On the Arc-Pancyclicity of Local Tournaments*

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ABSTRACT. Let \( T = (V, A) \) be a digraph with \( n \) vertices. \( T \) is called a local tournament if for every vertex \( x \in V \), \( T[O(x)] \) and \( T[I(x)] \) are tournaments. In this paper, we prove that every arc-3-cyclic connected local tournament \( T \) is arc-pancyclic except \( T \cong T_6^- \), \( T_8 \)-type digraphs or \( D_8 \).

1 Introduction

A digraph \( D = (V, A) \) consists a pair of \( V, A \), where \( V \) is a vertex set and \( A \) is an arc set. We say that \( x \) dominates \( y \) where \( x, y \in V \), denoted by \( x \rightarrow y \), if \( (x, y) \) is an arc of a digraph \( D \). Let \( S_1 \) and \( S_2 \) be two vertex subsets of \( V \). We say that \( S_1 \) dominates \( S_2 \), denoted by \( S_1 \rightarrow S_2 \), if there is a complete connection between \( S_1 \) and \( S_2 \) and all arcs between \( S_1 \) and \( S_2 \) are directed toward \( S_2 \). For convenience, we write \( x \rightarrow S_2 \) (resp., \( S_2 \rightarrow x \)) instead of \( \{x\} \rightarrow S_2 \) (resp., \( S_2 \rightarrow \{x\} \)). For any \( x \in V \) and any \( S \subseteq V \), We define

\[
O(x) = \{y | y \in V, (x, y) \in A\}, \quad I(x) = \{y | y \in V, (y, x) \in A\}
\]

\[
O_s(x) = O(x) \cap S, \quad I_s(x) = I(x) \cap S.
\]

A directed path of length \( k \) from \( x \) to \( y \) is denoted by \( P_k(x, y) \). A \( k \)-cycle containing arc \( (x, y) \) is denoted by \( C_k(x, y) \). The converse of \( D = (V, A) \)

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is defined as a digraph $\vec{D} = (V, \vec{A})$ such that $(x, y) \in \vec{A}$ if and only if $(y, x) \in A$.

A semicomplete digraph is a digraph without nonadjacent vertices. A locally semicomplete digraph is a digraph $D$ that satisfies the following condition for every vertex $x$ of $D$, $D[O(x)]$ and $D[I(x)]$ are semicomplete digraphs. A local tournament is a locally semicomplete digraph without directed cycles of length 2 and loops. A digraph $D$ is said to be arc-$k$-cyclic if each arc of $D$ is contained in a cycle of length $k(3 \leq k \leq n, n = |V|)$. An arc $e$ of $D$ is said to be pancyclic if it is contained in cycles of all length $m, 3 \leq m \leq n$. A digraph $D$ is said to be arc-pancyclic if each arc of $D$ is pancyclic.

Other notations and definitions not defined here can be found in [3].

2 The Main Results

The concept of locally semicomplete digraphs, which is a generalization of semicomplete digraphs or tournaments, was first introduced by J. Bang-Jensen [1]. Using this new concept, many classical theorems for tournaments have been generalized. For example:

Lemma 1 ([1] Theorems 3.2 and 3.3). A connected locally semicomplete digraph has a directed Hamiltonian path, and a strong locally semicomplete digraph has a directed Hamiltonian cyclic.

In this paper, we prove the following two theorems, which extend two theorems in [4] and [5] respectively. (See Corollaries 2 and 3 below)

Theorem 1. Every arc-3-cycle connected local tournament $T$ of order $n (n \geq 3)$ is arc-pancyclic, except $T \cong T_{6,7}, T_{8}$-type graphs or $D_8$. (See Figures 1, 2 and 3).

![Figure 1. $D_8$](image-url)
Figure 2.
$T_8$-type digraphs (The orientation of the edges without arrow can be chosen arbitrarily.)

Figure 3.
$T_6$-type digraphs ($T_1$ and $T_2$ both are arc-3-cyclic tournament. The orientation of the edges without arrow can be chosen arbitrarily.)

**Theorem 2.** At most one arc of an arc-3-cycle connected local tournament is not pancyclic.

**Corollary 1.** Let $T$ be a connected local tournament of order $n$. Then $T$ is arc-pancyclic if and only if $T$ is arc-3-cyclic and arc-$n$-cyclic.

**Corollary 2 ([5], Theorem 1).** Let $T$ be a tournament of order $n$. Then $T$ is arc-pancyclic if and only if $T$ is arc-3-cyclic and arc-$n$-cyclic.

**Corollary 3.** ([4], Theorem 1). Except for $T_6$-type digraphs and $T_8$-type digraphs, every arc-3-cyclic tournament is arc-pancyclic.
The proofs of our results are given in the next section.

3 The Proofs Of Theorems

In the following, we shall assume that $T = (V, A)$ is an arc-3-cyclic connected local tournament of order $n$. In order to prove Theorem 1, we need the following lemmas.

Lemma 2. ([2], Corollary 3.13). Let $P_1 = (x_1, x_2, \ldots, x_m)$ and $P_2 = (y_1, y_2, \ldots, y_t)$, $m \geq 2$, $t \geq 3$, be paths in $T$. If there exist $i, j, 1 \leq i < j \leq m$ such that $x_i = y_1$ and $x_j = y_t$ and $V(P_1) \cap V(P_2) - \{y_1, y_t\} = \emptyset$, then $T$ has an $(x_1, x_m)$-path $P$ such that $V(P) = V(P_1) \cup V(P_2)$.

If $T$ were not arc-pancyclic, then there is an arc $e = (k, 1)$ in $T$ such that $e$ is contained in one of $m$-cycles, $3 \leq m \leq k < n$, but $e$ is not contained in any $(k+1)$-cycle. i. e.

There does not exist any $P_k(1, k)$ in $T$. \hspace{1cm} (\ast)

Let $C = C_k(e) = (1, 2, \ldots, k, 1)$ be a $k$-cycle containing $e$. Without ambiguity, we also let $C$ be the set of itself's vertices. Let $W = V - C = V - \{1, 2, \ldots, k\}$, thus $|W| \geq 1$. If $O_c(w) \neq \emptyset$ and $I_c(w) \neq \emptyset$ for $w \in W$, we define:

$$a(w) = \max\{i|i \in O_c(w)\}, b(w) = \min\{i|i \in I_c(w)\}.$$

Lemma 3. If $T$ satisfies (\ast), then $T[W]$ is a tournament, and then $O_c(w) = \{1, 2, \ldots, a(w)\} \neq \emptyset$ and $I_c(w) = \{b(w), b(w)+1, \ldots, k\} \neq \emptyset$ for any $w \in W$.

**Proof:** We prove the following two assertions:

(a) $O_c(w) \neq \emptyset$ for $w \in W$ if and only if $I_c(w) \neq \emptyset$.

If $O_c(w) \neq \emptyset$, set $i = \min\{j|j \in O_c(w)\}$. Suppose that $i > 1$. By the definition of a local tournament and $\{w, i-1\} \subseteq I(i)$, we have that $i-1$ and $w$ are adjacent in $T$. Thus by the definition of $i$, we have $i-1 \rightarrow w$. Hence $T$ contains a $P_k(1, k) = (1, 2, \ldots, i-1, w, i, \ldots, k)$. This is a contradiction to (\ast). So $i = 1$.

From the above arguments, we also have $O_c(w) = \{1, 2, \ldots, a(w)\}$. If $a(w) = k$, then $w \rightarrow C$. Hence, since $T$ is arc-3-cyclic, there exists a 3-cyclic $C_3(w, 1) = (w, 1, x, w)$ with $x \in W$. Thus $T$ contains a $P_k(1, k) = (1, x, w, 3, \ldots, k)$. This is a contradiction to (\ast). So $a(w) < k$.

Similarly, we have $I_c(w) = \{b(w), \ldots, k\}$ and $b(w) > 1$ when $I_c(w) \neq \emptyset$.

Now if $O_c(w) \neq \emptyset$, then there is a $C_3(w, 1) = (w, 1, x, w)$. If $x \in W$, then $1 \in I_c(x)$ and $b(x) = 1$. This contradicts $b(w) > 1$ for any $w \in \{w|w \in W, I(w) \neq \emptyset\}$. Hence $x \in C$, i.e., $x \in I_c(w)$ and $I_c(w) \neq \emptyset$. Similarly, if $I_c(w) \neq \emptyset$, then $O_c(w) \neq \emptyset$ for $w \in W$.
(b) Let $W_1 = \{w|w \in W, O_c(w) \neq \emptyset\}$ and $W_2 = W - W_1$. Then $W_1 = W$.

Since $T$ is connected and arc-3-cyclic, we have $W_1 \neq \emptyset$. Suppose that $W_2 \neq \emptyset$. Thus for any $z \in W_2$, we have $O_c(z) = I_c(z) = \emptyset$ by (a). Since $T$ is connected, there exist $x \in W_1$ and $y \in W_2$ such that $x$ and $y$ are adjacent. Without loss of generality, we assume $x \rightarrow y$. (Otherwise, we consider the converse of $T$). Since $O_c(x) \neq \emptyset$ and $x \rightarrow 1, 1$ and $y$ are adjacent and $1 \in O_c(y)$ by (a), which is a contradiction. Hence $W_2 = \emptyset$. i.e., $W = W_1$.

From (a) and (b), we have that $W \subseteq I(1)$. Hence $T[W]$ is a tournament by the definition of a local tournament. So Lemma 3 is valid.

For any $w \in W$, we define:
\[ p(w) = \min\{i|i \in O(1) \cap I_c(w)\}, \quad q(w) = \max\{i|i \in I(k) \cap O_c(w)\}. \]

**Lemma 4.** If $T$ satisfies $\ast$, then $O(1) \cap I_c(w) \neq \emptyset$, $I(k) \cap O_c(w) \neq \emptyset$ and $2 \leq q(w) \leq a(w) < b(w) \leq p(w) \leq k - 1$ for any $w \in W$.

**Proof:** There is a $C_3(w, 1) = (w, 1, x, w)$. We have $x \notin W$ by Lemma 3. Thus $x \in O(1) \cap I_c(w)$ by $x \rightarrow w$. And $b(w) \leq x \leq k - 1$ since $k \rightarrow 1$. Similarly, we have $y \in I(k) \cap O_c(w)$ and $2 \leq y \leq a(w)$. By the definitions of $p(w)$ and $q(w)$, we have $2 \leq y \leq q(w) \leq a(w) < b(w) \leq p(w) \leq x \leq k - 1$ for every $w \in W$. So Lemma 4 is valid.

**Lemma 5.** If $T$ satisfies $\ast$, then $b(w) = b(w')$ and $a(w) = a(w')$ for every $w, w' \in W$. And $T[W]$ is an arc-3-cyclic tournament.

**Proof:** Suppose that there are two distinct vertices $w, w'$ in $W$ such that $b(w) \neq b(w')$. Let $w_0 \in W$ be chosen such that $b(w_0) = \min\{b(w)|w \in W\}$. Let $W_1 = \{w|w \in W, b(w) > b(w_0)\}$ and $W_2 = W - W_1$. Then $W_1 \neq \emptyset$, $W_2 \neq \emptyset$ and $b(w_0) = b(w) \rightarrow w$ for every $w \in W$. Suppose that there exist $w_1 \in W_1$ and $w_2 \in W_2$ such that $w_1 \rightarrow w$. Since $w_1 \rightarrow w_2$ and $b(w_2) \rightarrow w_2$, we know that $w_1$ and $b(w_2)$ are adjacent and $w_1 \rightarrow b(w_2)$ by $b(w_1) > b(w_0) = b(w_2)$. Hence $a(w_1) \geq b(w_2)$. By the definitions of $p(w_1)$ and $q(w_2)$, we have that
\[ 2 \leq q(w_2) \leq a(w_2) < b(w_2) \leq a(w_1) < b(w_1) \leq p(w_1) \leq k - 1 \quad (A) \]
and hence $q(w_2) + 1 \leq p(w_1) - 1$.

When $q(w_2) + 1 = p(w_1) - 1$, we have that $q(w_2) = a(w_2) = b(w_2) - 1$ and $b(w_2) - a(w_2) = b(w_1) - l = p(w_1) - 1$ from (A). Then we have $P_k(1, k) = (1, p(w_1), \ldots, k - 1, w_1, w_2, 2, \ldots, q(w_2), k)$ in $T$, a contradiction. Hence $q(w_2) + 1 \leq p(w_1) - 2$. Thus it follows that either $p(w_1) - 2 \geq b(w_2)$ or $q(w_2) + 2 \leq a(w_1)$ by (A). We have either $P_k(1, k) = (1, p(w_1), \ldots, k - 1, w_1, q(w_2) + 1, \ldots, p(w_1) - 2, w_2, 2, \ldots, q(w_2), k)$ if $p(w_1) - 2 \geq b(w_2)$ or $P_k(1, k) = (1, p(w_1), \ldots, k - 1, w_1, q(w_2) + 2, \ldots, p(w_1) - 1, w_2, 2, \ldots, q(w_2), k)$
if \(q(w_2) + 2 \leq a(w_1)\). These are contradictions. Hence no vertex of \(W_1\) dominates any vertex of \(W_2\). \(W_2 \rightarrow W_1\) since \(T[W]\) is a tournament.

Let \(w_1 \in W_1\) and \(w_2 \in W_2\), then \(w_2 \rightarrow w_1\) and \(b(w_0) = b(w_2) \rightarrow w_2\). There is a \(C_3(w_2, w_1, x, w_2)\). \(x \notin W\) since \(W_2 \rightarrow W_1\). Hence we have \(x \in C\) and \(a(w_1) \geq x \geq b(w_2)\). Thus we have that: \(q(w_2) \leq a(w_2) < b(w_2) \leq a(w_1) < b(w_1) \leq p(w_1)\). And hence \(q(w_2) + 1 \leq p(w_1) - 1\). As above, we can also prove that \(T\) contains a \(P_k(1, k)\), a contradiction. Therefore \(b(w_1) = b(w_2)\) for any \(w_1, w_2 \in W\). Similarly, we can prove \(a(w_1) = a(w_2)\) for any \(w_1, w_2 \in W\). Hence \(T[W]\) is an arc-3-cyclic tournament. So Lemma 5 is valid.

By Lemma 5, we denote \(a = a(w)\) and \(b = b(w)\) for each \(w \in W\). Thus by Lemma 3 and Lemma 4, We have \(2 \leq a < b \leq k - 1\), \(O_c(w) = \{1, 2, \ldots, a\}\), and \(I_c(w) = \{b, b+1, \ldots, k\}\). Hence \(T[\{1, 2, \ldots, a\}]\) and \(T[\{b, b+1, \ldots, k\}]\) both are tournaments.

**Lemma 6.** If there are \(a < \gamma < \delta\) in \(C\) such that \(1 \leq \alpha \leq a - 1\), \(a + 1 < \gamma \leq \delta \leq k\), \(b + 1 \leq \delta\), \((a, \gamma) \in A\) and \((\gamma - 1, \delta) \in A\). Then \(T\) contains a \(P_k(1, k)\).

**Proof:** Let \(\alpha, \gamma\) and \(\delta\) satisfy the conditions of Lemma 6 and \(w \in W\). Then there is \(P_k(1, k) = (1, 2, \ldots, \alpha, \gamma, \ldots, \delta - 1, w, \alpha + 1, \ldots, \gamma - 1, \delta, \ldots, k)\).

Furthermore, we shall use the following symbols. For \(1 \leq m \leq a\), \(b \leq l \leq k\), we denote:

\[
R(m) = \{i|b \leq i \leq k, (m, i) \in A\}, \quad L(l) = \{i|1 \leq i \leq a, (i, l) \in A\}.
\]

Thus for any \(w \in W\), \(1 \leq m \leq a\) and \(b \leq l \leq k\), since there exist \(C_3(w, m)\) and \(C_3(l, w)\), it is easy to see that \(R(m) \neq \emptyset\), \(L(l) \neq \emptyset\) and \(k \notin R(1)\), \(1 \notin L(k)\). Hence we can define:

\[
\psi(m) = \max\{i|i \in R(m)\}, \quad \varphi(l) = \min\{i|i \in L(l)\}, \quad p = \min\{i|b \leq i \leq k - 1, (1, i) \in A\}, \quad q = \max\{i|2 \leq i \leq a, (i, k) \in A\}.
\]

Then \((m, \psi(m)), (\varphi(l), l), (1, p), (q, k) \in A\) and \(b \leq \psi(m) \leq k\), \(1 \leq \varphi(l) \leq a\), \(2 \leq q \leq a < b \leq p \leq k - 1\) for any \(1 \leq m \leq a\) and \(b \leq l \leq k\).

**Lemma 7.** If \(T\) satisfies (*) and \(b > a + 1\), then \(T \cong D_8\).

**Proof:** First, we have \(\{a + 1, \ldots, b - 1\} \neq \emptyset\), and \(i\) and \(w\) are nonadjacent for any \(i \in \{a + 1, \ldots, b - 1\}\) and any \(w \in W\). There is a \(C_3(a, a + 1) = (a, a + 1, x, a)\) in \(T\). Obviously \(x \notin W\). If \(x \in \{a + 2, \ldots, b - 1\}\), then \(x\) and \(w\) are adjacent by \(x \rightarrow a\) and \(w \rightarrow a\), a contradiction. So \(x \notin \{a + 2, \ldots, b - 1\}\). Since \(w \rightarrow i\) for \(i \in \{1, 2, \ldots, a - 1\}\), \(a + 1 \rightarrow x\), \(a + 1\) and \(w\) are nonadjacent.
We have $x \notin \{1, 2, \ldots, a-1\}$. Thus, $x \in \{b, b+1, \ldots, k\}$. Suppose $x = b$, i.e., $b \rightarrow a$, then $\varphi(b) < a$ and $\psi(a) > b$. $\varphi(b)$ and $b-1$ are adjacent by $\varphi(b) \rightarrow b$ and $b-1 \rightarrow b$. Since $b-1$ and $w$ are nonadjacent and $w \rightarrow \varphi(b)$, $\varphi(b) \rightarrow b-1$. Similarly, we can get $\varphi(b) \rightarrow \{a+1, \ldots, b-1\}$. Let $\alpha = \varphi(b)$, $\gamma = a+1$ and $\delta = \psi(a)$, then by Lemma 6 there is a $P_k(1, k)$ in $T$. This is a contradiction to (x). Hence $x > b$. Similarly, using $C_3(b-1, b) = (b-1, b, y, b-1)$, we have $y < a$.

If $b > a + 2$, $x$ and $a + 2$ are adjacent since $a + 1 \rightarrow x$ and $a + 1 \rightarrow a + 2$. Since $a + 2$ and $w$ are nonadjacent and $x \rightarrow w$ for any $w \in W$, $a + 2 \rightarrow x$ by the definition of a local tournament. Similarly, $\{a+1, \ldots, b-1\} \rightarrow x$. Let $\alpha = y(<a)$, $\gamma = b - 1$ and $\delta = x(>b)$. By Lemma 6 there is a $P_k(1, k)$ in $T$, a contradiction. Hence $b = a + 2$.

Furthermore, $a + 1$ and $x - 1$ are adjacent since $a + 1 \rightarrow x$ and $x - 1 \rightarrow x$. Then $a + 1 \rightarrow x - 1$ by the fact that $x - 1 \rightarrow w$, $w$ and $a + 1$ are nonadjacent. Similarly, we have

$$b - 1 = a + 1 \rightarrow \{b + 1, \ldots, x - 1, x\}$$

where

(B)

Now the following three cases must be considered:

Case 1. $k > b + 1$ and $a > 2$.

If $\varphi(b) < a$, then we may choose $\alpha = \varphi(b)$, $\gamma = b$ and $\delta = x$. Hence there is a $P_k(1, k)$ in $T$ by Lemma 6. This is a contradiction. So $\varphi(b) = a$, i.e., $a \rightarrow b$. Since $1, a \in O(w)$, 1 and $a$ must be adjacent. Suppose $1 \rightarrow a$. If $\varphi(a - 1) > b$, then we may choose $\alpha = 1$, $\gamma = a$ and $\delta = \psi(a - 1)$. There is a $P_k(1, k)$ in $T$ by Lemma 6. This is a contradiction. So $\psi(a - 1) = b$ since $\psi(a - 1) \geq b$, i.e. $a - 1 \rightarrow b$. Now, let $\alpha = a - 1$, $\gamma = b$ and $\delta = x$, there is also a $P_k(1, k)$ in $T$ by Lemma 6, a contradiction. Hence we always assume that

$$a \rightarrow 1 \text{ and } a \rightarrow b$$

in the following arguments.

1) $\{1, 2, \ldots, a-1\} \rightarrow a + 1$.

1 $\rightarrow a + 1$ since $a + 1$ and $w$ are nonadjacent and 1, $a + 1 \in O(a)$.

Furthermore, $2 \rightarrow a + 1$ since $1 \rightarrow 2$ and $1 \rightarrow a + 1$. Similarly, we have $\{1, 2, \ldots, a - 1\} \rightarrow a + 1$.

2) $b \rightarrow 1$, $a + 1 \rightarrow k$ and $j \rightarrow b$ for each $j \in \{b + 2, \ldots, k\}$.

If there exists a $j \in \{b + 2, \ldots, k\}$ such that $b \rightarrow j$. Then $T$ contains a $P_k(1, k) = (1, 2, \ldots, a - 1, a + 1, b + 1, \ldots, j - 1, w, a, b, j, \ldots, k)$ by 1), (B) and (C). This is a contradiction. So $\{b + 2, \ldots, k\} \rightarrow b$.

Since $k, a + 1 \in I(b)$, we have that $k$ and $a + 1$ are adjacent. Furthermore, $a + 1 \rightarrow k$ since $k \rightarrow w$ and $a + 1$ and $w$ are nonadjacent.
Since $1, b \in O(k)$, $1$ and $b$ are adjacent. If $1 \rightarrow b$, then we may choose $\alpha = 1$, $\gamma = b$ and $\delta = k$. Then there is a $P_k(1, k)$ in $T$ by Lemma 6. This is a contradiction. Hence $b \not\rightarrow 1$.

3) $k = b + 2$ and $p = b + 1$.

$p > b$ since $b \rightarrow 1$. If $k - 1 \geq b + 2$, then $T$ contains a $P_k(1, k) = (1, p, \ldots, k - 1, b, \ldots, p - 1, w, 2, \ldots, b - 1 = a + 1, k)$ by 2). This is a contradiction. Hence $k = b + 2$ and $p = b + 1$.

4) $(a - 1, b) \notin A$, $a = 3$, $p = 6$ and $k = 7$.

Note that $\psi(a - 1) \in \{b, b + 1, b + 2 = k\}$. If $\psi(a - 1) = b$, then we may choose $\alpha = a - 1$, $\gamma = \psi(a - 1) = b = a + 2$ and $\delta = k$. By 2) and Lemma 6, there is a $P_k(1, k)$ in $T$. This is a contradiction. So $\psi(a - 1) > b$ and $(a - 1, b) \notin A$.

If $a - 1 > 2$, then we have either $P_k(1, k) = (1, 2, a, \ldots, \psi(a - 1) - 1, w, 3, \ldots, a - 1, \psi(a - 1), \ldots, k)$, if $2 \rightarrow a$ or $P_k(1, k) = (1, a + 1, \ldots, \psi(a - 1) - 1, w, a, 2, \ldots, a - 1, \psi(a - 1), \ldots, k)$ by 1), if $a \rightarrow 2$. These contradict to $(*)$. Hence $a \leq 3$. Thus $a = 3$ by the assumption that $a > 2$. Finally by 3) we have $p = b + 1 = a + 3 = 6$ and $k = b + 2 = a + 4 = 7$.

5) $x = k$ and $q = 2$ (hence $a + 1 = x = k \rightarrow a$).

Suppose $x < k = b + 2$. $x = b + 1$ since $x > b$. By 3) and the choice of $x$, we have $p = b + 1 = x \rightarrow a$. Hence there is a $P_k(1, k) = (1, p = x, a, b, w, 2, a + 1, k)$ by 1), 2) and (C). This is a contradiction to $(*)$. Hence $x = k$ and $q = 2$.

6) $b + 1 \rightarrow 2$.

2 and $b + 1$ are adjacent since $2, b + 1 = p \in O(1)$. If $2 \rightarrow b + 1$, then 2 and $b$ are adjacent by $b \rightarrow b + 1$. $b \rightarrow 2$ since $(2, b) = (a - 1, b) \notin A$ by 4). Then there is a $P_k(1, k) = (1, p = b + 1, w, a, a + 1, b, 2 = q, k)$. This is a contradiction. So $b + 1 \rightarrow 2$.

7) $|W| = 1$.

Suppose that there is a $w_0 \in W - \{w\}$. Without loss of generality, let $w \rightarrow w_0$. Then there is $P_k(1, k) = (1, p = b + 1, w, w_0, 2, a, a + 1, k)$ by 2). This is a contradiction.

8) 2 and 5, 3 and 6 are nonadjacent.

Otherwise, there is a $P_k(1, k)$ in $T$. For example if $(6, 3) \in A$, there exists a $P_k(1, k) = (1, p = 6, 3 = a, a + 1, b, w, 2 = q, k)$. This is a contradiction.

Up to now, we have proved that $T \cong D_8$ (See Figure 1) in this subbase.

Case 2. $k = b + 1$ and $a \geq 2$.

Since $b < x \leq k = b + 1$ and $b \leq p < k = b + 1$, $x = k$ and $p = b$, i.e. $a + 1 \rightarrow x = k$ and $1 \rightarrow p = b = a + 2$. Let $\alpha = 1$, $\gamma = a + 2$ and $\delta = k$, there is a $P_k(1, k)$ in $T$ by Lemma 6. This is a contradiction.

Case 3. $a = 2$ and $k \geq b + 2$.  

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Consider the converse of T. Note that Case 3 in T is Case 2 in T. So Lemma 7 is valid.

**Lemma 8.** If $T$ satisfies (*) and $b = a + 1$, then $T$ is a $T_6$- or $T_8$-type digraph.

**Proof:** We consider the following two cases.

**Case 1.** $|W| \geq 2$ (let $w, w' \in W$)

Suppose $p > a + 1$, $q < a$ and $k > 6$. Then there exists an $i \in \{1, 2, \ldots, k\} - \{1, q, a, a + 1, p, k\}$. If $1 < i < q$, then $q \geq 3$ and there is a $P_k(1, k) = (1, p, \ldots, k - 1, w, q + 1, \ldots, a, a + 1, \ldots, p - 1, w', 3, \ldots, q, k)$. Similarly, $T$ contains a $P_k(1, k)$ when $q < i < a$ or $a + 1 < i < p$ or $p < i < k$. These are contradictions. If $p > a + 1$, $q < a$ and $k = 6$, then $q = 2$, $a = 3$ and $p = 5$. Because $|W| \geq 2$ and $T[W]$ is an arc-3-cyclic tournament by Lemma 5, we have $|W| \geq 3$. Let $\{w_1, w_2, w_3\} \subseteq W$ and $w_1 \rightarrow w_2 \rightarrow w_3$. Then $T$ contains a $P_k(1, k) = (1, p, w_1, w_2, w_3, q, k)$. This is a contradiction. Hence we have $p = a + 1$ or $q = a$.

In the following we may assume that, without loss of generality, $p = a + 1$ (Otherwise $q = a$, we can consider the converse of $T$). Thus $1 \rightarrow a + 1 = b$. Now we can obtain the following assertions.

9) $q < a$ (therefore $(a, k) \notin A$).

If $q = a$, then $a = q \rightarrow k$. There is a $P_k(1, k) = (1, p = a + 1, \ldots, k - 1, w, 2, \ldots, a, k)$, a contradiction.

10) $k = a + 2$, $V_1 = \{q + 1, \ldots, a\} \rightarrow a + 1$ and $T[V_1]$ is a tournament.

Suppose $k > a + 2$. If $\varphi(a + 2) = a$, let $\alpha = 1$, $\gamma = a + 1$ and $\delta = a + 2$, then there is a $P_k(1, k)$ in $T$ by Lemma 6. This is a contradiction. Hence $\varphi(a + 2) < a$. Since $a + 1, k \in I(w)$, $a + 1$ and $k$ are adjacent. If $a + 1 \rightarrow k$, let $a = \varphi(a + 2)$, $\gamma = a + 2$ and $\delta = k$ in Lemma 6, then there is a $P_k(1, k)$ in $T$, a contradiction. Hence $k \rightarrow a + 1$. Thus $a$ and $k$ are adjacent by $a \rightarrow a + 1$. Hence $k \rightarrow a$ by 9). There is a $C_3(k, a) = (k, a, z, k)$. Obviously, $z \notin W$, $z \neq 1$, $z \neq a + 1$ and $z \notin \{q + 1, \ldots, a - 1\}$ by the definition of $q$. Let $P_1 = (1, p = a + 1, \ldots, k - 1, w, 2, \ldots, z, k)$ and $P_2 = (w, z + 1, \ldots, a, z)$. If $z \in \{2, \ldots, q\}$, then $P_1$ and $P_2$ satisfy the condition of Lemma 2, hence there is a $P_k(1, k)$ in $T$. This is a contradiction. So $z \in \{a + 2, \ldots, k - 1\}$. Thus there is a $P_k(1, k) = (1, p = a + 1, \ldots, z - 1, w, 2, \ldots, a, z, k)$ in $T$, a contradiction too. Hence $k = a + 2$.

Let $V_1 = \{q + 1, \ldots, a\}$. Then $T[V_1]$ is a tournament by $V_1 \subseteq O(w)$. Since $k = a + 2$ and by the definition of $q$, $\psi(j) = a + 1$ for each $j \in V_1$, that is $V_1 \rightarrow a + 1$.

11) $T[V_1]$ is a strong tournament.

If not, then $|V_1| \geq 2$ and $q + 1 \rightarrow a$. There is a $C_3(q + 1, a) = (q + 1, a, y, q + 1)$. Obviously $y \notin W$. By $q \rightarrow k$ and Lemma 6, $y \notin \{1, 2, \ldots, q - 1\}$. $y \neq k$ by 9). And $y \neq a + 1$ by 10). Since $T[V_1]$ is not strong, $y \notin V_1$. 

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Hence \( y = q \). i.e. \( a \to y = q \). Let \( P_1 = (1, p, a + 1, w, z, \ldots, q, k) \) and \( P_2 = (w, q + 1, \ldots, a, q) \). Then \( P_1 \) and \( P_2 \) satisfy the conditions of Lemma 2 and there is a \( P_k(1, k) \) in \( T \), a contradiction. So \( T[V_1] \) is a strong tournament.

12) \( q = 2 \) (therefore \( 2 \to k \)) and \( V_1 \to 1 \).

Suppose \( q \geq 3 \). By \( q \to k \) and Lemma 6, we have \( q + 1 \to 2 \). We may assume that \((q + 1, h, \ldots, q + 1)\) is a Hamilton cycle in \( T[V_1] \) by 11. Then there is a \( P_k(1, k) = (1, p = a + 1, w, h, \ldots, q + 1, 2, \ldots, q, k) \), a contradiction. Hence \( q = 2 \).

Now we show that \( V_1 \to 1 \). \( T[[1, 2] \cup V_1] \) is a tournament since \( \{1, 2\} \cup V_1 \subseteq O_0(w) \). Suppose there exists an \( x \in V_1 \) such that \( 1 \to x \). Let \((x, \ldots, h, x)\) be a Hamilton cycle in \( T[V_1] \). Then there is a \( P_k(1, k) = (1, x, \ldots, h, a + 1, w, 2, k) \) by 10. This is a contradiction. Hence \( V_1 \to 1 \).

13) \( T \) is a tournament.

In fact, \( T[[1, 2] \cup V_1] \) is a tournament since \( \{1, 2\} \cup V_1 \subseteq O(w) \). \( T[V_1 \cup \{k\}] \) is a tournament since \( V_1 \cup \{k\} \subseteq I(1) \), \( 2 \) and \( a + 1 \) are adjacent by \( 1 \to 2 \) and \( 1 \to p = a + 1 \). Hence \( T \) is a tournament by 10) and 12).

Therefore by 13 ) and \( p = a + 1 \), using the result of (9) case (i) in the proof of Theorem 1 of [4], we get that \( T \) is a \( T_6 \)-type digraph (See Figure 3) in this case.

Case 2. \(|W| = 1\).

Let \( W = \{w\} \). If \( p = a + 1 \) or \( q = a \), then we can obtain that \( T \) is a \( T_6 \)-type digraph by a similar argument in Case 1. Hence in the following we may assume that \( p > a + 1 \) and \( q < a \). And we can obtain the following assertions.

14) There is either \( \psi(a) = a + 1 \) or \( \varphi(a + 1) = a \).

Suppose \( \psi(a) > a + 1 \) and \( \varphi(a + 1) < a \), let \( \alpha = \varphi(a + 1) \), \( \gamma = a + 1 \) and \( \delta = \psi(a) \) in Lemma 6, then there is a \( P_k(1, k) \) in \( T \). This is a contradiction. Hence \( \psi(a) = a + 1 \) or \( \varphi(a + 1) = a \).

Without loss of generality, let \( \psi(a) = a + 1 \). Otherwise we can consider the converse of \( T \). Then \((a, j) \notin A \) and \( 1 \leq \varphi(j) < a \) for each \( j \in \{a + 2, \ldots, k\} \). Because \( \psi(\varphi(j)) \geq j \geq a + 2 \) for each \( j \in \{a + 2, \ldots, k\} \), we may define:

\[
m = \max\{i | 1 \leq i \leq a - 1, \psi(i) \geq a + 2\}, \quad V_2 = \{m + 1, \ldots, a\}.
\]

By the definition of \( m \) and \( q \), we have that \( 2 \leq q \leq m < a \), \( \psi(m) \geq a + 2 \) and \( i \) does not dominate any vertex of \( \{a + 2, \ldots, k\} \) for each \( i \in V_2 \). Hence \( \psi(i) = a + 1 \) for each \( i \in V_2 \).

15) \( m + 1 \to \{1, 2, \ldots, m - 1\} \) and \( q + 1 \to \{1, 2, \ldots, q - 1\} \).

Suppose there exists \( j \in \{1, 2, \ldots, m - 1\} \) such that \( j \to m + 1 \). Let \( \alpha = j \), \( \gamma = m + 1 \) and \( \delta = \psi(m) \). Then there is a \( P_k(1, k) \) in \( T \) by Lemma 6. This is
a contradiction. Hence $m+1 \rightarrow \{1, 2, \ldots, m-1\}$ since $T$ is a tournament. Similarly, we have that $q+1 \rightarrow \{1, 2, \ldots, q-1\}$.

16) $k = a + 3$, $p = a + 2$ and $(a + 1, 1) \in A$.

Since $a + 1 < p \leq k - 1$, $k \geq a + 3$. Suppose $k > a + 3$. Then we can obtain the following results.

(a) $k - 1 \rightarrow a + 1$.

Suppose $a + 1 \rightarrow k - 1$. By $\psi(a) = a + 1$, we have $\varphi(a + 2) \leq a$. Then let $\alpha = \varphi(a + 2)$, $\gamma = a + 2$ and $\delta = k - 1$, there is a $P_k(1, k)$ in $T$ by Lemma 6. This is a contradiction. Hence $k - 1 \rightarrow a + 1$ since $k - 1$, $a + 1 \in I(w)$.

(b) $k - 1 \rightarrow V_2 = \{m + 1, \ldots, a\}$.

$a$ and $k - 1$ are adjacent since $k - 1$, $a \in I(a + 1)$. So $k - 1 \rightarrow a$ since $\psi(a) = a + 1$. Similarly, by (D), we have $k - 1 \rightarrow V_2 = \{m + 1, \ldots, a\}$.

(c) $m = 2$.

Suppose $m \geq 3$. Since $m+1 \rightarrow 2$ by (15) and $k - 1 \rightarrow m+2$ by $a+1 \geq m+2$ and (a), (b), there is a $P_k(1, k) = (1, p, \ldots, k - 1, m + 2, \ldots, p - 1, w, q + 1, \ldots, m + 1, 2, q, k)$. This is a contradiction. So $m = 2$.

Now we have $2 = q \rightarrow k$ since $2 \leq q \leq m = 2$. And $k - 1 \rightarrow m + 1$ by (b). Thus there is a $P_k(1, k) = (1, p, \ldots, k - 1, m + 1 = 3, \ldots, p - 1, w, 2 = m = q, k)$. This is a contradiction. Hence $k = a + 3$ and $p = a + 2$.

Since $1 \rightarrow p = a + 2$ and $a + 1 \rightarrow a + 2$, 1 and $a + 1$ are adjacent. Thus $a + 1 \rightarrow 1$ by the definition of $p$.

17) $k \rightarrow \{q + 1, \ldots, a, a + 1\}$. Therefore $k \rightarrow V_2$.

Suppose $a + 1 \rightarrow k$, let $\alpha = 1$, $\gamma = p = a + 2$ and $\delta = k$ in Lemma 6, then there is a $P_k(1, k)$ in $T$. This is a contradiction. Hence $k \rightarrow a + 1$. Since $k \rightarrow a + 1$, $a \rightarrow a + 1$ and $\psi(a) = a + 1$, we have $k \rightarrow a$. Since $a - 1 \rightarrow a$, $a - 2 \rightarrow a - 1$, $q + 1 \rightarrow q + 2$ and the definition of $q$, we can obtain $k \rightarrow a - 1$, $\ldots$, $k \rightarrow q + 1$ one by one. That is, $k \rightarrow \{q + 1, \ldots, a, a + 1\}$ and $k \rightarrow V_2$.

18) $m \geq 3$.

Suppose $m < 3$. Then $m = 2$ and $q = 2$. By 17) $(k, a) \in A$. There is a $C_3(k, a) = (k, a, x, k)$. Obviously, $x \neq w$, $x \neq a + 1$ since $k \rightarrow a + 1$ in 17).

We also have $x \neq a + 2$ since $\psi(a) = a + 1$. $x \neq 1$ since $k \rightarrow 1$, and $x \notin V_2$ by 17). Hence $x = 2 = m$. That is, $a \rightarrow 2$.

By 16) we have $a + 2 = p \in O(1)$. So $2$ and $a + 2$ are adjacent. If $2 \rightarrow a + 2$, $a + 2$ and $m + 1$ are adjacent since $2 = m \rightarrow m + 1$. Thus $a + 2 \rightarrow m + 1$ by the definition of $m$. There is a $P_k(1, k) = (1, p = a + 2, m + 1 = 3, \ldots, a + 1, w, m = 2 = q, k)$, a contradiction. Hence $a + 2 \rightarrow 2$.

Since $a \rightarrow 2$, $a$ and $a + 2$ are adjacent. $a + 2 \rightarrow a$ since $\psi(a) = a + 1$. When $m + 1 \leq a - 1$, we have $a + 2 \rightarrow a - 1$ since $a - 1 \rightarrow a$ and the definition of $m$. Similarly, we have $a + 2 \rightarrow a - 2$, $\ldots$, $a + 2 \rightarrow m + 1$. Then there is a
19) $2 \rightarrow a + 1$ and $m \rightarrow a$.

In fact, we have $m + 1 \rightarrow a + 1$ by (D), and $m + 1 \rightarrow 2$ by 15) and 18). Hence 2 and $a + 1$ are adjacent. If $a + 1 \rightarrow 2$, then there is a $P_k(1, k) = (1, p = a + 2, w, q + 1, \ldots, a + 1, 2, \ldots, q, k)$. This is a contradiction. So $2 \rightarrow a + 1$.

Suppose $a \rightarrow m$. Since $\psi(m) \in \{a + 2, a + 3 = k\}$, we have $P_1 = (1, 2, a + 1, w, 3, \ldots, m, a + 2, k)$ (if $\psi(m) = a + 2$) or $P_1 = (1, 2, a + 1, a + 2, w, 3, \ldots, m, k)$ (if $\psi(m) = a + 3$) and $P_2 = (w, m + 1, \ldots, a, m)$. Clearly $P_1$ and $P_2$ satisfy the conditions of Lemma 2, hence $T$ contains a $P_k(1, k)$. This is a contradiction. So $m \rightarrow a$.

20) $T[V_2]$ is a strong tournament.

If not, then $|V_2| \geq 2$ and $m + 1 \rightarrow a$. There is a $C_3(m + 1, a) = (m + 1, a, x, m + 1)$ in $T$. Obviously, we have $x \neq w$, $x \notin \{1, 2, \ldots, m - 1\}$ by 15), $x \neq k$ by 17), $x \neq a + 1$ by (D), $x \neq a + 2$ by 14), $x \neq m$ by 19), and $x \notin V_2$ since $T[V_2]$ is not strong. Thus there is no $C_3(m + 1, a)$ in $T$. This contradicts the fact that $T$ satisfies the arc-3-cyclic property.

21) $m = 3$.

If $m \geq 4$, then $m + 1 \rightarrow 3$ by 15). Let $(m + 1, h, \ldots, m + 1)$ be a Hamilton cycle in $T[V_2]$. Since $\psi(m) \in \{a + 2, a + 3 = k\}$ and 19), we have $P_k(1, k) = (1, 2, a + 1, w, h, \ldots, m + 1, 3, \ldots, m, \psi(m) = a + 2, k)$ (if $\psi(m) = a + 2$) or $P_k(1, k) = (1, 2, a + 1, a + 2, w, h, \ldots, m + 1, 3, \ldots, m, k)$ (if $\psi(m) = k$). Hence $T$ contains a $P_k(1, k)$ by Lemma 2. This is a contradiction. Hence $m = 3$ by 18).

22) $q < m$ (that is, $q = 2$) and $(k, m) \in A$.

By 17) there is a $C_3(k, a) = (k, a, x, k)$. Using a similar proof of 18), we have $x \notin V_2 \cup \{1, a + 1, a + 2, w\}$, and $x \neq m$ by 19). So $x = 2$ and $2 = x \rightarrow k$.

If $q = m$, then $q = 3$ and $m = q \rightarrow k$. Since $a + 2 = k - 1 \rightarrow k$, $m$ and $a + 2$ are adjacent. If $a + 2 \rightarrow m$, then there is a $P_k(1, k) = (1, p = a + 2, m = 3, m + 1, \ldots, a + 1, w, 2, k)$, a contradiction. So $m \rightarrow a + 2$. We have $a + 2 \rightarrow m + 1$ since $m \rightarrow m + 1$ and the definition of $m$. Thus $T$ contains a $P_k(1, k) = (1, p = a + 2, m + 1, \ldots, a + 1, w, 2, 3 = q, k)$, a contradiction. Hence $q < m$ and $q = 2$. Thus $k \rightarrow m = 3$ by 17).

23) $m \rightarrow a + 2$, $a + 2 \rightarrow a$, $a = 4$ and $k = 7$.

$\psi(m) = a + 2$ since 22) and $a + 2 \leq \psi(m) \leq a + 3 = k$. That is, $m \rightarrow a + 2$. And note that $m \rightarrow a$ by 19), hence $a$ and $a + 2$ are adjacent. Then $a + 2 \rightarrow a$ since $\psi(a) = a + 1$.

By the definition of $m$ and $a - 1 \rightarrow a, a - 2 \rightarrow a - 1, \ldots, m + 1 \rightarrow m + 2$ and $a + 2 \rightarrow a$, we have that $a + 2 \rightarrow \{a - 1, a - 2, \ldots, m + 1\}$. If $a > 4$,
24) 6 and 2, 5 and 3 are adjacent respectively, where the orientation can be chosen arbitrarily.

Because $6 = a + 2 = p$, $2 \in O(1)$ and $3, 5 = a + 1 \in O(2)$ by 19), hence 24) is true.

Up to now, if $p > a + 1$, $q < a$ and $|W| = 1$, we have proved that $T$ is a $T_6$-type digraph (see Figure 2).

Note that the converse of $T_6$- ($T_8$-, resp.) type digraphs is a $T_6$- ($T_8$-, resp.) type digraphs. Therefore the proof of Lemma 8 is a completed.

**Proof of Theorem 1:** Let $T = (V, A)$ be an arc-3-cyclic connected local tournament of order $n$. If $T$ is not arc-pancyclic, then there is an arc $e$ in $T$ such that $e$ is not pancyclic. Thus $T$ satisfies $(\ast)$. By Lemmas 3, 4 and 5, we only consider the following two cases: $b > a + 1$ and $b = a + 1$. And then by Lemma 7 and Lemma 8, $T$ is a $T_6$- or $T_8$-type digraph or $D_8$. Therefore the proof of Theorem 1 is completed.

**Proof of Theorem 2:** By Theorem 1, if $T$ is an arc-3-cyclic connected local tournament and an arc $e$ is not pancyclic, then $T$ must be a $T_6$- or $T_8$-type digraph or $D_8$. It is easy to check that in each of $T_6$-, $T_8$-type digraph and $D_8$, there exists only one arc $(k, 1)$ which is not pancyclic.

**Proof of Corollary 1:** Note that for $T_6$- or $T_8$-type digraph or $D_8$, they are not arc-$n$-cyclic.

**References**


