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Pancyclic out-arcs of a vertex in tournaments ☆

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Abstract

Thomassen (J. Combin. Theory Ser. B 28, 1980, 142–163) proved that every strong tournament contains a vertex x such that each arc going out from x is contained in a Hamiltonian cycle. In this paper, we extend the result of Thomassen and prove that a strong tournament contains a vertex x such that every arc going out from x is pancyclic, and our proof yields a polynomial algorithm to find such a vertex. Furthermore, as another consequence of our main theorem, we get a result of Alspach (Canad. Math. Bull. 10, 1967, 283–286) that states that every arc of a regular tournament is pancyclic. © 2000 Elsevier Science B.V. All rights reserved.

1. Terminology and introduction

We denote the vertex set and the arc set of a digraph D by $V(D)$ and $E(D)$, respectively. A subdigraph induced by a subset $A \subseteq V(D)$ is denoted by $D[A]$. In addition, $D - A = D[V(D) - A]$.

If xy is an arc of a digraph D, then we say that x *dominates* y and write $x \to y$. We also say that xy is an *out-arc* of x or xy is an *in-arc* of y. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and write $A \to B$.

Let x be a vertex of D. The number of out-arcs of x is called the *out-degree* of x and denoted by $d_D^+(x)$, or simply $d^+(x)$. Note that a tournament T_n is regular if and only if all vertices of T_n have the same out-degree.

We consider only the directed paths and cycles. A digraph D is strong if for every pair of vertices x and y, D contains a path from x to y and a path from y to x, and

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D is k-connected if $|V(D)| \geqslant k+1$ and for any set $A \subset V(D)$ of at most $k-1$ vertices, $D-A$ is strong.

A k-cycle is a cycle of length k. An arc of a digraph on $n \geqslant 3$ vertices is said to be *pancyclic* if it is contained in a k-cycle for all k satisfying $3 \leqslant k \leqslant n$.

In [4], Thomassen proved that every strong tournament contains a vertex x such that each out-arc of x is contained in a Hamiltonian cycle and this extends the result of Goldberg and Moon [2] that every s-strong tournament has at least s distinct Hamiltonian cycles (a digraph D is s-strong if for any set $F \subset E(D)$ of at most $s-1$ arcs, $D-F$ is strong).

In this paper we extend the result of Thomassen and prove that every strong tournament contains a vertex x such that all out-arcs of x are pancyclic, and our proof yields a polynomial algorithm to find such a vertex. In addition, as another consequence of our main theorem, we get a result of Alspach [1] that states that every arc of a regular tournament is pancyclic.

2. Preliminaries

To prove our main theorem, we consider only the strong tournaments with minimum out-degree at least 2 in this section.

Definition 2.1. Let T be a strong tournament with minimum out-degree at least two. A vertex v of T is called a *bridgehead* if there is a partition (A,B) of $V(T)$ such that the following conditions are satisfied:
(1) $v \in A$, $T[A]$ is non-trivial and strong;
(2) $B \rightarrow A \backslash \{v\}$.

We denote a bridgehead v with respect to the partition (A,B) by $v = brd(A|B)$.

It is obvious that if a strong tournament T (with minimum out-degree at least two) has a bridgehead $v = brd(A|B)$, then $T-v$ is not strong and

$$|A| \geqslant 5. \tag{1}$$

Lemma 2.2. *Let T_n be a strong tournament on n vertices with minimum out-degree at least two and assume that the vertices of T_n are labeled $u_1, u_2, \ldots, u_n$ such that $d^+(u_1) \leqslant d^+(u_2) \leqslant \cdots \leqslant d^+(u_n)$. If T_n has a bridgehead $v = brd(A|B)$, then the following holds*:
(a) $\{u_1, u_2, u_3, u_4\} \subset A$.
(b) $\{u_1, u_2\}$ *contains at most one bridgehead.*

Proof. It is easy to see by (1) and Definition 2.1 that (a) is true.

To prove (b), we suppose, on the contrary, that u_1 and u_2 both are bridgeheads of T_n with $u_1 = brd(A_1|B_1)$ and $u_2 = brd(A_2|B_2)$. Then we see from (a) that $\{u_1, u_2\} \subset A_i$ for $i = 1, 2$. By Definition 2.1, there is a vertex $b_i \in B_i$ with $u_i \rightarrow b_i$ for $i = 1, 2$. Note that $\{u_1,u_2\}\cap\{b_1,b_2\}=\emptyset$. Since $u_i \in A_{3-i}$ and $u_i \to b_i$, we have $b_i \in A_{3-i}$, and hence, $b_{3-i} \to b_i$ for $i=1,2$. This contradicts the fact that T_n is a tournament. □

By Lemma 2.2, every strong tournament T_n with minimum out-degree at least two contains at least one vertex that is not a bridgehead.

3. Main results

Theorem 3.1. *Let T_n be a strong tournament on n vertices and assume that the vertices of T_n are labeled $u_1,u_2,\dots,u_n$ such that $d^+(u_1)\leqslant d^+(u_2)\leqslant\cdots\leqslant d^+(u_n)$. Let u be a vertex of T_n which can be chosen as follows:*
(1) *if $d^+(u_1)=1$ then $u=u_1$,*
(2) *if $d^+(u_1)\geqslant 2$ then*

$$d^+(u)=\min\{d^+(x)\,|\,x\in V(T_n) \text{ and } x \text{ is not a bridgehead}\}.$$

Then all out-arcs of u are pancyclic.

Proof. Suppose first that $d^+(u_1)=1$ and let uv be the only out-arc of $u=u_1$. By the well-known theorem of Moon [3], T_n is vertex pancyclic (i.e., every vertex of T_n is in a k-cycle for all k with $3\leqslant k\leqslant n$). Because uv is in all cycles through u, it is a pancyclic arc.

Suppose now that $d^+(u_1)\geqslant 2$ (i.e., the minimum out-degree of T_n is at least two). According to Lemma 2.2(b) and the choice of u, we have $u=u_1$ or u_2. In addition, we may assume, by relabelling if necessary, that $u=u_2$ only if u_1 is a bridgehead and $d^+(u_1)<d^+(u_2)$.

Let $u\to v$ be an arc of T_n. We prove by induction on the length of the cycle through uv that uv is pancyclic.

We first show that $u\to v$ is contained in a 3-cycle. If $d^+(u)\leqslant d^+(v)$, then there is a vertex $w\in V(T_n)$ such that $v\to w\to u$, and hence, $vwuv$ is a 3-cycle. If $d^+(u)>d^+(v)$, then $v=u_1$ must be a bridgehead and $u=u_2$. Let $v=brd(A|B)$. By Lemma 2.2(a), we see that $u\in A$. So, there is a vertex $w\in B$ such that $v\to w\to u$, i.e., uv is contained in a 3-cycle.

Let $C=v_1v_2\cdots v_kv_1$ be a k-cycle with $3\leqslant k<n$ which contains the arc uv and assume without loss of generality that $v_k=u$ and $v_1=v$. We show that there is a $(k+1)$-cycle through uv.

Let $W=V(T_n)\backslash V(C)$ and

$$M=\{x\in W \mid \text{there are } i \text{ and } j \text{ with } 1\leqslant i<j\leqslant k \text{ such that } v_i\to x\to v_j\},$$

$$M_0=\{x\in W\,|\,\{v_1,v_2,\dots,v_k\}\to x\},$$

$$M_i=\{x\in W\,|\,\{v_{i+1},\dots,v_k\}\to x\to\{v_1,\dots,v_i\}\} \quad \text{for } i=1,2,\dots,k-1,$$

$$M_k=\{x\in W\,|\,x\to\{v_1,v_2,\dots,v_k\}\}.$$

We consider the following cases:

Case 1: $M \neq \emptyset$. Let $x \in M$. It is easy to see that there is an integer t with $1 \leqslant t < k$ such that $v_t \to x \to v_{t+1}$. Thus, $v_1 v_2 \cdots v_t x v_{t+1} v_{t+2} \cdots v_k v_1$ is a $(k+1)$-cycle through uv.

Case 2: $M_0 \neq \emptyset$. Let $x \in M_0$. We shall show that there is a vertex $y \in W$ such that $x \to y \to u$, and hence, $uvv_2 \cdots v_{k-2} xyu$ will be a desired $(k+1)$-cycle. In fact, if $d^+(u) \leqslant d^+(x)$, then it is easy to find such a vertex y. Assume now that $d^+(u) > d^+(x)$. Then, x is a bridgehead. Let $x = brd(A|B)$. We see from Lemma 2.2(a) that $u \in A$. So, there is a vertex $y \in B$ such that $x \to y \to u$. Clearly, y belongs to W and it is the desired vertex.

Case 3: $k \geqslant 4$ and $M_2 \cup M_3 \cup \cdots \cup M_{k-2} \neq \emptyset$. Let $x \in M_j$ for some j satisfying $2 \leqslant j \leqslant k-2$. Then, $\{v_{k-1}, v_k\} \to x \to \{v_1, v_2\}$ holds.

Suppose that $d^+(u) > d^+(v)$. This implies that v is a bridgehead of T_n. Let $v = brd(A|B)$. Then, u belongs to A by Lemma 2.2(a) and there is a vertex $y \in B$ such that $v \to y \to u$. If $y \in W$, then $y \in M$ and we are done by Case 1. So, we assume that $y \in V(C)$ and $y = v_i$ for some i with $2 \leqslant i \leqslant k-1$. Since $u \to x \to v_2$ and $u \in A$, we have $\{x, v_2\} \subset A$. This implies that $i > 2$. Because of $v_{i-1} \neq v$ and $v_{i-1} \to v_i$, v_{i-1} belongs to B. It follows that $v_{i-1} \to u$. Thus, we see that $uvv_i v_{i+1} \cdots v_{k-1} x v_2 v_3 \cdots v_{i-1} u$ is a $(k+1)$-cycle.

Suppose now that $d^+(u) \leqslant d^+(v)$. If $d^+_C(u) < d^+_C(v)$, then it is not difficult to see that there is an integer i with $3 \leqslant i \leqslant k-1$ such that $v \to v_i$ and $v_{i-1} \to u$. Now we see that $uvv_i v_{i+1} \cdots v_{k-1} x v_2 v_3 \cdots v_{i-1} u$ is a $(k+1)$-cycle. So, we assume that $d^+_C(u) \geqslant d^+_C(v)$. Since $u \to x \to v$ and $d^+_W(u) \leqslant d^+_W(v)$, there is $y \in W$ such that $v \to y \to u$. Thus, $y \in M$ and we are done by Case 1.

Case 4: $W = M_1 \cup M_{k-1} \cup M_k$. If $M_1 = \emptyset$, then we see that $u = brd(V(C)|W)$, a contradiction.

Suppose that $M_{k-1} \cup M_k = \emptyset$ (i.e., $W = M_1$). It is obvious that $d^+(u) > d^+(x)$ for each $x \in W$. It follows that $|W| = 1$, and hence, $d^+(x) = 1$, a contradiction to the assumption that the minimum out-degree of T_n is at least two.

So, we assume that $M_1 \neq \emptyset$ and $M_{k-1} \cup M_k \neq \emptyset$. If $M_{k-1} \cup M_k \to M_1$, then we see that $u = brd(M_1 \cup V(C) \mid M_{k-1} \cup M_k)$, a contradiction. Therefore, there is a vertex $x \in M_1$ and a vertex $y \in M_{k-1} \cup M_k$ with $x \to y$. Note that $d^+(v) \geqslant 2$ yields $k \geqslant 4$.

In the case when $k = 4$, we have $d^+(v) = 2$ and $v \to v_3$. If $u \to y$, then $y \in M_{k-1}$, and hence, $d^+(u) > d^+(v_3) \geqslant d^+(v)$, a contradiction to the choice of u. Thus, $y \to u$. Now, we see that $uvv_2 xyu$ is a 5-cycle.

In the remaining case when $k \geqslant 5$, the cycle $v_k v_1 v_2 x y v_4 v_5 \cdots v_k$ is a $(k+1)$-cycle containing the arc uv.

The proof of the theorem is complete. □

As an immediate consequence of Theorem 3.1 and Lemma 2.2(b), we get the following results:

Corollary 3.2. *A strong tournament contains a vertex u such that every out-arc of u is pancyclic.*

Remark 3.3. Let $n \geqslant 5$ be an integer. Note that the tournament T_n with the vertex set $\{v_1, v_2, \ldots, v_n\}$ and the arc set

$$\{v_i v_j \mid 2 \leqslant i < j \leqslant n\} \cup \{v_{n-1}v_1, v_n v_1\} \cup \{v_1 v_j \mid 2 \leqslant j \leqslant n-2\},$$

contains exactly one vertex v_n whose out-arcs are pancyclic.

Remark 3.4. Using Theorem 3.1, it is easy to get a polynomial algorithm to find a vertex u in a strong tournament such that all out-arcs of u is pancyclic.

Corollary 3.5 (Thomassen [4]). *Every strong tournament contains a vertex u such that each arc going out from u lies on a Hamiltonian cycle.*

Corollary 3.6 (Alspach [1]). *All arcs of a regular tournament are pancyclic.*

Proof. It is a simple matter to verify that a regrular tournament contains no bridgehead. Since every vertex of a regular tournament has the same out-degree, this corollary holds by Theorem 3.1. □

Similarly, by showing that an almost regular tournament has no bridgehead, we obtain the next result:

Corollary 3.7. *In an almost regular tournament T on $2n$ vertices, all out-arcs of the vertices with out-degree $n-1$ are pancyclic.*

Corollary 3.8. *In a 2-connected tournament, all out-arcs of the vertices with minimum out-degree are pancyclic.*

Finally, we give the following conjecture:

Conjecture 3.9. *A k-connected tournament T_n has at least k vertices $v_1, v_2, \ldots, v_k$ such that all out-arcs of v_i are pancyclic for $i = 1, 2, \ldots, k$.*

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