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New Upper Bounds for Ramsey Numbers

HUANG YI RU† AND ZHANG KE MIN‡

The Ramsey number $R(G_1, G_2)$ is the smallest integer p such that for any graph G on p vertices either G contains G_1 or $\bar{G}$ contains G_2, where $\bar{G}$ denotes the complement of G. Let $R(m, n) = R(K_m, K_n)$. Some new upper bound formulas are obtained for $R(G_1, G_2)$ and $R(m, n)$, and we derive some new upper bounds for Ramsey numbers here.

The problem of determining Ramsey numbers is known to be very difficult. The few known exact values and several bounds for different G_1, G_2 or m, n are scattered among many technical papers (see [3]).

A graph G with order p is called a $(G_1, G_2; p)$-graph ($(m, n; p)$-graph, resp.) if G does not contain a G_1 and $\bar{G}$ does not contain a G_2 (K_m and K_n, resp.). It is easy to see that $R(G_1, G_2) = p_0 + 1$ iff $p_0 = \max\{p \mid \text{there exists a } (G_1, G_2; p)\text{-graph}\}$. In this paper, $f(G_1)$ ($g(G_2)$, resp.) denotes the number of G_1 (G_2, resp.) in G ($\bar{G}$, resp.) as a subgraph. The $(G_1, G_2; p)$-graph is called a $(G_1, G_2; p)$-Ramsey graph if $p = R(G_1, G_2) - 1$. Let d_i be the degree of vertex i in G of order p, and let $\bar{d}_i = p - 1 - d_i$, where $1 \le i \le p$. If G, H are graphs, $G \circ H$ denotes one of $\{G \vee H, G + H\}$-graph, where '$\vee$' is the join operation (see [1]). Let G_i^k ($i = 1, 2$) be a graph with order k and let $G_1 = G_1^{m-s} \circ G_1^s$, $G_2 = G_2^{n-t} \circ G_2^t$. Taking any vertex x (y, resp.), let $G_1^{s+1} = \{x\} \circ G_1^s$, $G_2^{t+1} = \{y\} \circ G_2^t$. The number of G_1^s (G_2^t, resp.) in G_1^{s+1} (G_2^{t+1}, resp.) as a subgraph is denoted by a_s (b_t, resp.). Thus we have:

THEOREM 1. *For any $(G_1, G_2; p)$-graph, the following inequalities must hold:*

$$a_s f(G_1^{s+1}) \le f(G_1^s)[R(G_1^{m-s}, G_2) - 1] \tag{1}$$

$$b_t g(G_2^{t+1}) \le g(G_2^t)[R(G_1, G_2^{n-t}) - 1]. \tag{2}$$

PROOF. In a $(G_1, G_2; p)$-graph G, by the definition of $R(G_1^{m-s}, G_2)$ and for any $G_1^s \subset G$, there are at most $R(G_1^{m-s}, G_2) - 1$ vertices x in $G - V(G_1^s)$ such that $\{x\} \circ G_1^s = G_1^{s+1}$, otherwise there is a G' ($\subset G - V(G_1^s)$) with order $R(G_1^{m-s}, G_2)$, either there is a $G_1^{m-s} \subset G'$ such that $G_1^{m-s} \circ G_1^s = G_1 \subset G$, or there is a $G_2 \subset \bar{G}' \subset \bar{G}$; a contradiction. Hence by the definition of $f(G_1^{s+1})$ and a_s, (1) follows.

Similarly, (2) is also true. □

Theorem 1 is a generalization of the theorem in [2].

COROLLARY 1. *If $G_1 = K_m$ or $K_m - e$, $G_2 = K_n$ or $K_n - e$, then for any $(G_1, G_2; p)$-graph G, the following inequalities must hold:*

$$(s + 1) f(K_{s+1}) \le f(K_s)[R(G_1^{m-s}, G_2) - 1] \tag{3}$$

$$(t + 1) g(K_{t+1}) \le g(K_t)[R(G_1, G_2^{n-t}) - 1] \tag{4}$$

where $G_1^{m-s} = K_{m-s}$ or $K_{m-s} - e$ and $G_2^{n-t} = K_{n-t}$ or $K_{n-t} - e$. In particular, if $G_1 = G_2 = K_n$, we have

$$f(K_{n-1}) + g(K_{n-1}) \le f(K_{n-2}) + g(K_{n-2}) \tag{5}$$

where $0 < s < m - 1$, $0 < t < n - 1$ and $3 \le m \le n$.

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PROOF. Note that for any K_{r+1}, it contains exactly $r+1$ K_r ($r \geq 1$). Hence, by (1) and (2), (3) and (4) follow. Furthermore, since $R(2,n) = R(n,2) = n$, (3) and (4), we obtain (5). □

COROLLARY 2. *For any* $(K_m - e, K_n - e; p)$*-graph, we have*

$$(s-1)f(K_{s+1}-e) \leq f(K_s - e)[R(K_{m-s}, K_n - e) - 1] \tag{6}$$

$$(t-1)g(K_{t+1}-e) \leq g(K_t - e)[R(K_m - e, K_{n-t}) - 1] \tag{7}$$

where $1 < s < m-1$, $1 < t < n-1$ *and* $4 \leq m \leq n$.

In particular, if $m = n$, we have:

$$f(K_4 - e) + g(K_4 - e) \leq \frac{1}{4}[R(K_{n-3}, K_n - e) - 1]\sum_{i=1}^{p} d_i \bar{d}_i. \tag{8}$$

PROOF. Note that for any $K_{r+1} - e$, it contains exactly $r-1$ $K_r - e$. Hence, by (1) and (2), (6) and (7) follow. On the other hand, since $f(K_3 - e) + g(K_3 - e) = \frac{1}{2}\sum_{i=1}^{p} d_i \bar{d}_i$, (6) and (7), we obtain (8). □

By the way, it is easy to obtain an analogous inequality as follows:

$$(n-3)[f(K_{n-1}-e) + g(K_{n-1}-e)] \leq (n-1)[f(K_{n-2}-e) + g(K_{n-2}-e)]. \tag{5$'$}$$

THEOREM 2. *For any graph* G_1 *with order* m (≥ 2) *and any graph* G_2 *with order* n (≥ 2),

$$R(G_1, G_2) \leq R(G_1^{m-1}, G_2) + R(G_1, G_2^{n-1}). \tag{9}$$

Furthermore, if $R(G_1^{m-1}, G_2)$ *and* $R(G_1, G_2^{n-1})$ *are both even, the strict inequality holds in (9).*

PROOF. Using Theorem 1 for $s = t = 1$ and $p = R(G_1, G_2) - 1$, we have

$$2f(K_2) \leq p[R(G_1^{m-1}, G_2) - 1] \tag{1'}$$

$$2g(K_2) \leq p[R(G_1, G_2^{n-1}) - 1]. \tag{2'}$$

Then $p(p-1) = 2\binom{p}{2} = 2[f(K_2) + g(K_2)] \leq p[R(G_1^{m-1}, G_2) + R(G_1, G_2^{n-1}) - 2]$. Thus we obtain (9).

If $R(G_1^{m-1}, G_2)$ and $R(G_1, G_2^{n-1})$ are both even, then (1′) and (2′) are strict when $p =$ odd, hence (9) is strict. When $p =$ even, $R(G_1, G_2)$ is odd, hence (9) is also strict. □

Clearly, (9) is a generalization of the classical inequality: $R(m,n) \leq R(m-1,n) + R(m,n-1)$.

Using Theorem 1 for $s = t = 2$, we can obtain a stronger theorem than (9). In the following, we only consider the cases: $G_1 = K_m$ or $K_m - e$ and $G_2 = K_n$ or $K_n - e$.

THEOREM 3. *Let* $G_1 = K_m$ *or* $K_m - e$ *and* $G_2 = K_n$ *or* $K_n - e$, *where* $3 \leq m \leq n$. *And let* $R(G_1^{m-2}, G_2) \leq \alpha + 1$, $R(G_1, G_2^{n-2}) \leq \beta + 1$; $R(G_1^{m-1}, G_2) \leq \gamma + 1$; $R(G_1, G_2^{n-1}) \leq \delta + 1$. *We have*

$$R(G_1, G_2) \leq \alpha + \beta + 4 + 2\sqrt{\alpha + \beta + 1 + \tfrac{1}{3}(\alpha^2 + \alpha\beta + \beta^2)}, \tag{10}$$

$$R(G_1, G_2) \leq \max\Big\{2r + 2 + \tfrac{1}{3}(\beta - \alpha), \tfrac{1}{2}(\beta + 3\gamma + 5) + \tfrac{1}{2}\sqrt{\gamma(4\alpha + 2\beta - 3\gamma + 6) + (\beta + 1)^2}\Big\}, \tag{11}$$

$$R(G_1, G_2) \leq \max\Big\{2\delta + 2 + \tfrac{1}{3}(\alpha - \beta), \tfrac{1}{2}(\alpha + 3\delta + 5) + \tfrac{1}{2}\sqrt{\delta(2\alpha + 4\beta - 3\delta + 6) + (\alpha + 1)^2}\Big\}. \tag{12}$$

PROOF. For any $(G_1, G_2; p)$-Ramsey graph, and letting $s = t = 2$, then by (3) + (4), we can obtain:

$$3\binom{p}{3} - \frac{3}{2}\sum_{i=1}^{p} d_i \bar{d}_i \le \alpha\binom{p}{2} + \frac{1}{2}(\beta - \alpha)\sum_{i=1}^{p} \bar{d}_i$$

i.e.

$$p(p-1)(p-2-\alpha) \le \sum_{i=1}^{p}(p-1-d_i)(3d_i+\beta-\alpha) \qquad (*)$$

Since $h(d) = (p-1-d)(3d+\beta-\alpha) \le h(d_0) = \frac{1}{12}(3p-3+\beta-\alpha)^2$ with $d_0 = \frac{1}{6}(3p-3+\alpha-\beta)$, by (*) we have $(p-1)(p-2-\alpha) \le h(d_0) = \frac{1}{12}(3p-3+\beta-\alpha)^2$. Thus we obtain (10).

In the following, we assume that $\gamma \le d_0$, i.e. $p \ge 2\gamma + 1 + \frac{1}{3}(\beta - \alpha)$. Since $d_i \le \gamma$ by the definition of γ, we obtain $h(d_i) \le h(\gamma)$. Hence we have $(p-1)(p-2-\alpha) \le h(\gamma) = (p-1-\gamma)(3\gamma+\beta-\alpha)$. Thus (11) follows.

Note that $R(G_1, G_2) = R(G_2, G_1)$. Hence (12) is true by (11). □

Using (10), when $G_1 = G_2$, we have a generalization formula from Walker [4]:

COROLLARY 3.

$$R(G_1, G_1) \le 4R(G_1^{n-2}, G_1) + 2. \qquad (13)$$

From the tables (1 and 2 here) in [3] we have the known nontrivial values and some upper bounds for $R(m, n)$ and two types of Ramsey number $R(G_1, G_2)$ including all known nontrivial values.

TABLE 1.
Known nontrivial values and some upper bounds for $R(m, n)$.

	m							
n	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	43
4		18	25	41	61	84	115	149
5			49	87	143	216	316	442
6				165	298	495	780	1171
7					540	1031	1713	2826
8						1870	3583	6090
9							6625	12715

TABLE 2.
Two types of Ramsey number $R(G_1, G_2)$ including all known nontrivial values.

	G_1							
G_2	$K_3 - e$	$K_4 - e$	$K_5 - e$	$K_6 - e$	$K_7 - e$	$K_8 - e$	$K_9 - e$	$K_{10} - e$
$K_3 - e$	3	5	7	9	11	13	15	17
K_3	5	7	11	17	21	25	31	36–39
$K_4 - e$	5	10	13	17	28			
K_4	7	11	19					
$K_5 - e$	7	13	22					
K_5	9	16	30–34					
$K_6 - e$	9	17						
K_6	11							

Now, by Theorems 1–3 and the formulas (9)–(13), and using Tables 1 and 2 we obtain the following 24 new upper bounds for the Ramsey number.

(1) $R(5,6) \leq 87$ since $(\alpha, \beta, \gamma) = (17, 24, 40)$ and (11);
(2) $R(5,7) \leq 143$ since $(\alpha, \beta, \gamma) = (22, 48, 60)$ and (11);
(3) $R(6,7) \leq 298$ since $(\alpha, \beta, \gamma) = (60, 86, 142)$ and (11);
(4) $R(7,8) \leq 1031$ since $(\alpha, \beta, \gamma) = (215, 297, 494)$ and (11);
(5) $R(7,9) \leq 1713$ since $(\alpha, \beta, \gamma) = (315, 539, 779)$ and (11);
(6) $R(8,10) \leq 6090$ since $(\alpha, \beta, \gamma) = (1170, 1869, 2825)$ and (11);
(7) $R(K_4, K_6 - e) \leq 36$ since $(\alpha, \beta, \gamma) = (5, 10, 16)$ and (11) or (9);
(8) $R(K_5 - e, K_6 - e) \leq 39$ by (9);
(9) $R(K_5 - e, K_6) \leq 59$ by Theorem 2;
(10) $R(K_3 - e, K_7) \leq 13$ by (9);
(11) $R(K_4 - e, K_7) \leq 36$ since $(\alpha, \beta, \gamma) = (1, 15, 12)$ and (11);
(12) $R(K_4 - e, K_8 - e) \leq 38$ since $(\alpha, \beta, \gamma) = (1, 16, 12)$ and (11);
(13) $R(K_4, K_7 - e) \leq 52$ since $(\alpha, \beta, \gamma) = (6, 18, 20)$ and (11);
(14) $R(K_4, K_8 - e) \leq 78$ since $(\alpha, \beta, \gamma) = (7, 35, 24)$ and (11);
(15) $R(K_5 - e, K_7 - e) \leq 66$ since $(\alpha, \beta, \gamma) = (10, 21, 27)$ and (11);
(16) $R(K_5 - e, K_7) \leq 92$ since $(\alpha, \beta, \gamma) = (12, 33, 35)$ and (11);
(17) $R(K_5, K_6 - e) \leq 67$ since $(\alpha, \beta, \gamma) = (16, 15, 35)$ and (11);
(18) $R(K_5, K_7 - e) \leq 112$ since $(\alpha, \beta, \gamma) = (20, 33, 51)$ and (11) or (10);
(19) $R(K_6 - e, K_6 - e) \leq 70$ by (13);
(20) $R(K_6 - e, K_7 - e) \leq 135$ by Theorem 2;
(21) $R(K_6 - e, K_6) \leq 125$ since $(\alpha, \beta, \gamma) = (25, 35, 58)$ and (11) or (10);
(22) $R(K_6 - e, K_7) \leq 207$ since $(\alpha, \beta, \gamma) = (35, 66, 91)$ and (11);
(23) $R(K_6, K_7 - e) \leq 224$ since $(\alpha, \beta, \gamma) = (51, 58, 111)$ and (11);
(24) $R(K_7 - e, K_7 - e) \leq 266$ by (13) etc.

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HUANG YI RU
Department of Mathematics,
Shanghai University,
Shanghai, 201800,
People's Republic of China

ZHANG KE MIN
Department of Mathematicas,
Nanjing University,
Nanjing, 210093,
People's Republic of China