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SOME LOCAL CONDITIONS FOR HAMILTON-CONNECTED GRAPHS*

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Abstract. In this paper, the following results are proved: Let G be a 3-connected graph of order $n(\geq 3)$. If $I(u,v) \geq \alpha_2(u,v)+3$ whenever $d(u,v)=2$ and $\max\{d(u), d(v)\} < (n+1)/2$, then G is Hamilton-connected. Here $I(u,v)=|N(u)\cap N(v)|$, $\alpha_2(u,v)=\alpha(G[N_2(u)\cap N_2(v)])$, and $N_2(u)=\{v\in V(G)|d(u,v)=2\}$.

Let G be a 3-connected graph of order $n(\geq 3)$. If $I(u,v)\geq S(u,v)+1$ whenever $d(u,v)=2$ and $\max\{d(u),d(v)\}<(n+1)/2$, then G is Hamilton-connected. Here $S(u,v)$ denotes the number of edges of maximum star containing u, v as an induced subgraph in G.

1. Introduction

We refer to [2] for terminology and notation not defined here and consider finite simple graphs only.

A path with x and y as end vertices is called an x-y path. A path is called a Hamilton path if it contains all the vertices of G. A graph G is Hamilton-connected if every two vertices of G are connected by a Hamilton path.

Let P be a path of G. For convenience, P will also denote the vertex set of the path P. We denote by $\overrightarrow{P}$ the path P with a given orientation, and $\overleftarrow{P}$ the path P with the reverse orientation. If $u,v\in V(P)$, then $u\overrightarrow{P}v$ denotes the consecutive vertices of $\overrightarrow{P}$ from u to v. We use u^+ to denote the successor of u on $\overrightarrow{P}$ and u^- to denote its predecessor. If $A\subseteq V(P)$, we define $A^+=\{v^+|v\in A\}$ and $A^-=\{v^-|v\in A\}$. The distance between vertices u and v is denoted by

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$d(u,v)$. For each vertex $u \in V(G)$, we denote by $N(u)$ the set of all vertices of G adjacent to u and by $N_2(u)$ the set of all vertices from u whose distance are two. The subgraph of G induced by $N(u) \cup \{u\}$ is denoted by $G(u)$. If $uv \notin E(G)$, we denote by $S(u,v)$ the number of edges of maximum star including u,v. We use $\alpha(G)$ to denote the independence number of G. Let x and y be two vertices in G with $d(x,y)=2$. We define $I(x,y) = |N(x) \cap N(y)|$ and

$$\alpha_2(x,y) = \begin{cases} \alpha(G[N_2(x) \cap N_2(y)]), & \text{if } N_2(x) \cap N_2(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

In this paper, we get the following results.

Theorem 1. *Let G be a 3-connected graph of order $n(\geq 3)$. If $I(u,v) \geq \alpha_2(u,v)+3$ whenever $d(u,v)=2$ and $\max\{d(u),d(v)\} < (n+1)/2$, then G is Hamilton-connected.*

Theorem 2. *Let G be a 3-connected graph of order $n(\geq 3)$. If $I(u,v) \geq S(u,v)+1$ whenever $d(u,v)=2$ and $\max\{d(u),d(v)\} < (n+1)/2$, then G is Hamilton-connected.*

2. Proof of Theorems 1 and 2

We first present the following lemma.

Lemma. *Let G be a 3-connected graph of order n, then for any distinct vertices u and v, there is an u-v path passing through all vertices of degree at least $(n+1)/2$.*

Proof. By contradiction. Let $P = v_1 v_2 \cdots v_k$ be a path containing as many vertices of degree at least $\frac{n+1}{2}$ as possible with $v_1 = u, v_k = v$. We give P an orientation from v_1 to v_k. Set $R = V(G) \backslash V(P)$. Since G is 3-connected, there exists a path P' connecting two vertices of P with internally disjointing from P and containing an internal vertex x of degree at least $\frac{n+1}{2}$. Assume $v_s, v_t \in V(P) \cap V(P')$ and $1 \leq s < t \leq k$. We may assume P and P' are chosen in such a way that $t-s$ is as small as possible. We give P' an orientation from v_s to v_t. By the choice of P' and P, there exists a vertex v_l with $s < l < t$ such that $d(v_l) \geq \frac{n+1}{2}$ and $d(v_i) < \frac{n+1}{2}$ for $s < i < l$. If $l > s+1$, then

$$(N_P(v_l)\backslash\{v_1\})^-\backslash\{v_s\}, N_P(x), N_R(v_l), N_R(x), \{x, v_{l-1}\}$$

are pairwise disjoint. For suppose $v_m \in ((N_P(v_l)\backslash\{v_1\}^-\backslash\{v_s\}) \cap N_P(x)$, then $v_m \in \{v_1, v_2, \ldots, v_{s-1}\}$ or $v_m \in \{v_t, v_{t+1}, \ldots, v_k\}$. There is $u-v$ path $v_1v_2\cdots v_m x \overleftarrow{P'} v_s \overleftarrow{P} v_{m+1}v_l v_{l+1}\cdots v_k$ or $v_1v_2\cdots v_s \overrightarrow{P'} x v_m \overleftarrow{P} v_l v_{m+1}\cdots v_k$. These two $u-v$ path all contain more vertices of degree at least $\frac{n+1}{2}$ than P, which contradict the choice of P. We can similarly prove other pairs of set are disjoint. This implies that

$$\begin{aligned} n &\geq |(N_P(v_l)\backslash\{v_1\})^-\backslash\{v_s\}| + |N_P(x)| + |N_R(v_l)| + |N_R(x)| + 2 \\ &\geq d_P(v_l) - 2 + d_P(x) + d_R(v_l) + d_R(x) + 2 \\ &\geq d(v_l) + d(x) \\ &\geq n+1, \end{aligned}$$

a contradiction. If $l = s+1$, then the sets

$$(N_P(v_l)\backslash\{v_1\})^-, N_P(x), N_R(v_l), N_R(x), \{x\}$$

are pairwise disjoint, yielding a similar contradiction.

Proof of Theorem 1. For any distinct vertices $u, v \in V(G)$, by Lemma there exists an u-v path containing all vertices of degree at least $\frac{n+1}{2}$. Among such paths, let $P = v_1v_2\cdots v_k$ be one of maximum length. We give P an orientation from v_1 to v_k. We assume $R = V(G)\backslash V(P) \neq \emptyset$.

There exists a path P' of length at least 2 connecting two vertices of P with internally disjointing from P since G is a 3-connected. Suppose $v_s, v_t \in V(P) \cap V(P')$, where $1 \leq s < t \leq k$. For G is 3-connected, we may assume $v_t \neq v_k$. We give P' an orientation from v_s to v_t. Let x be the successor of v_s on P' and x' the predecessor of v_t on P', (possibly $x = x'$). Clearly, $d(x) < (n+1)/2$, $d(x') < (n+1)/2$. We denote the vertices of $v_1\overrightarrow{P}v_{s+1}$ and $v_{t+2}\overrightarrow{P}v_k$ by S_1 and the vertices of $v_{s+2}\overrightarrow{P}v_{t+1}$ by S_2. By the choice of P' and P the sets $(N_{S_1}(v_{s+1})\backslash\{v_k\})^+$, $N_{S_1}(v_{t+1})$, $N_{S_2}(v_{s+1})$, $(N_{S_2}(v_{t+1}))^+$, $N_R(v_{s+1})$,

$N_R(v_{t+1}), \{x\}$ are pairwise disjoint. This implies that

$$\begin{aligned} n &\geq |(N_{S_1}(v_{s+1})\backslash\{v_k\})^+| + |N_{S_1}(v_{t+1})| + |N_{S_2}(v_{s+1})| \\ &\quad +|(N_{S_2}(v_{t+1}))^+| + |N_R(v_{s+1})| + |N_R(v_{t+1})| + 1 \\ &\geq d_P(v_{s+1}) - 1 + d_P(v_{t+1}) + d_R(v_{s+1}) + d_R(v_{t+1}) + 1 \\ &= d(v_{s+1}) + d(v_{t+1}). \end{aligned}$$

So we have $d(v_{s+1}) + d(v_{t+1}) < n + 1$. i.e. $\min\{d(v_{s+1}), d(v_{t+1})\} < \frac{n+1}{2}$, say $d(v_{s+1}) < \frac{n+1}{2}$. Then $I(x, v_{s+1}) \geq \alpha_2(x, v_{s+1}) + 3$.

It is obvious that $N(x) \cap N(v_{s+1}) \subseteq V(P)$. Let $N(x) \cap N(v_{s+1}) = \{v_s, v_{i_1}, v_{i_2}, \ldots, v_{i_l}\}$, where $1 \leq i_1 < i_2 < \cdots < i_l \leq k$. The choice of P implies that $d(x, v_{s+1}) = 2$ and $N(x) \cap N(v_{s+1}) \subseteq N(P)$. Thus $I(x, v_{s+1}) - 2 \leq \alpha_2(x, v_{s+1})$, a contradiction.

Proof of Theorem 2. By contradiction. Let G be a non-Hamilton-connected graph satisfying the condition of Theorem 2. Then there exist vertices $u, v \in V(G)$ such that there is no Hamilton path between u and v. By Lemma there is an u-v path passing through all vertices of degree at least $\frac{n+1}{2}$. Among such paths, let $P = v_1v_2\cdots v_k$ be one of maximum length. Let $R = V(G)\backslash V(P)$. Then $R \neq \emptyset$. There exists a path P' of length at least 2 connecting two vertices of P with internally disjointing from P since G is 3-connected. Suppose $v_s, v_t \in V(P) \cap V(P')$, where $1 \leq s < t \leq k$. For G is 3-connected, we may assume $v_t \neq v_k$. We give P' an orientation from v_s to v_t. Using an analogous method in the proof of Theorem 1, we have $d(v_{s+1}) + d(v_{t+1}) < n + 1$ i.e. $\min\{d(v_{s+1}), d(v_{t+1})\} < \frac{n+1}{2}$, say $d(v_{s+1}) < \frac{n+1}{2}$. So there is a vertex $v \in R$ such that $d(v_{i_j+1}) < \frac{n+1}{2}$, where $v_{i_j} \in N_P(v) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$, $1 \leq i_1 < i_2 < \cdots < i_p \leq k$. Since $d(v, v_{i_j+1}) = 2$, $I(v, v_{i_j+1}) \geq S(v, v_{i_j+1}) + 1 \geq 3$ and by the definition of P, $p \geq 3$, we can obtain the following two claims.

(1) $M = \{v, v_{i_1+1}, v_{i_2+1}, \ldots, v_{i_{p-1}+1}\}$ is an independent set.

(2) $d(v, v_{i_j+1}) = 2$ and $N(v) \cap N(v_{i_j+1}) \subseteq N_P(v)$, $j = 1, 2, \ldots, p-1$.

By the similar proof of Theorem 1, we have for any distinct vertices $x, y \in \{v_{i_1+1}, v_{i_2+1}, \ldots, v_{i_{p-1}+1}\}$, $d(x) + d(y) < n + 1$. So there is at most one vertex of degree at least $\frac{n+1}{2}$ in the set $\{v, v_{i_1+1}, \ldots, v_{i_{p-1}+1}\}$. If such vertex exists, we assume $d(v_{i_s+1}) \geq \frac{n+1}{2}$, $1 \leq s \leq p-1$. Now we consider the following two cases.

Case 1. $d(v_{i_j+1}) < \frac{n+1}{2}$ for any $j \in \{1, 2, \ldots, p-1\}$.

Consider the following iterated definition.

Let $A_j^1 = \{v, v_{i_j+1}\}$, $B_j^1 = \{v_{i_j}\}$, $j = 1, 2, \ldots, p-1$. Clearly $A_j^1 \subseteq N(v_{i_j}) \cap M$, $B_j^1 \subseteq N(v_{i_j+1}) \cap N_P(v)$ and $|A_j^1| > |B_j^1|$.

Assume A_j^k, B_j^k with $A_j^k \subseteq N(v_{i_j}) \cap M$, $B_j^k \subseteq N(v_{i_j+1}) \cap N_P(v)$ and $|A_j^k| \geq |B_j^k|$, $j = 1, 2, \ldots, p-1$, $k \geq 1$ are well defined, such that there exists t $(1 \leq t \leq p-1)$, $|A_t^k| > |B_t^k|$.

By (1),(2) and the condition of Theorem we have $I(v, v_{i_t+1}) \geq S(v, v_{i_t+1}) + 1 \geq |A_t^k| + 1 \geq |B_t^k| + 2$. So $|(N(v_{i_t+1}) \cap N_P(v)) \backslash B_t^k| \geq 2$. Thus there exists r $(1 \leq r \leq p-1)$ such that $v_{i_r} \in (N(v_{i_t+1}) \cap N_P(v)) \backslash B_t^k$.

Hence we can define $A_j^{k+1} = A_j^k, B_j^{k+1} = B_j^k$, when $j \neq r, t$ and $1 \leq j \leq p-1$; $A_t^{k+1} = A_t^k, B_t^{k+1} = B_t^k \cup \{v_{i_r}\}$ and $A_r^{k+1} = A_r^k \cup \{v_{i_t+1}\}, B_r^{k+1} = B_r^k$.

Clearly $A_j^{k+1} \subseteq N(v_{i_j}) \cap M$, $B_j^{k+1} \subseteq N(v_{i_j+1}) \cap N_P(v)$ and $|A_j^{k+1}| \geq |B_j^{k+1}|$, $j = 1, 2, \ldots, p-1$. Particularly $|A_r^{k+1}| > |B_r^{k+1}|$ and $|B_t^{k+1}| = |B_t^k| + 1$.

Clearly the above iterative process can be continued infinitely.

Set $b_k = \sum_{j=1}^{p-1} |B_j^k|$, $k = 1, 2, \ldots$, then $0 < b_1 < b_2 < \cdots < b_k < \cdots$. On the other hand, $b_k = \sum_{j=1}^{p-1} |B_j^k| \leq (p-1)p$, $k = 1, 2, \ldots$, since $B_j^k \subseteq N_P(v)$, a contradiction.

Case 2. $d(v_{i_s+1}) \geq \frac{n+1}{2}$.

Let $I = \{i_1 + 1, i_2 + 1, \ldots, i_{p-1} + 1\}$ and $J = \{j | j \in I \text{ and } v_{i_s} v_j \in E(G)\}$. In this case we define $A_j^1 = \{v, v_{i_j+1}\}$, $B_j^1 = \{v_{i_j}\}$ if $j \in \{1, 2, \ldots, s-1, s+1, \ldots, p-1\} \backslash J$; $A_j^1 = \{v, v_{i_j+1}, v_{i_s+1}\}$, $B_j^1 = \{v_{i_j}, v_{i_s}\}$ if $j \in \{1, 2, \ldots, s-1, s+1, \ldots, p-1\} \cap J$. A_j^k, B_j^k, $k > 1$, $1 \leq j \neq s \leq p-1$ can be defined as case 1. We will also deduce a similar contradiction as case 1. This completes the proof of Theorem.

Corollary 1.([1]) *Let G be a 3-connected graph of order n. If* $\max\{d(u), d(v)\} \geq \frac{n+1}{2}$ *for any vertices $u, v \in V(G)$ with $d(u, v) = 2$, then G is Hamilton-connected.*

Corollary 2.([3]) *Let G be a graph of order n. If $d(u) + d(v) \geq n + 1$ for each pair u, v of nonadjacent vertices, then G is Hamilton-connected.*

Corollary 3. *Let G be a connected graph of order n. If $d_{G(u)}(x)+d_{G(u)}(y) \geq d(u)+2$ for any $u \in V(G)$, $\{x,y\} \subseteq V(G(u))$, $xy \notin E(G)$, then G is Hamilton-connected.*

Proof. It is sufficient to prove that G satisfies the condition of Theorem 2. Let $u,v \in V(G)$, $d(u,v) = 2$. By the definition of $S(u,v)$, there exists $w \in V(G)$ such that $\{u,v,x_1,\dots,x_{s-2}\} \subseteq N(w)$ is an independent set, where $s = S(u,v) \geq 2$. By the condition of Corollary 3, we have

$$d_{G(w)}(u) + d_{G(w)}(v) \geq d(w) + 2 = |V(G(w))| + 1,$$
$$V(G(w)) = N(w) \cup \{w\} \supseteq (N_{G(w)}(u) \cup N_{G(w)}(v)) \cup \{u,v,x_1,\dots,x_{s-2}\},$$
$$(N_{G(w)}(u) \cup N_{G(w)}(v)) \cap \{u,v,x_1,\cdots,x_{s-2}\} = \emptyset.$$

So

$$\begin{aligned}|V(G(w))| &\geq |N_{G(w)}(u) \cup N_{G(w)}(v)| + s \\ &= |N_{G(w)}(u)| + |N_{G(w)}(v)| - |N_{G(w)}(u) \cap N_{G(w)}(v)| + s.\end{aligned}$$

Hence

$$\begin{aligned}I(u,v) = |N(u) \cap N(v)| &\geq |N_{G(w)}(u) \cap N_{G(w)}(v)| \\ &\geq |N_{G(w)}(u)| + |N_{G(w)}(v)| - |V(G(w))| + s \\ &\geq d_{G(w)}(u) + d_{G(w)}(v) - (d(w)+1) + s \\ &\geq s + 1 = S(u,v) + 1.\end{aligned}$$

On the other hand, $|N(u) \cap N(v)| \geq S(u,v) + 1 \geq 3$ implies that G is 3-connected. Therefore Corollary 3 follows from Theorem 2.

Note that Corollary 3 is localization of Corollary 2, it can be used to some sparse graphs with large diameter.

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