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THE TOTAL CHROMATIC NUMBER OF PSEUDO-OUTERPLANAR GRAPHS*

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Abstract. A planar graph G is called a i-pseudo-outerplanar graph if there is a subset $V_0 \subseteq V(G)$, $|V_0| = i$, such that $G - V_0$ is an outerplanar graph. In particular, when $G - V_0$ is a forest, G is called a i-pseudo-tree. In this paper, the following results are proved: (i) The conjecture on the total coloring is true for all 1-pseudo-outerplanar graphs; (ii) $\chi_t(G) = \Delta(G) + 1$ for any 1-pseudo-outerplanar graph G with $\Delta(G) \geq 6$ and for any 1-pseudo-tree G with $\Delta(G) \geq 3$, where $\chi_t(G)$ is the total chromatic number of a graph G.

1. Introduction

In this paper, we shall restrict ourselves to the simple planar graph $G(V,E,F)$, where V,E, and F are the set of vertices, the set of edges and the set of faces of a graph G, respectively. Let $p(G)$ and $q(G)$ denote the number of vertices and the number of edges of a graph G, respectively.

Definition 1.1. A graph G is k- total colorable if the elements of $V(G) \cup E(G)$ can be colored with k colors such that any two adjacent or incident elements receive different colors. The total chromatic number $\chi_t(G)$ of a graph G is defined as the minimum number k for which G is k- total colorable.

The bound $\chi_t(G) \geq \Delta(G) + 1$ is trivial. Behzad [1] and Vizing [6] conjectured independently that $\chi_t(G) \leq \Delta(G) + 2$ for any graph G. This conjecture was confirmed for $\Delta = 3$ by Vijayaditya [5] and for $4 \leq \Delta \leq 5$ by Kostochka [4]. In 1989, Borodin proved in [3] that

Theorem 1.1. *Let G be a planar graph, then*

(1) $\chi_t(G) \leq \Delta(G) + 2$ *for* $\Delta(G) \neq 6,7,8$;

(2) $\chi_t(G) \leq \Delta(G) + 3$ *for* $\Delta(G) = 6,7,8$;

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(3) $\chi_t(G)=\Delta(G)+1$ *for* $\Delta(G)\geq 14$.

The advance of the total coloring of graphs can be seen in [7]. The purpose of the present paper is to study the total coloring of pseudo-outerplanar graphs. First we write

$$V_k(G) = \{u \in V(G) \mid d_G(u) = k\},\ k = 0,1,\ldots,\Delta(G).$$

$$V_2^1(G) = \{u \in V_2(G) \mid u \text{ belongs to the boundary of some triangular face of } G\}.$$

Let $N_G(u)$ denote the neighbour set of a vertex u in G. A face f whose boundary contains the vertices $u_1, u_2, \ldots, u_n$ in some order is written as $f = u_1u_2\ldots u_n$. A k-total coloring σ of a graph G with a set C of k colors is denoted by (σ, C). Let $\sigma(y)$ denote the color assigned to an element $y \in V(G) \cup E(G)$ and $A_\sigma(u)$ denote the set of colors used on the vertex u and the edges incident to u under σ. The other statements and notations can be seen in [2].

2. The Characters of Pseudo-outerplanar Graphs

Definition 2.1. If all the vertices of a planar graph G are located on the boundary of the unbounded face, then this graph is called an outerplanar graph. The unbounded face is called outer face, other faces are called inner faces. The edges on the boundary of outer face are said to be outer edges; and other edges inner edges. Let $\Omega_k (k \geq 1)$ denote the set of simple outerplanar graphs with maximum degree k.

Definition 2.2. Let G be a planar graph. If there is a subset $V_0 \subseteq V(G)$, $|V_0| = i$, such that $G - V_0$ is an outerplanar graph (a forest, resp.), then G is called an i- pseudo-outerplanar graph (an i- pseudo-tree, resp.), where V_0 is called a base set (a root set, resp.) of G. Let $\Sigma_k(\Gamma_k$, resp$)$, $k \geq 1$, denote the set of 1-pseudo-outerplanar graphs (1-pseudo-trees, resp) with maximum degree k.

Lemma 2.1([8]) *Let G be a 2-connected outerplanar graph, then $\delta(G)=2$ and $|V_2(G)| \geq 2$.*

Lemma 2.2. *If $G \in \Omega_k (k \geq 1)$, then $\delta(G) \leq 2$.*

Theorem 2.1. *Let G be an outerplanar graph with $\delta(G)=2$, then at least one of the following statements is true:*

(i) *There are $u, v \in V_2(G)$ such that $uv \in E(G)$;*

(ii) *There are $u \in V_2(G)$, $x \in V_3(G)$, $y \in V_j(G)$, $j \geq 3$ such that $uxy \in F(G)$;*

(iii) *There is $w \in V_4(G)$, $N_G(w) = \{x, y, u, v\}$ such that $u, v \in V_2(G)$, $x \in V_i(G)$, $y \in V_j(G)$, $i, j \geq 4$ and $ux, uy \in E(G)$.*

Proof. The theorem is trivial for the case $\Delta(G) = 2$. Thus suppose that $\Delta(G) \geq 3$. Let G be a counter example of Theorem 2.1 with vertices as few as possible. Choose a block H of G which contains at most one cut vertex, say t, of G. So $p(H) \geq 5$ and $\Delta(H) \geq 3$, since otherwise the theorem holds obviously. We first prove the following Claim 2.1:

Claim 2.1. At least one of (i)-(iii) in Theorem 2.1 is true for H. And all vertices with given degree (i.e. u and v in (i), u and x in (ii) and w, u and v in (iii)) differ from t.

By Lemma 2. 1, $|V_2(H)| \geq 2$, so $|V_2(H)\setminus\{t\}| \geq 1$. Moreover, $V_2(H)\setminus(V_2^1(H) \cup \{t\}) = \emptyset$. In fact, if there is $w \in V_2(H)\setminus(V_2^1(H) \cup \{t\})$ with $N_G(w) = \{w_1, w_2\}$, then $G - w + w_1w_2$ is another counter example of Theorem 2. 1 with even fewer vertices. Thus $V_2^1(H)\setminus\{t\} \neq \emptyset$. For any $v \in V_2^1(H)\setminus\{t\}$, letting $N_G(v) = \{x, y\}$, we have $xy \in E(G)$. Then one of the following cases is true:

Case A. $t \neq x, y$. Thus $d_H(x) \geq 5$ and $d_H(y) \geq 5$; or $x \in V_4(H), d_H(y) \geq 5$ and $N_H(x) \cap (V_2^2(H)\setminus\{v\}) = \emptyset$; or $y \in V_4(H), d_H(x) \geq 5$ and $N_H(y) \cap (V_2^1(H)\setminus\{v\}) = \emptyset$; or $x, y \in V_4(H)$ and $(N_H(x) \cup N_H(y)) \cap (V_2^1(H)\setminus\{v\}) = \emptyset$.

Case B. $t = x$ (we have the similar discussion for $t = y$). In this case, $d_H(x) \geq 3$ and $d_H(y) \geq 5$; or $d_H(x) \geq 3, y \in V_4(H)$ and $N_H(y) \cap (V_2^1(H)\setminus\{v\}) = \emptyset$.

Consider the graph $H_1 = H - V_2^1(H)\setminus\{t\}$. Obviously, $d_{H_1}(z) \geq 3$ for every $z \in V(H_1)\setminus\{t\}$ and $d_{H_1}(t) \geq 2$. It follows that $|V_2(H_1)\setminus\{t\}| = 0$. On the other hand, by Lemma 2. 1, we have $|V_2(H_1)\setminus\{t\}| \geq 1$, a contradiction. Hence Claim 2. 1 holds for H. Further it follows that Theorem 2. 1 is also true for G, which contradicts the choice of G. □

Corollary 2. 1. *Let G be a 2-connected outerplanar graph with $\Delta(G) \geq 4$, then*

(i) *There are $u \in V_2(G)$ and $v \in V_k(G), k \leq \Delta(G) - 1$ such that $uv \in E(G)$; or*

(ii) *There is $w \in V_4(G)$ with $N_G(w) = \{x, y, u, v\}$ such that $u, v \in V_2(G), x \in V_i(G), y \in V_j(G), i, j \geq 4$, and $ux, vy \in E(G)$.*

Lemma 2. 3. *If G is a 1-pseudo-outerplanar graph, then $\delta(G) \leq 3$.*

Proof. Let u be a base of G. Since $G - u$ is an outerplanar graph, then it follows form Lemma 2. 2 that $\delta(G - u) \leq 2$. Hence $\delta(G) \leq \delta(G - u) + 1 \leq 3$. □

Corollary 2. 2. *If G is a 1-pseudo-tree, then $\delta(G) \leq 2$.*

Theorem 2. 2. *Let G be a 1-pseudo-outerplanar graph with $\Delta(G) \geq 6$ and $\delta(G) \geq 2$, then at least one of the following is true:*

(1) *There are $v \in V_2(G), w \in V_k(G), k \leq \Delta(G) - 1$ such that $vw \in E(G)$.*

(2) *There are $v, w \in V_2(G), x, y, z \in V_\Delta(G)$ such that $vx, vy, wy, wz, xy, yz \in E(G)$, where $V_\Delta(G) = V_{\Delta(G)}(G)$.*

(3) *There are $v, w \in V_2(G)$ such that $N_G(w) = N_G(v) = \{x, y\}$, and $x, y \in V_\Delta(G)$.*

(4) *There are $v \in V_2(G), w \in V_3(G)$, and $N_G(v) = \{x, y\}, N_G(w) = \{x, y, z\}$ such that $x, y \in V_\Delta(G), xy \in E(G)$.*

(5) *There are $v \in V_3(G)$ and $w \in V_j(G)$, $3 \leq j \leq 4$ such that $vw \in E(G)$.*

(6) *There are $v, w \in V_3(G)$, $x \in V_j(G), j \geq 3$, $y \in V_5(G)$ such that $xv, xw, yw, yv, xy \in E(G)$.*

Proof. Let $H = G - u$, where u is a base of G. By the definition, H is an outerplanar graph with $5 \leq \Delta(H) \leq \Delta(G)$, and $1 \leq \delta(H) \leq 2$.

Case 1. If $\delta(H)=2$, by Theorem 2.1, we have two subcases:

1.1. There are $v\in V_2(H), w\in V_j(H), 2\le j\le 3$ such that $vw\in E(H)$. If $uv\notin E(G)$, then (1) holds for G. If $uv\in E(G)$, when $uw\in E(G)$ or $uw\notin E(G)$ and $w\in V_3(G)$, then (5) holds for G; when $uw\notin E(G)$ and $w\in V_2(G)$, (1) holds for G.

1.2. There are $w\in V_4(G), N_G(w)=\{x,y,z,v\}$ such that $v,z\in V_2(G), x\in V_i(G)$, $y\in V_j(G), i,j\ge 4, xz, vy\in E(G)$. In this case, if $uz\notin E(G)$ or $uv\notin E(G)$, then (1) holds. If $uz, uv\in E(G)$ and $uw\notin E(G)$, then (5) holds; If $uv, uz, uw\in E(G)$, then (6) holds.

Case 2. If $\delta(H)=1$, it follows from $\delta(G)\ge 2$ that $V_1(H)\subseteq N_G(u)$. Let $x\in V_1(H)$ and $xy\in E(H)$, then $x\in V_2(G)$. If $d_G(y)<\Delta(G)$, or $d_G(u)<\Delta(G)$, (1) holds. If $d_G(y)=d_G(u)=\Delta(G)\ge 6$, there are two subcases:

2.1. If there is $z\in V_1(H)\backslash\{x\}$ such that $yz\in E(H)$, (3) holds.

2.2. If $N_H(y)\cap(V_1(H)\backslash\{x\})=\emptyset$, there are three subcases:

2.2.1. If $uy\notin E(G)$, remove the vertex x;

2.2.2. If $uy\in E(G)$, and there is $z\in V_2(H)$ such that $yz\in E(H)$, let $t\in N_H(z)\backslash\{y\}$. If $uz\in E(G)$, then (4) holds. If $uz\notin E(G)$, then when $d_G(t)<\Delta(G)$, (1) holds; when $d_G(t)=\Delta(G)$, and $ty\in E(H)$, (2) holds; when $d_G(t)=\Delta(G)$, and $ty\notin E(H)$, remove the vertex z and add the edge ty to H.

2.2.3. If $uy\in E(G)$, and $N_G(y)\cap V_2(H)=\emptyset$, remove the vertex x and the edge uy.

After the above procedure, if the theorem has not been proved, we obtain finally a graph H_0 from H. Clearly, H_0 is an outerplanar graph with $\Delta(H_0)\ge 4, \delta(H_0)=2$ and $d_{H_0}(y)\ge 4$. By virtue of Theorem 2.1, there are $v\in V_2(H_0)$, and $w\in V_j(H_0), 2\le j\le 3$ such that $vw\in E(H_0)$; or there is $z\in V_4(H_0), N_{H_0}(z)=\{v_1,v_2,s,t\}$ such that $v_1,v_2\in V_2(H_0), sv_1, tv_2\in E(H_0)$. In view of the structure of H_0, it follows that $y\notin\{v,w,z,v_1,v_2,s,t\}$. Thus by Theorem 2.1 for H_0 and as in Case 1, (1) or (5) or (6) is true for G. □

Lemma 2.4. *If F is a forest, then $|V_1(F)|\ge\Delta(F)$.*

Proof. It is obvious even without any proof. □

Lemma 2.5. *If F is a forest with $\Delta(F)\ge 3$, then*

(1) *There are $v\in V_1(F), w\in V_k(F)$, $k\le\Delta(F)-1$ such that $vw\in E(F)$; or*

(2) *There are $x,y\in V_1(F), w\in V_\Delta(F)$ such that $wx, wy\in E(F)$.*

Proof. Let F be a forest with $\Delta(F)\ge 3$. It follows from Lemma 2.4 that $|V_1(F)|\ge 3$. If both (1) and (2) are not true for G, we deduce that in F every vertex with maximum degree is adjacent to at most one vertex in $V_1(F)$, and every vertex in $V_1(F)$ is adjacent to some vertex of maximum degree. Consider the graph $F_1=F-V_1(F)$. On the one hand, $|V_1(F_1)|\ge 2$

for F_1 is a forest. On the other hand, for any $x \in V(F_1), d_{F_1}(x) \geq 2$, thus $|V_1(F_1)| = 0$, a contradiction. □

Theorem 2.3. *Let G be a 1-pseudo-tree with $\Delta(G) \geq 4$ and $\delta(G) = 2$, then*

(1) *There are $v \in V_2(G), w \in V_k(G), k \leq \Delta(G) - 1$ such that $vw \in E(G)$; or*

(2) *There are $v, w \in V_2(G)$ such that $N_G(v) = N_G(w) = \{x, y\}$ and $x, y \in V_\Delta(G)$.*

Proof. Let u be a root of G. Then $H = G - u$ is a forest, and thus $\delta(H) = 1$. Since $\delta(G) = 2, V_1(H) \subseteq N_G(u)$. Hence, by Lemma 2.4, we have

$$\Delta(G) \geq d_G(u) \geq |V_1(H)| \geq \Delta(H) \geq \Delta(G) - 1 \geq 3.$$

Case 1. If $d_G(u) = \Delta(G) - 1$, then (1) holds for G.

Case 2. If $d_G(u) = \Delta(G)$ and $\Delta(H) = \Delta(G) - 1$, by Lemma 2.5, we have

2.1. There are $v \in V_1(H), w \in V_j(H), j \leq \Delta(H) - 1$ such that $vw \in E(H)$. We deduce

$$d_G(w) \leq d_H(w) + 1 \leq \Delta(H) - 1 + 1 = \Delta(H) < \Delta(G)$$

which implies that (1) holds for G.

2.2. There are $x, y \in V_1(H), w \in V_\Delta(H)$ such that $wx, wy \in E(H)$. Obviously, $x, y \in V_2(G)$. If $d_G(w) < \Delta(G)$, then (1) holds; if $d_G(w) = \Delta(G)$, then (2) holds.

Case 3. If $d_G(u) = \Delta(G)$, and $\Delta(H) = \Delta(G)$, then $|V_1(H)| = \Delta(H) = \Delta(G)$ and $|V_2(G)| = \Delta(G)$. By Lemma 2.5, either (1) or (2) holds for G. □

3. Main Results

Lemma 3.1. *Let $G = G_1 \cup G_2, V(G_1) \cap V(G_2) = \{v\}$, then*

$$\chi_t(G) = \max\{\chi_t(G_1), \chi_t(G_2), d_G(v) + 1\}.$$

Lemma 3.2. *Let $P = u_0 e_1 u_1 e_2 \cdots u_k e_{k+1} u_{k+1}$ be a path of length $k+1 (\geq 3)$, where $e_i = u_{i-1}u_i, i = 1, 2, \ldots, k+1$, and let C denote a set of 4 colors. Then P can be totally colored under the condition that the colors appearing in two vertices u_0, u_{k+1} and two edges e_1, e_{k+1} are assigned properly in advance.*

Lemma 3.3. ([8]) *If $G \in \Omega_3$, then $\chi_t(G) = 4$.*

For the following Theorem 3.1, a proof has been given in [8]. Now we provide a shorter proof for it.

Theorem 3.1. *If $G \in \Omega_k (k \geq 4)$, then $\chi_t(G) = k+1$.*

Proof. By Lemma 3.1, it suffices to consider G to be a 2-connected outerplanar graph. The lower bound $\chi_t(G) \geq \Delta(G) + 1 = k + 1$ is trivial. Let us prove the upper bound $\chi_t(G) \leq k +$

1. First consider the case $k=4$, by the induction on $p(G)$. When $p=5, G\in\Omega_4$ is a fan F_3 and $\chi_t(G)=\chi_t(F_5)=5$. Assume the theorem is true for equal to or less than $p-1$, and let G be a 2-connected outerplanar graph of order $p(\geq 6)$ with $\Delta(G)=4$. By Corollary 2.1, we have two possibilties:

Case 1. There is $v\in V_2(G)$ with $N_G(v)=\{x,w\}, w\in V_j(G), 2\leq j\leq 3$. Consider the graph $H=G-v$ if $wx\in E(G)$, or $H=G-v+wx$ if $wx\notin E(G)$. By the inductive assumption or Lemma 3.3, H has a 5-totally coloring (λ,C). Let (σ,C) be a 5-totally coloring of G defind as follows:

When $wx\in E(G)$, we put $\sigma(vx)\in C-A_\lambda(x), \sigma(vw)\in C-(A_\lambda(w)\cup\{\sigma(vx)\})$. When $wx\notin E(G)$, we put $\sigma(vx)=\lambda(wx), \sigma(vw)\in C-A_\lambda(w)$. Then we put $\sigma(v)\in C-\{\lambda(x),\lambda(w),\sigma(vx),\sigma(vw)\}$.

Case 2. There are $w,x,y\in V_4(G), N_G(w)=\{u,v,x,y\}$ such that $u,v\in V_2(G), ux, vy \in E(G)$. Let us consider the graph $H=G-u-v$, which has a 5-totally coloring (λ,C). Write $C-A_\lambda(x)=\{c_1\}, C-A_\lambda(y)=\{c_2\}, C-A_\lambda(w)=\{a,b\}, a\neq b$.

2.1. If $c_1\neq a,b$ (for $c_2\neq a,b$, the proof is similar), we put $\sigma(vy)=c_2, \sigma(vw)\in C-A_\lambda(w)\cup\{c_2\}, \sigma(uw)\in A_\lambda(w)\cup\{\sigma(vw)\}, \sigma(ux)=c_1$.

2.2. If $\{c_1,c_2\}\subseteq\{a,b\}$, we first suppose that $c_1\neq c_2$, and $c_1=a, c_2=b$. Then we put $\sigma(ux)=\sigma(vw)=a, \sigma(vy)=\sigma(uw)=b$. If $c_1=c_2$, we assume that $c_1=c_2=a$ and put $\sigma(ux) =\sigma(vw)=\lambda(wx), \sigma(vy)=\sigma(wx)=a, \sigma(uw)=b$. Then we put $\sigma(u)\in C-\{\lambda(x),\lambda(w), \sigma(ux),\sigma(uw)\}, \sigma(v)\in C-\{\lambda(y),\lambda(w),\sigma(vy),\sigma(vw)\}$.

In Case 1 and Case 2, the other uncolored elements of $V(G)\cup E(G)$ are colored with the same colors as in (λ,C) of H. Thus $\chi_t(G)\leq 5$. Consequently, we have $\chi_t(G)=5$ for $G\in\Omega_4$. Further, it is easily seen that if $G\in\Omega_k(k\geq 5)$, Case 1 in Corollary 3.1 holds. Therefore we can similarly prove $\chi_t(G)=k+1$. □

Theorem 3.2. *If* $G\in\Sigma_k(k\geq 6)$, *then* $\chi_t(G)=k+1$.

Proof. At first we prove $\chi_t(G)=7$ for $G\in\Sigma_6$. Let $G\in\Sigma_6$ be a graph with the minimum number of edges which can not be colored with 7 colors. Note that $\delta(G)\geq 2$. In fact, if there is $v\in V_1(G)$, then $G-v$ can be totally colored by the definition of G, afterward we can color the pendant edge incident to v, after all we can color v. By Theorem 2.2, we have six cases and can deal with the first two cases by a similar method of Theorem 3.1. Now we consider the remaining cases.

Case 3. There are $v,w\in V_2(G), N_G(v)=N_G(w)=\{x,y\}, x,y\in V_6(G)$. Consider the graph $H=G-v-w$. Since $q(H)<q(G), 4\leq\Delta(H)\leq 6$, by the definition of G and Theorem 1.1, H has a 7-totally coloring (λ,C). A 7-totally coloring (σ,C) of G is formed as follows: If $A_\lambda(x)=A_\lambda(y)$, we put $\sigma(vx)=\sigma(wy)=a\in C-A_\lambda(x), \sigma(wx)=\sigma(vy)=b\in C-A_\lambda(x)\cup\{a\}$. If $A_\lambda(x)\neq A_\lambda(y)$, then we put $\sigma(vy)=a\in A_\lambda(x)-A_\lambda(y), \sigma(wx)= b\in A_\lambda(y)-A_\lambda(x), \sigma(vx)\in C-A_\lambda(x)\cup\{b\}$ and $\sigma(wy)\in C-A_\lambda(y)\cup\{a\}$. Then we put $\sigma(v)\in C-\{\lambda(x),\lambda(y),\sigma(vx),\sigma(vy)\}, \sigma(w)\in C-\{\lambda(x),\lambda(y),\sigma(wx),\sigma(wy)\}$.

Case 4. There are $v \in V_2(G), w \in V_3(G)$ with $N_G(v) = \{x, y\}, N_G(w) = \{x, y, z\}, x, y \in V_6(G)$ and $xy \in E(G)$. Set $H = G - v - wx - wy$, and suppose that (λ, C) is a 7-totally coloring of H.

4. 1. If $\lambda(wz) \notin A_\lambda(x) \cup A_\lambda(y)$, we put $\sigma(vx) = \sigma(wy) = \lambda(xy), \sigma(xy) = \lambda(wz), \sigma(wx) \in C - A_\lambda(x) \cup \{\lambda(wz)\}$ and $\sigma(vy) \in C - A_\lambda(y) \cup \{\lambda(wz)\}$.

4. 2. If $\lambda(wz) \in A_\lambda(x)$ (For $\lambda(wz) \in A_\lambda(y)$, the proof is similar), we put $\sigma(wy) = c_1 \in C - A_\lambda(y) \cup \{\lambda(wz)\}, \sigma(wx) = c_2 \in C - A_\lambda(x) \cup \{c_1\}$. When $A_\lambda(x) \cup \{c_2\} \neq A_\lambda(y) \cup \{c_1\}$, the edges vx and vy can be easily colored. When $A_\lambda(x) \cup \{c_2\} = A_\lambda(y) \cup \{c_1\}$, we let $a \in C - A_\lambda(x) \cup \{c_2\}$. If $a \neq \lambda(wz)$, then put $\sigma(vx) = c_2, \sigma(wx) = \sigma(vy) = a$. If $a = \lambda(wz)$, put $\sigma(vy) = \sigma(wx) = \lambda(xy), \sigma(xy) = a, \sigma(vx) = c_2$. Then we put $\sigma(v) \in C - \{\lambda(x), \lambda(y), \sigma(vx), \sigma(vy)\}, \sigma(w) \in C - \{\lambda(x), \lambda(y), \lambda(z), \sigma(wx), \sigma(wy), \sigma(wz)\}$.

Case 5. There is $v \in V_3(G), N_G(v) = \{w, x, y\}, w \in V_j(G), 3 \leq j \leq 4$. Set $H = G - vw$, and let (λ, C) be a 7-totally coloring of H. For G we put: $\sigma(vw) \in C - (A_\lambda(v) \cup A_\lambda(w) - \{\lambda(v)\}), \sigma(v) \in C - \{\lambda(x), \lambda(y), \lambda(w), \sigma(vx), \sigma(vy), \sigma(vw)\}$.

Case 6. There are $v, w \in V_3(G), x \in V_j(G), 3 \leq j \leq 6, y \in V_5(G), vx, wx, vy, wy, xy \in E(G)$. Assume that $N_G(v) = \{x, y, u_1\}, N_G(w) = \{x, y, u_2\}$. Obviously, $H = G - vy - wy$ has a 7-totally coloring (λ, C).

6. 1. If $A_\lambda(y) \cap \{\lambda(wx), \lambda(wu_2)\} \neq \emptyset$ (For the case $A_\lambda(y) \cap \{\lambda(vx), \lambda(vu_1)\} \neq \emptyset$, we shall have a similar proof), we put $\sigma(vy) \in C - A_\lambda(y) \cup \{\lambda(vx), \lambda(vu_1)\}, \sigma(wy) \in C - A_\lambda(y) \cup \{\lambda(wx), \lambda(wu_2), \sigma(vy)\}$.

6. 2. If $A_\lambda(y) \cap \{\lambda(vx), \lambda(wx), \lambda(vu_1), \lambda(wu_2)\} = \emptyset$, then when $\{\lambda(vx), \lambda(vu_1)\} \neq \{\lambda(wx), \lambda(wu_2)\}$, we put $\sigma(vy) \in C - A_\lambda(y) \cup \{\lambda(vx), \lambda(vu_1)\}, \sigma(wy) \in C - A_\lambda(y) \cup \{\lambda(wx), \lambda(wu_2)\}$; when $\{\lambda(vx), \lambda(vu_1)\} = \{\lambda(wx), \lambda(wu_2)\}$, we put $\sigma(vy) \in C - A_\lambda(y) \cup \{\lambda(vx), \lambda(vu_1)\}, \sigma(vx) = \sigma(wy) = \lambda(xy), \sigma(xy) = \lambda(vx)$.

Afterward we put $\sigma(v) \in C - \{\lambda(x), \lambda(y), \lambda(u_1), \lambda(vx), \sigma(vy), \sigma(vu_1)\}, \sigma(w) \in C - \{\lambda(x), \lambda(y), \lambda(u_2), \sigma(wx), \sigma(wy), \sigma(wu_2)\}$.

In Cases 3, 4, 5 and 6, the other uncolored elements of $V(G) \cup E(G)$ are colored with the same colors as in (λ, C) of H. We can always obtain a 7-totally coloring of G in all cases, and provide a contradiction with the choice of G. Thus $\chi_t(G) = 7$ for each $G \in \Sigma_6$. Similary we can prove that $\chi_t(G) = k + 1$ for $G \in \Sigma_k, k \geq 7$. □

By Theorem 3. 2 and the results of [4] and [5], we have

Theorem 3. 3. *The conjecture on the totally coloring is true for all 1-pseudo-outerplanar graphs.*

Theorem 3. 4. *If* $G \in \Gamma_k (k \geq 3)$, *then* $\chi_t(G) = k+1$.

Proof. Let us first prove that $\chi_t(G) = 4$ for $G \in \Gamma_3$. By Lemma 3. 1 we may suppsoe that G is 2-connected, thus $\delta(G) = 2$. Let u be a root of G and set $H = G - u$. Then $2 \leq \Delta(H) \leq 3$ and $\delta(H) = 1$.

If $\Delta(H) = 2, H$ is a path and G is an outerplanar graph. It follows from Lemma 3. 3 that

$\chi_t(G) = 4$.

If $\Delta(H) = 3$, in view of Lemma 2.4. $|V_1(H)| \geq \Delta(H) = 3$. Further it follows from $\delta(G) = 2$ and $\delta(H) = 1$ that each vertex in $V_1(H)$ is adjacent to u. Thus $|V_1(H)| \leq d_G(u) \leq \Delta(G) = 3$. So $|V_1(H)| = 3$. If $|V_3(H)| \geq 2$, then

$$\begin{aligned} 2p(H) - 2 &= |V_1(H)| + 3|V_3(H)| + 2(p(H) - |V_1(H)| - |V_3(H)|) \\ &= 3 + 3|V_3(H)| + 2(p(H) - 3 - |V_3(H)|) \\ &= 2p(H) + |V_3(H)| - 3 \geq 2p(H) - 1. \end{aligned}$$

This implies that $|V_3(H)| = 1$. Suppose $V_1(H) = \{v_1, v_2, v_3\}$ and $V_3(H) = \{w\}$. It is easily seen that $H = P_1 \cup P_2 \cup P_3$, where $P_i(i = 1,2,3)$ is a (w, v_i)- path satisfying $P_i \cap P_j = \{w\}$, $i \neq j$ and $G = Q_1 \cup Q_2 \cup Q_3$, where $Q_i = P_i \cup \{uv_i\}, i = 1,2,3$. In other words, G is the union of three internally-disjoint (u,w)- paths Q_1, Q_2 and Q_3 and $l(Q_i) \geq 2, i = 1,2,3$, where $l(Q_i)$ denotes the length of Q_i. If max $\{l(Q_1), l(Q_2), l(Q_3)\} \leq 3$, the theorem can be easily proved by the method of enumeration. If max $\{l(Q_1), l(Q_2), l(Q_3)\} \geq 4$, by Lemma 3.2, we can prove that $\chi_t(G) = 4$.

Second, we prove that $\chi_t(G) = 5$ for $G \in \Gamma_4$. Let $G \in \Gamma_4$ be a 1-pseudo-tree with the minimum number of edges which can not be colored with 5 colors. Obviously, $\delta(G) = 2$, and by Theorem 2.3, there are $v \in V_2(G), w \in V_k(G), k \leq 3$ such that $vw \in E(G)$, or there are $v, w \in V_2(G), N_G(w) = N_G(v) = \{x, y\}$ and $x, y \in V_4(G)$. With the similar discussion for Case 1 in Theorem 3.1 and for Case 3 in Theorem 3.2, we can form a 5-totally coloring of G in the above two cases, which contradicts the definition of G. Similarly, we can complete the proof for the case $k \geq 5$. □

Conjecture 3.1. *For* $G \in \Sigma_k \backslash \Omega_k (k \geq 4), \chi_t(G) = k+1$.

Since $K_4 \in \Sigma_3 \backslash \Omega_3$ and $\chi_t(K_4) = 5 = \Delta(K_4) + 2$, it follows that this conjecture is not true for $k = 3$.

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