On the Arc-Pancyclicity of Local Tournaments*

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ABSTRACT. Let $T = (V, A)$ be a digraph with n vertices. T is called a local tournament if for every vertex $x \in V$, $T[O(x)]$ and $T[I(x)]$ are tournaments. In this paper, we prove that every arc-3-cyclic connected local tournament T is arc-pancyclic except $T \cong T_6$-, T_8-type digraphs or D_8.

1 Introduction

A digraph $D = (V, A)$ consists a pair of V, A, where V is a vertex set and A is an arc set. We say that x dominates y where $x, y \in V$, denoted by $x \to y$, if (x, y) is an arc of a digraph D. Let S_1 and S_2 be two vertex subsets of V. We say that S_1 dominates S_2, denoted by $S_1 \to S_2$, if there is a complete connection between S_1 and S_2 and all arcs between S_1 and S_2 are directed toward S_2. For convenience, we write $x \to S_2$ (resp., $S_2 \to x$) instead of $\{x\} \to S_2$ (resp., $S_2 \to \{x\}$). For any $x \in V$ and any $S \subseteq V$, We define

$$O(x) = \{y | y \in V, (x, y) \in A\}, \ I(x) = \{y | y \in V, (y, x) \in A\}$$
$$O_s(x) = O(x) \cap S, I_s(x) = I(x) \cap S.$$

A directed path of length k from x to y is denoted by $P_k(x, y)$. A k-cycle containing arc (x, y) is denoted by $C_k(x, y)$. The converse of $D = (V, A)$

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is defined as a digraph $\overleftarrow{D} = (V, \overleftarrow{A})$ such that $(x, y) \in \overleftarrow{A}$ if and only if $(y, x) \in A$.

A semicomplete digraph is a digraph without nonadjacent vertices. A locally semicomplete digraph is a digraph D that satisfies the following condition for every vertex x of D, $D[O(x)]$ and $D[I(x)]$ are semicomplete digraphs. A local tournament is a locally semicomplete digraph without directed cycles of length 2 and loops. A digraph D is said to be arc-k-cyclic if each arc of D is contained in a cycle of length $k (3 \leq k \leq n, n = |V|)$. An arc e of D is said to be pancyclic if it is contained in cycles of all length m, $3 \leq m \leq n$. A digraph D is said to be arc-pancyclic if each arc of D is pancyclic.

Other notations and definitions not defined here can be found in [3].

2 The Main Results

The concept of locally semicomplete digraphs, which is a generalization of semicomplete digraphs or tournaments, was first introduced by J. Bang-Jensen [1]. Using this new concept, many classical theorems for tournaments have been generalized. For example:

Lemma 1 ([1] Theorems 3.2 and 3.3). *A connected locally semicomplete digraph has a directed Hamiltonian path, and a strong locally semicomplete digraph has a directed Hamiltonian cyclic.*

In this paper, we prove the following two theorems, which extend two theorems in [4] and [5] respectively. (See Corollaries 2 and 3 below)

Theorem 1. *Every arc-3-cycle connected local tournament T of order n $(n \geq 3)$ is arc-pancyclic, except $T \cong T_6$-, T_8-type graphs or D_8. (See Figures 1, 2 and 3).*

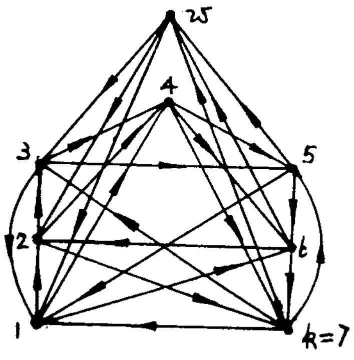

Figure 1. D_8

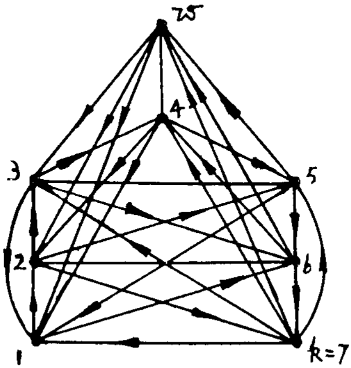

Figure 2.
T_8-type digraphs (The orientation of the edges without arrow can be chosen arbitrarily.)

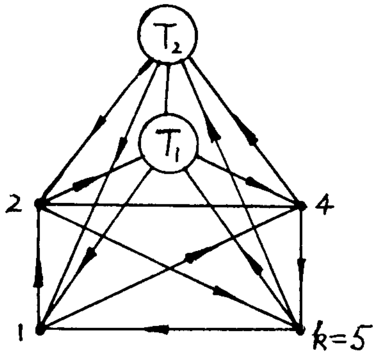

Figure 3.
T_6-type digraphs (T_1 and T_2 both are arc-3-cyclic tournament. The orientation of the edges without arrow can be chosen arbitrarily.)

Theorem 2. *At most one arc of an arc-3-cycle connected local tournament is not pancyclic.*

Corollary 1. *Let T be a connected local tournament of order n. Then T is arc-pancyclic if and only if T is arc-3-cyclic and arc-n-cyclic.*

Corollary 2 ([5], Theorem 1). *Let T be a tournament of order n. Then T is arc-pancyclic if and only if T is arc-3-cyclic and arc-n-cyclic.*

Corollary 3. ([4], Theorem 1). *Except for T_6-type digraphs and T_8-type digraphs, every arc-3-cyclic tournament is arc-pancyclic.*

The proofs of our results are given in the next section.

3 The Proofs Of Theorems

In the following, we shall assume that $T = (V, A)$ is an arc-3-cyclic connected local tournament of order n. In order to prove Theorem 1, we need the following lemmas.

Lemma 2. **([2], Corollary 3.13).** *Let $P_1 = (x_1, x_2, \ldots, x_m)$ and $P_2 = (y_1, y_2, \ldots, y_t)$, $m \geq 2$, $t \geq 3$, be paths in T. If there exist i, j, $1 \leq i < j \leq m$ such that $x_i = y_1$ and $x_j = y_t$ and $V(P_1) \cap V(P_2) - \{y_1, y_t\} = \emptyset$, then T has an (x_1, x_m)-path P such that $V(P) = V(P_1) \cup V(P_2)$.*

If T were not arc-pancyclic, then there is an arc $e = (k, 1)$ in T such that e is contained in one of m-cycles, $3 \leq m \leq k < n$, but e is not contained in any $(k+1)$-cycle. i. e.

$$\textbf{There does not exist any } P_k(1,k) \textbf{ in } T. \qquad (\star)$$

Let $C = C_k(e) = (1, 2, \ldots, k, 1)$ be a k-cycle containing e. Without ambiguity, we also let C be the set of itself's vertices. Let $W = V - C = V - \{1, 2, \ldots, k\}$, thus $|W| \geq 1$. If $O_c(w) \neq \emptyset$ and $I_c(w) \neq \emptyset$ for $w \in W$, we define:

$$a(w) = \max\{i | i \in O_c(w)\}, b(w) = \min\{i | i \in I_c(w)\}.$$

Lemma 3. *If T satisfies $(\star)$, then $T[W]$ is a tournament, and then $O_c(w) = \{1, 2, \ldots, a(w)\} \neq \emptyset$ and $I_c(w) = \{b(w), b(w)+1, \ldots, k\} \neq \emptyset$ for any $w \in W$.*

Proof: We prove the following two assertions:

(a) $O_c(w) \neq \emptyset$ for $w \in W$ if and only if $I_c(w) \neq \emptyset$.

If $O_c(w) \neq \emptyset$, set $i = \min\{j | j \in O_c(w)\}$. Suppose that $i > 1$. By the definition of a local tournament and $\{w, i-1\} \subseteq I(i)$, we have that $i-1$ and w are adjacent in T. Thus by the definition of i, we have $i - 1 \rightarrow w$. Hence T contains a $P_k(1,k) = (1, 2, \ldots, i-1, w, i, \ldots, k)$. This is a contradiction to $(\star)$. So $i = 1$.

From the above arguments, we also have $O_c(w) = \{1, 2, \ldots, a(w)\}$. If $a(w) = k$, then $w \rightarrow C$. Hence, since T is arc-3-cyclic, there exists a 3-cyclic $C_3(w, 1) = (w, 1, x, w)$ with $x \in W$. Thus T contains a $P_k(1, k) = (1, x, w, 3, \ldots, k)$. This is a contradiction to $(\star)$. So $a(w) < k$.

Similarly, we have $I_c(w) = \{b(w), \ldots, k\}$ and $b(w) > 1$ when $I_c(w) \neq \emptyset$.

Now if $O_c(w) \neq \emptyset$, then there is a $C_3(w, 1) = (w, 1, x, w)$. If $x \in W$, then $1 \in I_c(x)$ and $b(x) = 1$. This contradicts $b(w) > 1$ for any $w \in \{w | w \in W, I(w) \neq \emptyset\}$. Hence $x \in C$, i.e., $x \in I_c(w)$ and $I_c(w) \neq \emptyset$. Similarly, if $I_c(w) \neq \emptyset$, then $O_c(w) \neq \emptyset$ for $w \in W$.

(b) Let $W_1 = \{w|w \in W, O_c(w) \neq \emptyset\}$ and $W_2 = W - W_1$. Then $W_1 = W$.

Since T is connected and arc-3-cyclic, we have $W_1 \neq \emptyset$.. Suppose that $W_2 \neq \emptyset$. Thus for any $z \in W_2$, we have $O_c(z) = I_c(z) = \emptyset$ by (a). Since T is connected, there exist $x \in W_1$ and $y \in W_2$ such that x and y are adjacent. Without loss of generality, we assume $x \to y$. (Otherwise, we consider the converse of T). Since $O_c(x) \neq \emptyset$ and $x \to 1$, 1 and y are adjacent and $1 \in O_c(y)$ by (a), which is a contradiction. Hence $W_2 = \emptyset$. i.e, $W = W_1$.

From (a) and (b), we have that $W \subseteq I(1)$. Hence $T[W]$ is a tournament by the definition of a local tournament. So Lemma 3 is valid.

For any $w \in W$, we define:

$$p(w) = \min\{i|i \in O(1) \cap I_c(w)\}, \; q(w) = \max\{i|i \in I(k) \cap O_c(w)\}.$$

Lemma 4. *If T satisfies $(\star)$, then $O(1) \cap I_c(w) \neq \emptyset$, $I(k) \cap O_c(w) \neq \emptyset$ and $2 \leq q(w) \leq a(w) < b(w) \leq p(w) \leq k-1$ for any $w \in W$.*

Proof: There is a $C_3(w, 1) = (w, 1, x, w)$. We have $x \notin W$ by Lemma 3. Thus $x \in O(1) \cap I_c(w)$ by $x \to w$. And $b(w) \leq x \leq k-1$ since $k \to 1$. Similarly, we have $y \in I(k) \cap O_c(w)$ and $2 \leq y \leq a(w)$. By the definitions of $p(w)$ and $q(w)$, we have $2 \leq y \leq q(w) \leq a(w) < b(w) \leq p(w) \leq x \leq k-1$ for every $w \in W$. So Lemma 4 is valid.

Lemma 5. *If T satisfies $(\star)$, then $b(w) = b(w')$ and $a(w) = a(w')$ for every $w, w' \in W$. And $T[W]$ is an arc-3-cyclic tournament.*

Proof: Suppose that there are two distinct vertices w, w' in W such that $b(w) \neq b(w')$. Let $w_0 \in W$ be chosen such that $b(w_0) = \min\{b(w)|w \in W\}$. Let $W_1 = \{w|w \in W, b(w) > b(w_0)\}$ and $W_2 = W - W_1$. Then $W_1 \neq \emptyset$, $W_2 \neq \emptyset$ and $b(w_0) = b(w) \to w$ for every $w \in W$. Suppose that there exist $w_1 \in W_1$ and $w_2 \in W_2$ such that $w_1 \to w$. Since $w_1 \to w_2$ and $b(w_2) \to w_2$, we know that w_1 and $b(w_2)$ are adjacent and $w_1 \to b(w_2)$ by $b(w_1) > b(w_0) = b(w_2)$. Hence $a(w_1) \geq b(w_2)$. By the definitions of $p(w_1)$ and $q(w_2)$, we have that

$$2 \leq q(w_2) \leq a(w_2) < b(w_2) \leq a(w_1) < b(w_1) \leq p(w_1) \leq k-1 \quad \text{(A)}$$

and hence $q(w_2) + 1 \leq p(w_1) - 1$.

When $q(w_2) + 1 = p(w_1) - 1$, we have that $q(w_2) = a(w_2) = b(w_2) - 1$ and $b(w_2) = a(w_1) = b(w_1) - l = p(w_1) - 1$ from (A). Then we have $P_k(1, k) = (1, p(w_1), \ldots, k-1, w_1, w_2, 2, \ldots, q(w_2), k)$ in T, a contradiction. Hence $q(w_2) + 1 \leq p(w_1) - 2$. Thus it follows that either $p(w_1) - 2 \geq b(w_2)$ or $q(w_2) + 2 \leq a(w_1)$ by (A). We have either $P_k(1, k) = (1, p(w_1), \ldots, k-1, w_1, q(w_2) + 1, \ldots, p(w_1) - 2, w_2, 2, \ldots, q(w_2), k)$ if $p(w_1) - 2 \geq b(w_2)$ or $P_k(1, k) = (1, p(w_1), \ldots, k-1, w_1, q(w_2)+2, \ldots, p(w_1)-1, w_2, 2, \ldots q(w_2), k)$

if $q(w_2) + 2 \le a(w_1)$. These are contradictions. Hence no vertex of W_1 dominates any vertex of W_2. $W_2 \to W_1$ since $T[W]$ is a tournament.

Let $w_1 \in W_1$ and $w_2 \in W_2$, then $w_2 \to w_1$ and $b(w_0) = b(w_2) \to w_2$. There is a $C_3(w_2, w_1, x, w_2)$. $x \notin W$ since $W_2 \to W_1$. Hence we have $x \in C$ and $a(w_1) \ge x \ge b(w_2)$. Thus we have that: $q(w_2) \le a(w_2) < b(w_2) \le a(w_1) < b(w_1) \le p(w_1)$. And hence $q(w_2) + 1 \le p(w_1) - 1$. As above, we can also prove that T contains a $P_k(1, k)$, a contradiction. Therefore $b(w_1) = b(w_2)$ for any $w_1, w_2 \in W$. Similarly, we can prove $a(w_1) = a(w_2)$ for any $w_1, w_2 \in W$. Hence $T[W]$ is an arc-3-cyclic tournament. So Lemma 5 is valid.

By Lemma 5, we denote $a = a(w)$ and $b = b(w)$ for each $w \in W$. Thus by Lemma 3 and Lemma 4, We have $2 \le a < b \le k-1$, $O_c(w) = \{1, 2, \dots, a\}$, and $I_c(w) = \{b, b+1, \dots, k\}$. Hence $T[\{1, 2, \dots, a\}]$ and $T[\{b, b+1, \dots, k\}]$ both are tournaments.

Lemma 6. *If there are $\alpha < \gamma < \delta$ in C such that $1 \le \alpha \le a-1$, $a+1 < \gamma < \delta \le k$, $b+1 \le \delta$, $(\alpha, \gamma) \in A$ and $(\gamma - 1, \delta) \in A$. Then T contains a $P_k(1,k)$.*

Proof: Let α, γ and δ satisfy the conditions of Lemma 6 and $w \in W$. Then there is $P_k(1,k) = (1, 2, \dots, \alpha, \gamma, \dots, \delta - 1, w, \alpha + 1, \dots, \gamma - 1, \delta, \dots, k)$.

Furthermore, we shall use the following symbols. For $1 \le m \le a$, $b \le l \le k$, we denote:

$$R(m) = \{i | b \le i \le k, (m, i) \in A\}, \ L(l) = \{i | 1 \le i \le a, (i, l) \in A\}.$$

Thus for any $w \in W$, $1 \le m \le a$ and $b \le l \le k$, since there exist $C_3(w, m)$ and $C_3(l, w)$, it is easy to see that $R(m) \ne \emptyset$, $L(l) \ne \emptyset$ and $k \notin R(1)$, $1 \notin L(k)$. Hence we can define:

$$\begin{aligned} \psi(m) &= \max\{i | i \in R(m)\}, \\ \varphi(l) &= \min\{i | i \in L(l)\}, \\ p &= \min\{i | b \le i \le k-1, (1, i) \in A\}, \\ q &= \max\{i | 2 \le i \le a, (i, k) \in A\}. \end{aligned}$$

Then $(m, \psi(m))$, $(\varphi(l), l)$, $(1, p)$, $(q, k) \in A$ and $b \le \psi(m) \le k$, $1 \le \varphi(l) \le a$, $2 \le q \le a < b \le p \le k-1$ for any $1 \le m \le a$ and $b \le l \le k$.

Lemma 7. *If T satisfies $(\star)$ and $b > a+1$, then $T \cong D_8$.*

Proof: First, we have $\{a+1, \dots, b-1\} \ne \emptyset$, and i and w are nonadjacent for any $i \in \{a+1, \dots, b-1\}$ and any $w \in W$. There is a $C_3(a, a+1) = (a, a+1, x, a)$ in T. Obviously $x \notin W$. If $x \in \{a+2, \dots, b-1\}$, then x and w are adjacent by $x \to a$ and $w \to a$, a contradiction. So $x \notin \{a+2, \dots, b-1\}$. Since $w \to i$ for $i \in \{1, 2, \dots, a-1\}$, $a+1 \to x$, $a+1$ and w are nonadjacent.

We have $x \notin \{1, 2, \ldots, a-1\}$. Thus, $x \in \{b, b+1, \ldots, k\}$. Suppose $x = b$, i.e, $b \to a$, then $\varphi(b) < a$ and $\psi(a) > b$. $\varphi(b)$ and $b-1$ are adjacent by $\varphi(b) \to b$ and $b-1 \to b$. Since $b-1$ and w are nonadjacent and $w \to \varphi(b)$, $\varphi(b) \to b-1$. Similarly, we can get $\varphi(b) \to \{a+1, \ldots, b-1\}$. Let $\alpha = \varphi(b)$, $\gamma = a+1$ and $\delta = \psi(a)$, then by Lemma 6 there is a $P_k(1, k)$ in T. This is a contradiction to ($\star$). Hence $x > b$. Similarly, using $C_3(b-1, b) = (b-1, b, y, b-1)$, we have $y < a$.

If $b > a+2$, x and $a+2$ are adjacent since $a+1 \to x$ and $a+1 \to a+2$. Since $a+2$ and w are nonadjacent and $x \to w$ for any $w \in W$, $a+2 \to x$ by the definition of a local tournament. Similarly, $\{a+1, \ldots, b-1\} \to x$. Let $\alpha = y(< a)$, $\gamma = b-1$ and $\delta = x(> b)$. By Lemma 6 there is a $P_k(1, k)$ in T, a contradiction. Hence $b = a+2$.

Furthermore, $a+1$ and $x-1$ are adjacent since $a+1 \to x$ and $x-1 \to x$. Then $a+1 \to x-1$ by the fact that $x-1 \to w$, w and $a+1$ are nonadjacent. Similarly, we have

$$b-1 = a+1 \to \{b+1, \ldots, x-1, x\} \tag{B}$$

Now the following three cases must be considered:

Case 1. $k > b+1$ and $a > 2$.

If $\varphi(b) < a$, then we may choose $\alpha = \varphi(b)$, $\gamma = b$ and $\delta = x$. Hence there is a $P_k(1, k)$ in T by Lemma 6. This is a contradiction. So $\varphi(b) = a$, i.e, $a \to b$. Since $1, a \in O(w)$, 1 and a must be adjacent. Suppose $1 \to a$. If $\varphi(a-1) > b$, then we may choose $\alpha = 1$, $\gamma = a$ and $\delta = \psi(a-1)$. There is a $P_k(1, k)$ in T by Lemma 6. This is a contradiction. So $\psi(a-1) = b$ since $\psi(a-1) \geq b$, i.e. $a-1 \to b$. Now, let $\alpha = a-1$, $\gamma = b$ and $\delta = x$, there is also a $P_k(1, k)$ in T by Lemma 6, a contradiction. Hence we always assume that

$$a \to 1 \text{ and } a \to b \tag{C}$$

in the following arguments.

1) $\{1, 2, \ldots, a-1\} \to a+1$.

$1 \to a+1$ since $a+1$ and w are nonadjacent and $1, a+1 \in O(a)$. Furthermore, $2 \to a+1$ since $1 \to 2$ and $1 \to a+1$. Similarly, we have $\{1, 2, \ldots, a-1\} \to a+1$.

2) $b \to 1$, $a+1 \to k$ and $j \to b$ for each $j \in \{b+2, \ldots, k\}$.

If there exists a $j \in \{b+2, \ldots, k\}$ such that $b \to j$. Then T contains a $P_k(1, k) = (1, 2, \ldots, a-1, a+1, b+1, \ldots, j-1, w, a, b, j, \ldots, k)$ by 1), (B) and (C). This is a contradiction. So $\{b+2, \ldots, k\} \to b$.

Since k, $a+1 \in I(b)$, we have that k and $a+1$ are adjacent. Furthermore, $a+1 \to k$ since $k \to w$ and $a+1$ and w are nonadjacent.

Since 1, $b \in O(k)$, 1 and b are adjacent. If $1 \to b$, then we may choose $\alpha = 1$, $\gamma = b$ and $\delta = k$. Then there is a $P_k(1,k)$ in T by Lemma 6. This is a contradiction. Hence $b \to 1$.

3) $k = b+2$ and $p = b+1$.

$p > b$ since $b \to 1$. If $k-1 \geq b+2$, then T contains a $P_k(1,k) = (1, p, \ldots, k-1, b, \ldots, p-1, w, 2, \ldots, b-1 = a+1, k)$ by 2). This is a contradiction. Hence $k = b+2$ and $p = b+1$.

4) $(a-1, b) \notin A$, $a = 3$, $p = 6$ and $k = 7$.

Note that $\psi(a-1) \in \{b, b+1, b+2 = k\}$. If $\psi(a-1) = b$, then we may choose $\alpha = a-1$, $\gamma = \psi(a-1) = b = a+2$ and $\delta = k$. By 2) and Lemma 6, there is a $P_k(1,k)$ in T. This is a contradiction. So $\psi(a-1) > b$ and $(a-1, b) \notin A$.

If $a-1 > 2$, then we have either $P_k(1,k) = (1, 2, a, \ldots, \psi(a-1)-1, w, 3, \ldots, a-1, \psi(a-1), \ldots, k)$, if $2 \to a$ or $P_k(1,k) = (1, a+1, \ldots, \psi(a-1)-1, w, a, 2, \ldots, a-1, \psi(a-1), \ldots, k)$ by 1), if $a \to 2$. These contradict to $(\star)$. Hence $a \leq 3$. Thus $a = 3$ by the assumption that $a > 2$. Finally by 3) we have $p = b+1 = a+3 = 6$ and $k = b+2 = a+4 = 7$.

5) $x = k$ and $q = 2$ (hence $a+1 \to x = k \to a$).

Suppose $x < k = b+2$. $x = b+1$ since $x > b$. By 3) and the choice of x, we have $p = b+1 = x \to a$. Hence there is a $P_k(1,k) = (1, p = x, a, b, w, 2, a+1, k)$ by 1), 2) and (C). This is a contradiction to $(\star)$. Hence $x = k$ and $q = 2$.

6) $b+1 \to 2$.

2 and $b+1$ are adjacent since 2, $b+1 = p \in O(1)$. If $2 \to b+1$, then 2 and b are adjacent by $b \to b+1$. $b \to 2$ since $(2, b) = (a-1, b) \notin A$ by 4). Then there is a $P_k(1,k) = (1, p = b+1, w, a, a+1, b, 2 = q, k)$. This is a contradiction. So $b+1 \to 2$.

7) $|W| = 1$.

Suppose that there is a $w_0 \in W - \{w\}$. Without loss of generality, let $w \to w_0$. Then there is $P_k(1,k) = (1, p = b+1, w, w_0, 2, a, a+1, k)$ by 2). This is a contradiction.

8) 2 and 5, 3 and 6 are nonadjacent.

Otherwise, there is a $P_k(1,k)$ in T. For example if $(6,3) \in A$, there exists a $P_k(1,k) = (1, p = 6, 3 = a, a+1, b, w, 2 = q, k)$. This is a contradiction.

Up to now, we have proved that $T \cong D_8$ (See Figure 1) in this subbase.

Case 2. $k = b+1$ and $a \geq 2$.

Since $b < x \leq k = b+1$ and $b \leq p < k = b+1$, $x = k$ and $p = b$, i.e. $a+1 \to x = k$ and $1 \to p = b = a+2$. Let $\alpha = 1$, $\gamma = a+2$ and $\delta = k$, there is a $P_k(1,k)$ in T by Lemma 6. This is a contradiction.

Case 3. $a = 2$ and $k \geq b+2$.

Consider the converse of T. Note that Case 3 in T is Case 2. in $\overleftarrow{T}$. So Lemma 7 is valid.

Lemma 8. *If T satisfies ($\star$) and $b = a + 1$, then T is a T_6- or T_8-type digraph.*

Proof: We consider the following two cases.

Case 1. $|W| \geq 2$ (let w, $w' \in W$)

Suppose $p > a + 1$, $q < a$ and $k > 6$. Then there exists an $i \in \{1, 2, \ldots, k\} - \{1, q, a, a + 1, p, k\}$. If $1 < i < q$, then $q \geq 3$ and there is a $P_k(1, k) = (1, p, \ldots, k-1, w, q+1, \ldots, a, a+1, \ldots, p-1, w', 3, \ldots, q, k)$. Similarly, T contains a $P_k(1, k)$ when $q < i < a$ or $a + 1 < i < p$ or $p < i < k$. These are contradictions. If $p > a + 1$, $q < a$ and $k = 6$, then $q = 2$, $a = 3$ and $p = 5$. Because $|W| \geq 2$ and $T[W]$ is an arc-3-cyclic tournament by Lemma 5, we have $|W| \geq 3$. Let $\{w_1, w_2, w_3\} \subseteq W$ and $w_1 \to w_2 \to w_3$. Then T contains a $P_k(1, k) = (1, p, w_1, w_2, w_3, q, k)$. This is a contradiction. Hence we have $p = a + 1$ or $q = a$.

In the following we may assume that, without loss of generality, $p = a+1$ (Otherwise $q = a$, we can consider the converse of T). Thus $1 \to a+1 = b$. Now we can obtain the following assertions.

9) $q < a$ (therefore $(a, k) \notin A$).

If $q = a$, then $a = q \to k$. There is a $P_k(1, k) = (1, p = a + 1, \ldots, k - 1, w, 2, \ldots, a, k)$, a contradiction.

10) $k = a + 2$, $V_1 = \{q + 1, \ldots, a\} \to a + 1$ and $T[V_1]$ is a tournament.

Suppose $k > a + 2$. If $\varphi(a + 2) = a$, let $\alpha = 1$, $\gamma = a + 1$ and $\delta = a + 2$, then there is a $P_k(1, k)$ in T by Lemma 6. This is a contradiction. Hence $\varphi(a+2) < a$. Since $a+1$, $k \in I(w)$, $a+1$ and k are adjacent. If $a+1 \to k$, let $\alpha = \varphi(a + 2)$, $\gamma = a + 2$ and $\delta = k$ in Lemma 6, then there is a $P_k(1, k)$ in T, a contradiction. Hence $k \to a + 1$. Thus a and k are adjacent by $a \to a+1$. Hence $k \to a$ by 9). There is a $C_3(k, a) = (k, , a, z, k)$. Obviously, $z \notin W$, $z \neq 1$, $z \neq a + 1$ and $z \notin \{q + 1, \ldots, a - 1\}$ by the definition of q. Let $P_1 = (1, p = a+1, \ldots, k-1, w, 2, \ldots, z, k)$ and $P_2 = (w, z+1, \ldots, a, z)$. If $z \in \{2, \ldots, q\}$, then P_1 and P_2 satisfy the condition of Lemma 2, hence there is a $P_k(1, k)$ in T. This is a contradiction. So $z \in \{a + 2, \ldots, k - 1\}$. Thus there is a $P_k(1, k) = (1, p = a + 1, \ldots, z - 1, w, 2, \ldots, a, z, k)$ in T, a contradiction too. Hence $k = a + 2$.

Let $V_1 = \{q + 1, \ldots, a\}$. Then $T[V_1]$ is a tournament by $V_1 \subseteq O(w)$. Since $k = a + 2$ and by the definition of q, $\psi(j) = a + 1$ for each $j \in V_1$, that is $V_1 \to a + 1$.

11) $T[V_1]$ is a strong tournament.

If not, then $|V_1| \geq 2$ and $q+1 \to a$. There is a $C_3(q+1, a) = (q+1, a, y, q+1)$. Obviously $y \notin W$. By $q \to k$ and Lemma 6, $y \notin \{1, 2, \ldots, q - 1\}$. $y \neq k$ by 9). And $y \neq a + 1$ by 10). Since $T[V_1]$ is not strong, $y \notin V_1$.

Hence $y = q$. i.e. $a \to y = q$. Let $P_1 = (1, p = a+1, w, z, \ldots, q, k)$ and $P_2 = (w, q+1, \ldots, a, q)$. Then P_1 and P_2 satisfy the conditions of Lemma 2 and there is a $P_k(1,k)$ in T, a contradiction. So $T[V_1]$ is a strong tournament.

12) $q = 2$ (therefore $2 \to k$) and $V_1 \to 1$.

Suppose $q \geq 3$. By $q \to k$ and Lemma 6, we have $q+1 \to 2$. We may assume that $(q+1, h, \ldots, q+1)$ is a Hamilton cycle in $T[V_1]$ by 11). Then there is a $P_k(1,k) = (1, p = a+1, w, h, \ldots, q+1, 2, \ldots, q, k)$, a contradiction. Hence $q = 2$.

Now we show that $V_1 \to 1$. $T[\{1,2\} \cup V_1]$ is a tournament since $\{1,2\} \cup V_1 \subseteq O_c(w)$. Suppose there exists an $x \in V_1$ such that $1 \to x$. Let $(x, \ldots, h, x)$ be a Hamilton cycle in $T[V_1]$. Then there is a $P_k(1,k) = (1, x, \ldots, h, a+1, w, 2, k)$ by 10). This is a contradiction. Hence $V_1 \to 1$.

13) T is a tournament.

In fact, $T[\{1,2\} \cup V_1]$ is a tournament since $\{1,2\} \cup V_1 \subseteq O(w)$. $T[V_1 \cup \{k\}]$ is a tournament since $V_1 \cup \{k\} \subseteq I(1)$, 2 and $a+1$ are adjacent by $1 \to 2$ and $1 \to p = a+1$. Hence T is a tournament by 10) and 12).

Therefore by 13) and $p = a+1$, using the result of (9) case (i) in the proof of Theorem 1 of [4], we get that T is a T_6-type digraph (See Figure 3) in this case.

Case 2. $|W| = 1$.

Let $W = \{w\}$. If $p = a+1$ or $q = a$, then we can obtain that T is a T_6-type digraph by a similar argument in Case 1. Hence in the following we may assume that $p > a+1$ and $q < a$. And we can obtain the following assertions.

14) There is either $\psi(a) = a+1$ or $\varphi(a+1) = a$.

Suppose $\psi(a) > a+1$ and $\varphi(a+1) < a$, let $\alpha = \varphi(a+1)$, $\gamma = a+1$ and $\delta = \psi(a)$ in Lemma 6, then there is a $P_k(1,k)$ in T. This is a contradiction. Hence $\psi(a) = a+1$ or $\varphi(a+1) = a$.

Without loss of generality, let $\psi(a) = a+1$. Otherwise we can consider the converse of T. Then $(a, j) \notin A$ and $1 \leq \varphi(j) < a$ for each $j \in \{a+2, \ldots, k\}$. Because $\psi(\varphi(j)) \geq j \geq a+2$ for each $j \in \{a+2, \ldots, k\}$, we may define:

$$m = \max\{i | 1 \leq i \leq a-1, \psi(i) \geq a+2\}, \; V_2 = \{m+1, \ldots, a\}.$$

By the definition of m and q, we have that $2 \leq q \leq m < a$, $\psi(m) \geq a+2$ and i does not dominate any vertex of $\{a+2, \ldots, k\}$ for each $i \in V_2$. Hence

$$\psi(i) = a+1 \text{ for each } i \in V_2. \qquad \text{(D)}$$

15) $m+1 \to \{1, 2, \ldots, m-1\}$ and $q+1 \to \{1, 2, \ldots, q-1\}$.

Suppose there exists $j \in \{1, 2, \ldots, m-1\}$ such that $j \to m+1$. Let $\alpha = j$, $\gamma = m+1$ and $\delta = \psi(m)$. Then there is a $P_k(1,k)$ in T by Lemma 6. This is a contradiction. Hence $m+1 \to \{1,2,\dots,m-1\}$ since $T[\{1,2,\dots,q,\dots,a\}]$ is a tournament. Similarly, we have that $q+1 \to \{1,2,\dots,q-1\}$.

16) $k = a+3$, $p = a+2$ and $(a+1,1) \in A$.

Since $a+1 < p \leq k-1$, $k \geq a+3$. Suppose $k > a+3$. Then we can obtain the following results.

(a) $k-1 \to a+1$.

Suppose $a+1 \to k-1$. By $\psi(a) = a+1$, we have $\varphi(a+2) < a$. Then let $\alpha = \varphi(a+2)$, $\gamma = a+2$ and $\delta = k-1$, there is a $P_k(1,k)$ in T by Lemma 6 . This is a contradiction. Hence $k-1 \to a+1$ since $k-1$, $a+1 \in I(w)$.

(b) $k-1 \to V_2 = \{m+1,\dots,a\}$.

a and $k-1$ are adjaçent since $k-1$, $a \in I(a+1)$. So $k-1 \to a$ since $\psi(a) = a+1$. Similarly, by (D). we have $k-1 \to V_2 = \{m+1,\dots,a\}$.

(c) $m = 2$.

Suppose $m \geq 3$. Since $m+1 \to 2$ by 15) and $k-1 \to m+2$ by $a+1 \geq m+2$ and (a), (b), there is a $P_k(1,k) = (1,p,\dots,k-1,m+2,\dots,p-1,w,q+1,\dots,m+1,2,q,k)$. This is a contradiction. So $m = 2$.

Now we have $2 = q \to k$ since $2 \leq q \leq m = 2$. And $k-1 \to m+1$ by (b). Thus there is a $P_k(1,k) = (1,p,\dots,k-1,m+1 = 3,\dots,p-1,w,2 = m = q,k)$. This is a contradiction. Hence $k = a+3$ and $p = a+2$.

Since $1 \to p = a+2$ and $a+1 \to a+2$, 1 and $a+1$ are adjacent. Thus $a+1 \to 1$ by the definition of p.

17) $k \to \{q+1,\dots,a,a+1\}$. Therefore $k \to V_2$.

Suppose $a+1 \to k$, let $\alpha = 1$, $\gamma = p = a+2$ and $\delta = k$ in Lemma 6, then there is a $P_k(1,k)$ in T. This is a contradiction. Hence $k \to a+1$. Since $k \to a+1$, $a \to a+1$ and $\psi(a) = a+1$, we have $k \to a$. Since $a-1 \to a$, $a-2 \to a-1,\dots,q+1 \to q+2$ and the definition of q, we can obtain $k \to a-1,\dots,k \to q+1$ one by one. That is, $k \to \{q+1,\dots,a,a+1\}$ and $k \to V_2$.

18) $m \geq 3$.

Suppose $m < 3$. Then $m = 2$ and $q = 2$. By 17) $(k,a) \in A$. There is a $C_3(k,a) = (k,a,x,k)$. Obviously, $x \neq w$, $x \neq a+1$ since $k \to a+1$ in 17). We also have $x \neq a+2$ since $\psi(a) = a+1$. $x \neq 1$ since $k \to 1$, and $x \notin V_2$ by 17). Hence $x = 2 = m$. That is, $a \to 2$.

By 16) we have $a+2 = p \in O(1)$. So 2 and $a+2$ are adjacent. If $2 \to a+2$, $a+2$ and $m+1$ are adjacent since $2 = m \to m+1$. Thus $a+2 \to m+1$ by the definition of m.There is a $P_k(1,k) = (1,p = a+2,m+1 = 3,\dots,a+1,w,m = 2 = q,k)$, a contradiction. Hence $a+2 \to 2$. Since $a \to 2$, a and $a+2$ are adjacent. $a+2 \to a$ since $\psi(a) = a+1$. When $m+1 \leq a-1$, we have $a+2 \to a-1$ since $a-1 \to a$ and the definition of m. Similarly, we have $a+2 \to a-2,\dots,a+2 \to m+1$. Then there is a

$P_k(1,k) = (1, p = a+2, m+1, \ldots, w+1, w, 2, \ldots q), \ldots$
$m \geq 3$.

19) $2 \to a+1$ and $m \to a$.

In fact, we have $m+1 \to a+1$ by (D), and $m+1 \to 2$ by 15) and 18). Hence 2 and $a+1$ are adjacent. If $a+1 \to 2$, then there is a $P_k(1,k) = (1, p = a+2, w, q+1, \ldots, a+1, 2, \ldots, q, k)$. This is a contradiction. So $2 \to a+1$.

Suppose $a \to m$. Since $\psi(m) \in \{a+2, a+3 = k\}$, we have $P_1 = (1, 2, a+1, w, 3, \ldots, m, a+2, k)$ (if $\psi(m) = a+2$) or $P_1 = (1, 2, a+1, a+2, w, 3, \ldots, m, k)$ (if $\psi(m) = a+3$) and $P_2 = (w, m+1, \ldots, a, m)$. Clearly P_1 and P_2 satisfy the conditions of Lemma 2, hence T contains a $P_k(1,k)$. This is a contradiction. So $m \to a$.

20) $T[V_2]$ is a strong tournament.

If not, then $|V_2| \geq 2$ and $m+1 \to a$. There is a $C_3(m+1, a) = (m+1, a, x, m+1)$ in T. Obviously, we have $x \neq w$, $x \notin \{1, 2, \ldots, m-1\}$ by 15), $x \neq k$ by 17), $x \neq a+1$ by (D), $x \neq a+2$ by 14), $x \neq m$ by 19), and $x \notin V_2$ since $T[V_2]$ is not strong. Thus there is no $C_3(m+1, a)$ in T. This contradicts the fact that T satisfies the arc-3-cyclic property.

21) $m = 3$.

If $m \geq 4$, then $m+1 \to 3$ by 15). Let $(m+1, h, \ldots, m+1)$ be a Hamilton cycle in $T[V_2]$. Since $\psi(m) \in \{a+2, a+3 = k\}$ and 19), we have $P_k(1,k) = (1, 2, a+1, w, h, \ldots, m+1, 3, \ldots, m, \psi(m) = a+2, k)$ (if $\psi(m) = a+2$) or $P_k(1,k) = (1, 2, a+1, a+2, w, h, \ldots, m+1, 3, \ldots, m, k)$ (if $\psi(m) = k$). Hence T contains a $P_k(1,k)$ by Lemma 2. This is a contradiction. Hence $m = 3$ by 18).

22) $q < m$ (that is, $q = 2$) and $(k, m) \in A$.

By 17) there is a $C_3(k, a) = (k, a, x, k)$. Using a similar proof of 18), we have $x \notin V_2 \cup \{1, a+1, a+2, w\}$, and $x \neq m$ by 19). So $x = 2$ and $2 = x \to k$.

If $q = m$, then $q = 3$ and $m = q \to k$. Since $a+2 = k-1 \to k$, m and $a+2$ are adjacent. If $a+2 \to m$, then there is a $P_k(1,k) = (1, p = a+2, m = 3, m+1, \ldots, a+1, w, 2, k)$, a contradiction. So $m \to a+2$. We have $a+2 \to m+1$ since $m \to m+1$ and the definition of m. Thus T contains a $P_k(1,k) = (1, p = a+2, m+1, \ldots, a+1, w, 2, 3 = q, k)$, a contradiction. Hence $q < m$ and $q = 2$. Thus $k \to m = 3$ by 17).

23) $m \to a+2$, $a+2 \to a$, $a = 4$ and $k = 7$.

$\psi(m) = a+2$ since 22) and $a+2 \leq \psi(m) \leq a+3 = k$. That is, $m \to a+2$. And note that $m \to a$ by 19), hence a and $a+2$ are adjacent. Then $a+2 \to a$ since $\psi(a) = a+1$.

By the definition of m and $a-1 \to a, a-2 \to a-1, \ldots, m+1 \to m+2$ and $a+2 \to a$, we have that $a+2 \to \{a-1, a-2, \ldots, m+1\}$. If $a > 4$, then $a + 2$ [illegible] $m + 2$ since $m + 2 = 5 \le a$. So, there is a $P_k(1, k) = (1, p - a + 2, m + 2, \ldots, a, a + 1, w, m = 3, m + 1, m - 1 = 2 = q, k)$ by 15). This is a contradiction. Hence $a \le 4$. So we have $a = 4$ since $3 = m < a$. And then $k = a + 3 = 7$.

24) 6 and 2, 5 and 3 are adjacent respectively, where the orientation can be chosen arbitrarily.

Because $6 = a + 2 = p$, $2 \in O(1)$ and 3, $5 = a + 1 \in O(2)$ by 19), hence 24) is true.

Up to now, if $p > a + 1$, $q < a$ and $|W| = 1$, we have proved that T is a T_8-type digraph (see Figure 2).

Note that the converse of T_6- (T_8-, resp.) type digraphs is a T_6- (T_8- resp.) type digraphs. Therefore the proof of Lemma 8 is a completed.

Proof of Theorem 1: Let $T = (V, A)$ be an arc-3-cyclic connected local tournament of order n. If T is not arc-pancyclic, then there is an arc e in T such that e is not pancyclic. Thus T satisfies $(\star)$. By Lemmas 3, 4 and 5, we only consider the following two cases: $b > a+1$ and $b = a+1$. And then by Lemma 7 and Lemma 8, T is a T_6- or T_8-type digraph or D_8. Therefore the proof of Theorem 1 is completed.

Proof of Theorem 2: By Theorem 1, if T is an arc-3-cyclic connected local tournament and an arc e is not pancyclic, then T must be a T_6- or T_8-type digraph or D_8. It is easy to check that in each of T_6-, T_8-type digraph and D_8, there exists only one arc $(k, 1)$ which is not pancyclic.

Proof of Corollary 1: Note that for T_6- or T_8-type digraph or D_8, they are not arc-n-cyclic.

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