1. $s = 1$. For any $A \in \mathrm{IBM}(n, p)$ with $n = p + 1$, $D(A)$ must be isomorphic to the digraph D_8 in Figure 8. So we have

$$h_A(k) = \begin{cases} 1, & k = 1, \\ p, & 2 \leqslant k \leqslant n. \end{cases}$$

Case 2. $2 \leqslant s \leqslant p - 1$. For any $A \in \mathrm{IBM}(n, p)$ with $n = p + s$, in the normal form of A, there must exist $n_t = 1$. By Lemma 2.7 we have $\gamma_i = 1$ for any $i = 1, 2, \dots, p$. By Lemma 2.8, we have $h_A(k) \leqslant p$ for $1 \leqslant k \leqslant p$. If $\max\{n_i\} = 2$, then there are s subsets of the partition containing two vertices. Without loss of generality, we assume $n_1 = 2$ and $n_p = 1$. It's obvious that $h_A(1) = 1$. If $\max\{n_i\} \geqslant 3$, we assume $n_1 = \max\{n_i\}$. There exists with $p - s + 2 \leqslant t \leqslant p$ such that $n_t = 1$. Hence the vertex of V_t can reach all vertices of V_1 through some paths with length $p - t + 1$. So $h_A(1) \leqslant p - t + 1 \leqslant s - 1$. Hence we have

$$h_A(k) = \begin{cases} s - 1, & k = 1, \\ p, & 2 \leqslant k \leqslant n. \end{cases} \tag{C}$$

Consider the digraph D_9 in Figure 9. Its adjacency matrix $A \in \mathrm{IBM}(n, p)$ with $n = p + s$, $2 \leqslant s \leqslant p - 1$ and

$$h_A(k) = \begin{cases} s - 1, & k = 1, \\ p, & 2 \leqslant k \leqslant n. \end{cases}$$

So the upper bounds of (C) can be attained.

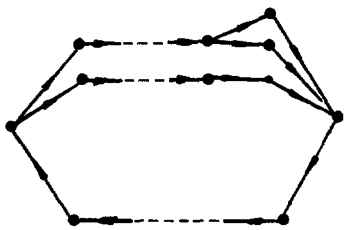

FIG. 9. D_9.

Combining cases 1 and 2, we have

$$h(n, p, k) = \begin{cases} \max\{1, s - 1\}, & k = 1, \\ p, & 2 \leqslant k \leqslant n. \end{cases} \qquad \blacksquare$$

The problem of determining $h(n, p, k)$ is completely solved.

REFERENCES

1 A. L. Dulmage and N. S. Mendelsohn, Graphs and matrices, in *Graph Theory and Theoretical Physics* (F. Harary, Ed.), Academic, New York, 1967, Chapter 6, pp. 167–227.
2 R. A. Brualdi and Bolian Liu, Generalized exponents of primitive directed graphs, *J. Graph Theory* 14(4):483–499 (1990).
3 B. R. Heap and M. S. Lynn, The structure of powers of non-negative matrices II, the index of maximum density, *SIAM J. Appl. Math.* 14:762–777 (1996).
4 Shao Jiayu and Li Qiao, On the index of maximum density for irreducible Boolean matrices, *Discrete Appl. Math.* 21:147–156 (1988).

Received 3 August 1995; final manuscript accepted 27 November 1995