

A New Upper Bound Formula for Two Color Classical Ramsey Numbers

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ABSTRACT. The two color Ramsey number $R(k, l)$ is the smallest integer p such that for any graph G on p vertices either G contains a K_k or \overline{G} contains a K_l , where \overline{G} denotes the complement of G . A new upper bound formula is given for two color Ramsey numbers. For example, we get $R(7, 9) \leq 1713$, $R(8, 10) \leq 6090$ etc.

The problem of determining the Ramsey numbers is known to be very difficult and so we are often satisfied with partial results, e. g. upper or lower bounds.

An $(m, n; p)$ -graph is a graph with order p which has no K_m and no \overline{K}_n as a subgraph. If p is unspecified, the graph will be called an (m, n) -graph. The Ramsey number $R(m, n)$ is the smallest integer p such that for any graph with order p , either G contains a K_m or \overline{G} contains a K_n . It is easy to see that $R(m, n) = p$ iff the largest (m, n) -graph has $p - 1$ vertices. In

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this paper, we will use $N(K_i)$ ($N(\overline{K}_j)$ resp.) to denote the number of K_i (\overline{K}_j resp.) in G .

Theorem 1. For any (m, n) -graph G , the following inequalities must hold:

$$(s + 1)N(K_{s+1}) \leq N(K_s)[R(m - s, n) - 1] \quad 0 < s < m - 1, \quad (1.1)$$

$$(t + 1)N(\overline{K}_{t+1}) \leq N(\overline{K}_t)[R(m, n - t) - 1] \quad 0 < t < n - 1. \quad (1.2)$$

In particular for any (n, n) -graph, then

$$N(K_{n-1}) + N(\overline{K}_{n-1}) \leq N(K_{n-2}) + N(\overline{K}_{n-2}). \quad (1.3)$$

Proof: By the definition of Ramsey number, for any K_s , there are at most $R(m - s, n) - 1$ vertices in G which form a K_{s+1} in combination with the K_s . Otherwise there exists either a K_m or a \overline{K}_n as a subgraph of G , a contradiction. On the other hand, for any K_{s+1} , it contains exactly $s + 1$ K_s . Hence (1.1) is true.

Similarly, we can prove that (1.2) is true.

Using (1.1), (1.2) and $R(2, n) = R(n, 2) = n$, it is easy to prove that (1.3) is true. \square

Note that (1.3) and the following facts:

$$\begin{cases} N(K_2) + N(\overline{K}_2) = \frac{1}{2}p(p - 1) > 2p = N(K_1) + N(\overline{K}_1) & \text{if } p > R(3, 3) - 1 = 5, \\ N(K_2) + N(\overline{K}_2) \leq N(K_1) + N(\overline{K}_1) & \text{if } p \leq R(3, 3) - 1 = 5; \end{cases}$$

$$\begin{cases} N(K_3) + N(\overline{K}_3) = \frac{1}{6}p(p - 1)(p - 2) - \frac{1}{2} \sum_i d_i(p - 1 - d_i) \\ \geq \frac{1}{6}p(p - 1)(p - 2) - \frac{1}{8}p(p - 1)^2 > N(K_2) + N(\overline{K}_2) & \text{if } p > R(4, 4) - 1 = 17, \\ N(K_3) + N(\overline{K}_3) \leq N(K_2) + N(\overline{K}_2) & \text{if } p \leq R(4, 4) - 1 = 17. \end{cases}$$

where $\{d_1, d_2, \dots, d_p\}$ is a degree sequence of G .

So, we raise a conjecture as follows:

Conjecture 2: Let n and p be natural numbers, $p > R(n, n) - 1$. Then

$$N(K_{n-1}) + N(\overline{K}_{n-1}) > N(K_{n-2}) + N(\overline{K}_{n-2}).$$

Now, let $s = t = l$ and $p = R(m, n) - 1$, by Theorem 1, then

$$2N(K_2) \leq N(K_1)[R(m - 1, n) - 1], \quad (2.1)$$

$$2N(\overline{K}_2) \leq N(\overline{K}_1)[R(m, n - 1) - 1]. \quad (2.2)$$

Thus we have $p(p - 1) = 2(N(K_2) + N(\overline{K}_2)) \leq p[R(m - 1, n) + R(m, n - 1) - 2]$. i.e. $R(m, n) \leq R(m - 1, n) + R(m, n - 1)$.

Note that when $R(m-1, n)$, $R(m, n-1)$ are even, then $R(m, n) \leq R(m-1, n) + R(m, n-1) - 1$. In fact, if $R(m, n)$ is odd, clearly it is true. If $R(m, n)$ is even, i.e. $N(K_1) = N(\overline{K}_1)$ is odd, thus, by Theorem 1, $2N(K_2) \leq N(K_1)[R(m-1, n) - 1] - 1$ and $2N(\overline{K}_2) \leq N(\overline{K}_1)[R(m, n-1) - 1] - 1$. Therefore we have $p(p-1) = 2[N(K_2) + N(\overline{K}_2)] \leq p[R(m-1, n) + R(m, n-1) - 2] - 2$, i.e. it is also true.

For the case $s = t = 2$, we can obtain a deeper result.

Theorem 3. Let $a + 1$, $b + 1$ and $c + 1$ be upper bounds on $R(m-2, n)$, $R(m, n-2)$ and $R(m-1, n)$ respectively. If $p = R(m, n) - 1 \geq 2c + 1 + \frac{1}{3}(b-a)$ and $m \leq n$, then

$$R(m, n) \leq \frac{1}{2}(b + 3c + 5) + \frac{1}{2}\sqrt{(b + 3c + 3)^2 - 8 - 4a - 4(1 + c)(3c + b - a)}.$$

Proof: Let $s = t = 2$, $p = R(m, n) - 1$ and G be an $(m, n; p)$ -graph by Theorem 1, we have that $3N(K_3) + 3N(\overline{K}_3) \leq aN(K_2) + bN(\overline{K}_2)$. Note that $N(K_3) + N(\overline{K}_3) = \frac{1}{6}p(p-1)(p-2) - \frac{1}{2}\sum_i d_i(p-1-d_i)$ and $N(K_2) + N(\overline{K}_2) = \frac{1}{2}p(p-1)$, thus we have

$$p(p-1)(p-2-a) \leq \sum_i (p-1-d_i)(3d_i + b-a) \quad (*)$$

Let $f(d) = (p-1-d)(3d+b-a)$. There is an unique maximum value of $f(d)$ at $d_0 = \frac{1}{6}(3p-3-b+a)$. Since G is $(m, n; p)$ -graph and $p \geq 2c+1 + \frac{1}{3}(b-a)$, $d_i \leq c \leq d_0$. Thus $f(d_i) \leq f(c)$. We substitute $f(c)$ for $f(d_i)$ in $(*)$. Hence $p(p-1)(p-2-a) \leq p(p-1-c)(3c+b-c)$, and then $[p - \frac{1}{2}(b+3c+3)]^2 \leq \frac{1}{4}(b+3c+3)^2 - 2-a - (1+c)(3c+b-a)$.

This completes the proof. □

Corollary. ([1], Theorem 2.4) $R(n, n) \leq 4R(n-2, n) + 2$.

Proof: Let G be an $(m, n; p)$ -graph, where $p = R(n, n) - 1$. Since $f(d_i) \leq f(d_0) = \frac{1}{12}(3p-3+b-a)^2$, $a = b$ and $(*)$, $p(p-1)(p-2-a) \leq \frac{1}{12}(3p-3)^2 p$. i.e. $p \leq 5 + 4a$. Let $a = R(n-2, n) - 1$. Thus we have $R(n, n) \leq 4R(n-2, n) + 2$. □

$m \quad n$	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	43
4		18	25	41	61	84	115	149
5			49			216	316	442
6				165		495	780	1171
7					540			2826
8						1870		
9							6625	

Table 1.

Known nontrivial values and some upper bounds for $R(m, n)$

Using Theorem 3 and Table 1 in [2], we can obtain Table 2 as follows:

$m \quad n$	5	6	7	8	9	10
5		87	143			
6			298			
7				1031	1713	
8					3583*	6090
9						12715*

* Using $R(m, n) \leq R(m - 1, n) + R(m, n - 1)$.

Table 2. Some new upper bounds for $R(m, n)$

References

- [1] F.R.K. Chung and C.M. Grinstead, A survey of bounds for classical Ramsey numbers, *J. Graph Theory* 7 (1983), 25–37.
- [2] S.P. Radziszowski, Small Ramsey numbers, *The Electronic J. of Combinatorics* 1 (1994), DS1.