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The local exponent sets of primitive digraphs ☆

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Abstract

Let $D = (V, E)$ be a primitive digraph. The local exponent of D at a vertex $u \in V$, denoted by $\exp_D(u)$, is defined to be the least integer k such that there is a directed walk of length k from u to v for each $v \in V$. Let $V = \{1, 2, \ldots, n\}$. The vertices of V can be ordered so that $\exp_D(1) \leqslant \exp_D(2) \leqslant \cdots \leqslant \exp_D(n) = \gamma(D)$. We define the kth local exponent set $E_n(k) := \{\exp_D(k) \mid D \in PD_n\}$, where PD_n is the set of all primitive digraphs of order n. It is known that $E_n(n) = \{\gamma(D) \mid D \in PD_n\}$ has been completely settled by K. Zhang [Linear Algebra Appl. 96 (1987) 102–108]. In 1998, $E_n(1)$ was characterized by J. Shen and S. Neufeld [Linear Algebra Appl. 268 (1998) 117–129]. In this paper, we describe $E_n(k)$ for all n, k with $2 \leqslant k \leqslant n - 1$. So the problem of local exponent sets of primitive digraphs is completely solved. © 2000 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction and notations

Let $D = (V, E)$ be a digraph on n vertices. We permit loops but no multiple arcs in D. A $u \to v$ walk of length p in D is a sequence of vertices $u, u_1, \ldots, u_p = v$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_{p-1}, v)$, where the vertices and the arcs are not necessarily distinct. A closed walk is a $u \to v$ walk, where $u = v$. A path is

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a walk with distinct vertices. A cycle is a closed $u \to v$ walk with distinct vertices except for $u = v$. An r-cycle is a cycle of length r. $L(D)$ is the length set of all cycles of D.

Let $u, v \in V$. We use the notation $u \to v[k]$ ($u \not\to v[k]$, resp.) to mean that there is a $u \to v$ walk (no $u \to v$ walk, resp.) with length k in D . If $D_0 \subset D$, we use $u \to v[D_0]$ such that any $u \to v$ path in D_0. If $N^+(v) = N^+(u)$ and $N^-(v) = N^-(u)$, we call v a copy of u. A digraph D is primitive if there exists an integer k such that $u \to v[k]$ for every pair $u, v \in V$. The least such k is called the exponent of D, denoted $\gamma(D)$. Let PD_n be the set of all primitive digraphs of order n. Let $L(n) := \{(p, q) \mid 2 \leqslant p < q \leqslant n, p + q > n, \gcd(p, q) = 1\}$ and $L'(n) := \{(p, q) \mid 1 \leqslant p < q \leqslant n, p + q > n, \gcd(p, q) = 1\}$.

Wielandt [8] found that $\gamma(D) \leqslant w_n = (n-1)^2 + 1$ and showed that there is a unique digraph $W(n)$ that attains this bound, where $W(n) = (V, E)$ is defined as follows: $V = \{v_i \mid 1 \leqslant i \leqslant n\}$ and $E = \{(v_i, v_{i+1}) \mid 1 \leqslant i \leqslant n-1\} \cup \{(v_{n-1}, v_1), (v_n, v_1)\}$. Dulmage and Mendelsohn [2] observed that there are gaps in the exponent set $E_n = \{\gamma(D) \mid D \in PD_n\}$. Each gap is a set S of consecutive integers below w_n such that no $D \in PD_n$ has an exponent in S. Lewin and Vitek [3] found a general method for determining all the gaps between $\lfloor \frac{1}{2} w_n \rfloor + 1$ and w_n, and they conjectured that there is no gap in $\{1, 2, \ldots, \lfloor \frac{1}{2} w_n \rfloor + 1\}$. Jiayu [5] proved that the conjecture is true when n is sufficiently large and gave a counterexample to show that the conjecture is not true when $n = 11$. Zhang [9] proved that the conjecture is true except 48 for $n = 11$. Therefore, the problem of determining the exponent set is completely solved.

Let $D \in PD_n$ with $L(D) = \{s_1, s_2, \ldots, s_\lambda\}$. Let $u, v \in V(D)$. The relative distance $d_{L(D)}(u, v)$ from u to v is the length of the shortest walk from u to v which meets at least one s_i-cycle for $i = 1, 2, \ldots, \lambda$. The exponent from u to v, denoted by $\exp_D(u, v)$, is the least integer k such that $u \to v[k]$ for all $m \geqslant k$. Clearly, $\gamma(D) = \max_{u,v \in V(D)} \exp_D(u, v)$.

Now let $\{s_1, s_2, \ldots, s_\lambda\}$ be a set of distinct positive integers. Then $\phi(s_1, s_2, \ldots, s_\lambda)$ is defined to be the least integer m such that every integer $k \geqslant m$ can be expressed in the form $k = \sum_{i=1}^{\lambda} a_i s_i$, where a_i $(i = 1, 2, \ldots, \lambda)$ are nonnegative integers. A result due to Schur shows that $\phi(s_1, s_2, \ldots, s_\lambda)$ is well-defined if $\gcd(s_1, s_2, \ldots, s_\lambda) = 1$. It is known that $\phi(s_1, s_2) = (s_1 - 1)(s_2 - 1)$ if $\gcd(s_1, s_2) = 1$, and $\phi(s_1, s_2, s_3) = \phi(s_1, s_2)$ if $s_1 \mid s_3$.

Lemma 1 [1]. *Let $D \in PD_n$ with $L(D) = \{s_1, s_2, \ldots, s_\lambda\}$. Let $u, v \in V(D)$. Then* $\exp_D(u, v) \leqslant d_{L(D)}(u, v) + \phi(s_1, s_2, \ldots, s_\lambda)$. *Furthermore*, $\gamma(D) \leqslant \max d_{L(D)}(u, v) + \phi(s_1, s_2, \ldots, s_\lambda)$.

The local exponent of D at vertex $u \in V$, denoted by $\exp_D(u)$, is the least integer k such that $u \to v[k]$ for each $v \in V$. Clearly, $\exp_D(u) = \max_{v \in V(D)} \exp_D(u, v)$ and $\gamma(D) = \max_{u \in V(D)} \exp_D(u)$. Let $V = \{1, 2, \ldots, n\}$. Then the vertices can be ordered so that $\exp_D(1) \leqslant \exp_D(2) \leqslant \cdots \leqslant \exp_D(n) = \gamma(D)$.

Brualdi and Liu [1] obtained $\max\{\exp_D(k) \mid D \in PD_n\} = n^2 - 3n + 2 + k$. Shao et al. [6] characterized the extremal digraphs of $\exp_D(k)$. Let $E_n(k) := \{\exp_D(k) \mid D \in PD_n\}$ for each k $(1 \leqslant k \leqslant n)$. Clearly, $E_n(n) = E_n$. Zhang [9] obtained $E_n(n)$, and Shen and Neufeld [7] obtained $E_n(1)$ for all n. Let $m(p, q, k) = (p-1)(q-1) + \max\{k + q - n - 1, 0\}$ and

$$M(p,q,k) = \begin{cases} p(q-2) + k + (n-q) & \text{if } 1 \leqslant k \leqslant p+q-n+1, \\ p(q-2) + k + (n-q-1) & \text{if } p+q-n+2 \leqslant k \leqslant p+q-n+3, \\ \vdots & \vdots \\ p(q-2) + k + 1 & \text{if } n-q+p-2 \leqslant k \leqslant n-q+p-1, \\ p(q-2) + k & \text{if } n-q+p \leqslant k \leqslant n. \end{cases}$$

In this paper, we consider $E_n(k)$ for all n, k with $2 \leqslant k \leqslant n-1$ and obtain the following:

Main Theorem. *Let n, k be integers with* $2 \leqslant k \leqslant n-1$. *Then*

$$E_n(k) = \left\{1, 2, \ldots, \lfloor \tfrac{1}{2}(n-2)^2 \rfloor + k\right\} \cup \bigcup_{(p,q)\in L(n)} \{m \mid m(p,q,k) \leqslant m \leqslant M(p,q,k)\}$$

for all n, k with $2 \leqslant k \leqslant n-1$ *except* $n = 11$ *and* $9 \leqslant k \leqslant 10$. *And*

$$E_{11}(k) = (\{1, 2, \ldots, 40 + k\} \backslash \{37 + k\}) \cup \bigcup_{(p,q)\in L(11)} \{m \mid m(p,q,k) \leqslant m \leqslant M(p,q,k)\}$$

for $9 \leqslant k \leqslant 10$.

2. Bounds of the kth local exponent $\exp_D(k)$

Let $D = (V, E)$ be a digraph, and $u \in V$. For $i \geqslant 1$, let $R_i(u) := \{v \in V \mid u \to v[i]\}$. Also we define $R_0(u) := \{u\}$. In this section, we always assume that $D \in PD_n$.

Lemma 2 [1]. $\exp_D(k) \leqslant \exp_D(k-1) + 1$ *for* $2 \leqslant k \leqslant n$.

Lemma 3 [6]. *If s be the girth of D, then* $\exp_D(k) \leqslant s(n-2) + k$ *for* $1 \leqslant k \leqslant n$.

Lemma 4 [7]. *If* $L(D)=\{p,q\}$ *with* $(p,q)\in L'(n)$, *then* $\exp_D(1)\geqslant \max\{(p-1)(q-1),q-1\}$.

Lemma 5 [7]. *Suppose that* $\{p,q\}\subset L(D)$, $\gcd(p,q)=1$ *and* $p<q$. *If D contains a p-cycle which intersects a q-cycle, then* $\exp_D(k)\leqslant p(q-2)+n-q+k$ *for* $1\leqslant k\leqslant n$.

Corollary 1. *If* $L(D)=\{p,q\}$ *with* $(p,q)\in L'(n)$, *then* $\exp_D(k)\leqslant p(q-2)+n-q+k$ *for* $1\leqslant k\leqslant n$.

Lemma 6 [4]. *If* $L(D)=\{p,q\}$ *with* $p+q\leqslant n$, *then* $\exp_D(k)\leqslant \lfloor\frac{1}{4}(n-1)^2\rfloor+k$ *for* $1\leqslant k\leqslant n$.

Lemma 7 [4]. *If* $L(D)=\{p,q,n\}$, *then* $\exp_D(1)\leqslant\phi(p,q,n)+n-q$.

Corollary 2. *If* $L(D)=\{p,q,n\}$, *then* $\exp_D(k)\leqslant\phi(p,q,n)+n-q+k-1$ *for* $1\leqslant k\leqslant n$.

Lemma 8 [4]. *If* $|L(D)|\geqslant 3$, *then* $\exp_D(k)\leqslant\lfloor\frac{1}{2}(n-2)^2\rfloor+k$ *for* $n\geqslant 6$ *and* $1\leqslant k\leqslant n$.

Theorem 1. *If* $|L(D)|\geqslant 3$, *then* $\exp_D(k)\leqslant\lfloor\frac{1}{2}(n-2)^2\rfloor+k$ *for* $n\geqslant 4$ *and* $1\leqslant k\leqslant n$. *Moreover, it is a sharp bound in a sense.*

Proof. By Lemma 8, it is enough to check that $\exp_D(k)\leqslant\lfloor\frac{1}{2}(n-2)^2\rfloor+k$ for $n=4$ or 5.

Case 1. $n=4$. If $1\in L(D)$, then $\exp_D(k)\leqslant n-2+k=\lfloor\frac{1}{2}(n-2)^2\rfloor+k$ by Lemma 3. If $1\notin L(D)$, then $L(D)=\{2,3,4\}$. Thus $\exp_D(k)\leqslant 2+k=\lfloor\frac{1}{2}(n-2)^2\rfloor+k$ by Corollary 2.

Case 2. $n=5$. If $1\in L(D)$, then $\exp_D(k)\leqslant n-2+k\leqslant\lfloor\frac{1}{2}(n-2)^2\rfloor+k$ by Lemma 3. If $2\in L(D)$, then $\exp_D(k)\leqslant 4+k=\lfloor\frac{1}{2}(n-2)^2\rfloor+k$ by Lemmas 5 and 7. If 1 and $2\notin L(D)$, then $L(D)=\{3,4,5\}$. Thus $\exp_D(k)\leqslant\phi(3,4,5)+n-4+k-1=n-2+k\leqslant\lfloor\frac{1}{2}(n-2)^2\rfloor+k$ by Corollary 2.

Let D^* consist of n-cycle $(v_1,v_2,\ldots,v_n,v_1)$ and the arcs $\{(v_n,v_2),(v_n,v_{n-\lfloor\frac{1}{2}n\rfloor+1})\}$. Then $L(D^*)=\{\lfloor\frac{1}{2}n\rfloor,n-1,n\}$ and $\exp_{D^*}(1)\leqslant\lfloor\frac{1}{2}(n-2)^2\rfloor+1$. Since $R_1(v_i)=\{v_{i+1}\}$ for $1\leqslant i\leqslant n-1$, $\exp_{D^*}(v_n)<\exp_{D^*}(v_{n-1})<\cdots<\exp_{D^*}(v_1)$. If $v_n\to v_1[\lfloor\frac{1}{2}(n-2)^2\rfloor]$, then there exist integers k_1,k_2 and k_3 such that $1+k_1\lfloor\frac{1}{2}n\rfloor+k_2(n-1)+k_3n=\lfloor\frac{1}{2}(n-2)^2\rfloor$, i.e. $k_1\lfloor\frac{1}{2}n\rfloor+k_2(n-1)+k_3n=\lfloor\frac{1}{2}(n-2)^2\rfloor-1$. On the other hand, it is easy to check that $\phi(\lfloor\frac{1}{2}n\rfloor,n-1,n)=\lfloor\frac{1}{2}(n-2)^2\rfloor$. This is a contradiction. Thus $\exp_{D^*}(v_n)=\exp_{D^*}(1)=\lfloor\frac{1}{2}(n-2)^2\rfloor+1$. Hence $\exp_{D^*}(k)=\lfloor\frac{1}{2}(n-2)^2\rfloor+k$ for $1\leqslant k\leqslant n$. $\square$

Lemma 9 [3]. *If $L(D) = \{p, q\}$ with $(p, q) \in L'(n)$, then $p(q-1) \leqslant \gamma(D) \leqslant p(q-2) + n$.*

Theorem 2. *If $L(D) = \{p, q\}$ with $(p, q) \in L(n)$, then $\exp_D(k) \geqslant m(p, q, k)$ for $1 \leqslant k \leqslant n$. Moreover, it is a sharp bound in a sense.*

Proof. By the definition of $\exp_D(k)$ and Lemma 4, $\exp_D(k) \geqslant (p-1)(q-1)$, it is enough to prove $\exp_D(k) \geqslant (p-1)(q-1) + k + q - n - 1$. Suppose that $\exp_D(k) < (p-1)(q-1) + k + q - n - 1$. By Lemma 2, we have $\exp_D(n) \leqslant \exp_D(k) + n - k < p(q-1)$, i.e. $\gamma(D) < p(q-1)$. This contradicts Lemma 9. So $\exp_D(k) \geqslant (p-1)(q-1) + \max\{k + q - n - 1, 0\}$.

Let $V(D^*) = \{v_1, v_2, \ldots, v_n\}$. Consider $C_q = (v_1, v_2, \ldots, v_q, v_1)$ with additional arcs $\{(v_{p+i}, v_{1+i}) \mid 0 \leqslant i \leqslant q - p\}$. Further let v_j be a copy of v_q for $q + 1 \leqslant j \leqslant n$. If $v_1 \to v_q[p(q-1) - 1]$, then there exist integers k_1, k_2 such that $q - 1 + k_1 p + k_2 q = p(q-1) - 1$, i.e. $k_1 p + k_2 q = (p-1)(q-1) - 1$, a contradiction. Hence $v_1 \not\to v_q[p(q-1) - 1]$. Thus $\gamma(D^*) \geqslant p(q-1)$. On the other hand we have $\max d_{L(D^*)}(u, v) = q - 1$. So $\gamma(D^*) \leqslant q - 1 + \phi(p, q) = p(q-1)$. It follows that $\gamma(D^*) = \exp(v_1) = p(q-1)$. Since each vertex in D^* is at a distance of $q-1$ or less from v_1, we have $\exp_{D^*}(1) \geqslant \exp_{D^*}(v_1) - q + 1 = (p-1)(q-1)$. We now show that $\exp_{D^*}(v_q) \leqslant (p-1)(q-1)$. For each i, where $p \leqslant i \leqslant q-1$, let $r_i = \lfloor (i-p)/(p-1) \rfloor$. It is easy to check that $R_{r_i+2}(v_i) = R_{r_i+1}(v_{i+1}) \cup R_{r_i+1}(v_{i-(p-1)}) = R_{r_i+1}(v_{i+1}) \cup \{v_{i-(p-1)-j(p-1)+(r_i+1-j)} \mid 0 \leqslant j \leqslant r_i\} = R_{r_i+1}(v_{i+1}) \cup \{v_{i+1-j(p-1)+(r_i+1-j)} \mid 1 \leqslant j \leqslant r_i + 1\} = R_{r_i+1}(v_{i+1})$. Thus $R_l(v_i) = R_{l-1}(v_{i+1})$ for all $l \geqslant r_i + 2$. Let $r = \max\{r_i \mid p \leqslant i \leqslant q-1\} = \lfloor (q-p-1)/(p-1) \rfloor$. Then $R_{q-p+r+1}(v_p) = R_{q-p+r}(v_{p+1}) = R_{q-p+r-1}(v_{p+2}) = \cdots = R_{r+2}(v_{q-1}) = R_{r+1}(\{v_q, v_{q+1}, \ldots, v_n\})$. But $R_{r+1}(\{v_q, v_{q+1}, \ldots, v_n\}) = R_{r+1}(v_q)$, hence we have $R_{(p-1)(q-1)}(v_q) = R_{pq-2p+1}(v_p) = R_{pq-2p+1}(R_{p-1}(v_1)) = R_{p(q-1)}(v_1) = V(D)$ since $(p-1)(q-1) > r + 1$ and $\exp_{D^*}(v_1) = p(q-1)$. So $\exp_{D^*}(1) \leqslant \exp_{D^*}(v_q) \leqslant (p-1)(q-1)$ whence we conclude $\exp_{D^*}(1) = \exp_{D^*}(v_q) = (p-1)(q-1)$. Since $R_l(v_i) = R_{l-1}(v_{i+1})$ for all $l \geqslant r_i + 2$, then $\exp_{D^*}(v_q) < \exp_{D^*}(v_{q-1}) < \cdots < \exp_{D^*}(v_2) < \exp_{D^*}(v_1)$. Since $\exp_{D^*}(v_i) = \exp_{D^*}(v_q)$ for all $q + 1 \leqslant i \leqslant n$, we have

$$\exp_{D^*}(k) = \begin{cases} (p-1)(q-1) & \text{if } 1 \leqslant k \leqslant n - q + 1, \\ (p-1)(q-1) + k - (n - q + 1) & \text{if } n - q + 2 \leqslant k \leqslant n, \end{cases}$$

i.e. $\exp_{D^*}(k) = m(p, q, k)$. □

Let D_0 consist of q-cycle $(v_1, v_2, \ldots, v_q, v_1)$ and the walk $(v_q, v_{q+1}, \ldots, v_{p+q-l-1}, v_{q-l})$, where p, q, l are integers with $\gcd(p, q) = 1$, $0 \leqslant l \leqslant p - 1$ and $2 \leqslant p < q$.

Lemma 10. $\exp_{D_0}(v_q, v_y) = (p-1)(q-1) + y - q = p(q-2) + y + p - 2q + 1$ *for* $q - l \leqslant y \leqslant p + q - l - 1$ *and* $\exp_{D_0}(v_q, v_y) = (p-1)(q-1) + y = p(q-2) + y + p - q + 1$ *for* $1 \leqslant y \leqslant q - l - 1$.

Proof. Since $\phi(p,q)=(p-1)(q-1)$ and $2\leqslant p<q$, there exist nonnegative integers k_1, k_2 such that $k_1p+k_2q=m$ $(\geqslant (p-1)(q-1))$. Then $k_1>0$ or $k_2>0$. Clearly, $v_q\to v_{q-l}[p-l]$ and $v_q\to v_{q-l}[q-l]$. Thus $v_q\to v_{q-l}[p-l+(k_1-1)p+k_2q]$ or $v_q\to v_{q-l}[q-l+k_1p+(k_2-1)q]$, i.e. $v_q\to v_{q-l}[m-l]$. Hence $\exp(v_q,v_{q-l})\leqslant (p-1)(q-1)-l$. If $v_q\to v_{q-l}[(p-1)(q-1)-l-1]$, then there exist nonnegative integers l_1, l_2 such that $(p-1)(q-1)-l-1=p-l+l_1p+l_2q$ or $(p-1)(q-1)-l-1=q-l+l_1p+l_2q$, i.e. $(p-1)(q-1)-1=(l_1+1)p+l_2q$ or $(p-1)(q-1)-1=l_1p+(l_2+1)q$. This contradicts to $\phi(p,q)=(p-1)(q-1)$. Thus $v_q\not\to v_{q-l}[(p-1)(q-1)-l-1]$. Hence $\exp_{D_0}(v_q,v_{q-l})=(p-1)(q-1)-l$. So, $\exp_{D_0}(v_q,v_y)=(p-1)(q-1)+y-q$ for $q-l\leqslant y\leqslant p+q-l-1$ and $\exp_{D_0}(v_q,v_y)=(p-1)(q-1)+y$ for $1\leqslant y\leqslant q-l-1$. $\square$

Lemma 11. *Let $D_0\subset D\in PD_n$. If $\exp_D(v_q)\leqslant p(q-2)+p-l$, then $\exp_D(k)\leqslant M(p,q,k)$.*

Proof. Since $R_1(v_y)\supset\{v_{y+1}\}$ for $1\leqslant y\leqslant q-1$ or $q+1\leqslant y\leqslant p+q-l-2$ and $R(v_{p+q-l-1})\supset\{v_{q-l}\}$ in D, $\exp_D(v_y)\leqslant\exp_D(v_{y+1})+1$ for $1\leqslant y\leqslant q-1$ or $q+1\leqslant y\leqslant p+q-l-2$ and $\exp_D(v_{p+q-l(D)-1})\leqslant\exp_D(v_{q-l(D)})+1$. Since $\exp_D(1)\leqslant\exp_D(v_q)$,

$$\exp_D(k)\leqslant\begin{cases} p(q-2)+k+(p-l-1) & \text{if } 1\leqslant k\leqslant l+2,\\ p(q-2)+(p+1) & \text{if } l+3\leqslant k\leqslant 2p-l-1,\\ \quad +\lfloor\frac{1}{2}(k-l-2)\rfloor & \\ p(q-2)+k & \text{if } 2p-l\leqslant k\leqslant p+q-l+t-1.\end{cases}$$

Hence

$$\exp_D(k)\leqslant F(p,q,k,l(D)) := \begin{cases} p(q-2)+k+(p-l-1) & \text{if } 1\leqslant k\leqslant l+2,\\ p(q-2)+(p+1)+\lfloor\frac{1}{2}(k-l-2)\rfloor & \text{if } l+3\leqslant k\leqslant 2p-l-1,\\ p(q-2)+k & \text{if } 2p-l\leqslant k\leqslant n.\end{cases}$$

Since $F(p,q,k,l)\leqslant F(p,q,k,l-1)$ and $l\geqslant p+q-n-1$, $\exp_D(k)\leqslant F(p,q,k,p+q-n-1)=M(p,q,k)$. $\square$

Let $D\in PD_n$ with $L(D)=\{p,q\}$. Then there exist p-cycle C_1 and q-cycle C_2 such that $C_1\cap C_2\neq\emptyset$. Clearly, $C_1\cap C_2$ is the union of some paths. Suppose that $l(C_1\cap C_2)$ is the length of the longest path in $C_1\cap C_2$. Let $l(D)=\max_{C_1,C_2} l(C_1\cap C_2)$, where the maximum is taken over all p-cycle C_1 and q-cycle C_2 with $C_1\cap C_2\neq\emptyset$. It is easy to see that $l(D)$ is well-defined if D is a primitive digraph on n vertices with $L(D)=\{p,q\}$ and that $0\leqslant l(D)\leqslant p-1$.

Theorem 3. *If $L(D)=\{p,q\}$ with $(p,q)\in L(n)$, then $\exp_D(k)\leqslant M(p,q,k)$ for $1\leqslant k\leqslant n$. Moreover, it is a sharp bound in a sense.*

Proof. It is enough to consider $n \geqslant 3$. Let C_1 and C_2 be a p-cycle and a q-cycle, respectively, such that $l(C_1 \cap C_2) = l(D)$. We claim that $C_1 \cap C_2$ is a path of length $l(D)$. Otherwise, suppose $Q_1 = (u_0, u_1, \ldots, u_a)$, $Q_2 = (v_0, v_1, \ldots, v_b)$ are two paths of $C_1 \cap C_2$ with $a = l(D)$, where $u_a \neq v_0, v_b \neq u_0$. Then $C_i = Q_1 \cup Q_2 \cup (u_a \to v_0[C_i]) \cup (v_b \to u_0[C_i])$, $i = 1, 2$. Let the length of $u_a \to v_0[C_i]$ ($v_b \to u_0[C_i]$, resp.) be equal to x_{i1} (x_{i2}, resp.), $i = 1, 2$. Then $a + b + x_{11} + x_{12} = p, a + b + x_{21} + x_{22} = q$. Since $L(D) = \{p, q\}$ and $p < q$, then $x_{11} \leqslant x_{21}$ and $x_{12} \leqslant x_{22}$. If $x_{11} < x_{21}$ and $x_{12} < x_{22}$, then $Q_1 \cup Q_2 \cup (u_a \to v_0[C_1]) \cup (v_b \to u_0[C_2])$ is a cycle of length $a + b + x_{11} + x_{22}$. But $p < a + b + x_{11} + x_{22} < q$, this contradicts $L(D) = \{p, q\}$. If $x_{11} = x_{21}$ or $x_{12} = x_{22}$, then one of $Q_1 \cup Q_2 \cup (u_a \to v_0[C_2]) \cup (v_b \to u_0[C_1])$ and $Q_1 \cup Q_2 \cup (u_a \to v_0[C_1]) \cup (v_b \to u_0[C_2])$ is a p -cycle, say $C_3 = Q_1 \cup Q_2 \cup (u_a \to v_0[C_2]) \cup (v_b \to u_0[C_1])$ is a p-cycle. But it is easy to see that $l(C_2 \cap C_3) \geqslant a + b + x_{11} > a$. This contradicts $l(D) = a$.

Let D_0 be a subdigraph of D induced by the set of vertices $V(C_1) \cup V(C_2)$. Then $|V(D_0)| = p + q - (l(D) + 1)$. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$. Without loss of generality, let D_0 consist of q-cycle $(v_1, v_2, \ldots, v_q, v_1)$ and the walk $(v_q, v_{q+1}, \ldots, v_{q-l(D)+p-1}, v_{q-l(D)})$, where $v_{q-l(D)}$ belongs to q-cycle. Then $\exp_{D_0}(v_q, v_y) = (p - 1)(q - 1) + y - q$ for $q - l(D) \leqslant y \leqslant p + q - l(D) - 1$ and $\exp_{D_0}(v_q, v_y) = (p - 1)(q - 1) + y$ for $1 \leqslant y \leqslant q - l(D) - 1$ by Lemma 10.

Let $v \in V(D) \backslash V(D_0)$. Since D is primitive and $D_0 \subset D$, there exist $v_i, v_j \in V(D_0)$ such that $P_1 := v \to v_i[d(v, D_0)]$ and $P_2 := v_j \to v[d(D_0, v)]$. Also $(P_1 \cap P_2) \subset \{v, v_i, v_j\}$ since $p + q > n$. Suppose v_i, v_j are not on same cycle of D_0, say $v_j \in V(C_1) \backslash V(C_2)$ and $v_i \in V(C_2) \backslash V(C_1)$. Let $C_4 = P_1 \cup P_2 \cup (v_i \to v_q[C_2]) \cup (v_q \to v_j[C_1])$. Then $|E(C_1) \cap E(C_4)| > l(D)$ and $|E(C_2) \cap E(C_4)| > l(D)$. Since $L(D) = \{p, q\}$, C_4 is p-cycle or q-cycle. Hence this contradicts the choice of C_1 and C_2. Thus v_i, v_j are on the same cycle of D_0. Also since $p + q > n$, the length of $P_1 \cup P_2$ is not greater than $n - (p + q - l(D) - 1) + 1 \leqslant l(D) + 1 \leqslant p$. Thus the length of $P_1 \cup P_2$ is less than p except for $n = p + q - 1$ and $l(D) = p - 1$. Hence $i \neq j$ except for $l(D) = p - 1 = n - q$. We estimate the upper bound of $\exp_D(k)$ according to the following two cases.

(I) $i = j$. Then D has a subdigraph D_1 which consists of D_0 and p-cycle $(v_j, v_{q+1}, \ldots, v_n, v_j)$. It is easy to check that $\exp_{D_1}(v_q, v_y) = \exp_{D_0}(v_q, v_y)$ for $1 \leqslant y \leqslant q$ and $\exp_{D_1}(v_q, v_y) = \exp_{D_0}(v_q, v_j) + (y - q)$ for $q + 1 \leqslant y \leqslant n$. We divide into the following cases.

1. $q \geqslant 2p - 1$. Then $q - p \geqslant p - 1$. Let $q - p = c(p - 1) + r, 0 \leqslant r < p - 1$.

1.1. $j \in \{1, 2, \ldots, (c - 1)(p - 1) + r, q - p + 1, q - p + 2, \ldots, q\}$. Since $\exp_{D_1}(v_q, v_y) = (p - 1)(q - 1) + (y - q)$ for $q - p + 1 \leqslant y \leqslant q$ and $\exp_{D_1}(v_q, v_y) = (p - 1)(q - 1) + y$ for $1 \leqslant y \leqslant q - p$, $\exp_{D_1}(v_q, v_y) \leqslant (p - 1)(q - 1) + (q - p) = p(q - 2) + 1$ for $1 \leqslant y \leqslant q$. Also since $\exp_{D_1}(v_q, v_y) = \exp_{D_0}(v_q, v_j) + (y - q)$ for $q + 1 \leqslant y \leqslant n$, $\exp_{D_1}(v_q, v_y) \leqslant (p - 1)(q - 1) + (c - 1)(p - 1) + r + (p - 1) = (p - 1)(q - 1) + (q - p) = p(q - 2) + 1$ for $q + 1 \leqslant y \leqslant n$.

Thus $\exp_{D_1}(1) \leqslant \exp_{D_1}(v_q) \leqslant p(q-2)+1$. Hence $\exp_D(k) \leqslant \exp_{D_1}(k) \leqslant p(q-2)+k \leqslant M(p,q,k)$.

1.2. $j \in \{(c-1)(p-1)+r+1, (c-1)(p-1)+r+2, \ldots, q-p\}$. Since $\exp_{D_1}(v_q, v_y) = \exp_{D_0}(v_q, v_j) + y - q$ for $q+1 \leqslant y \leqslant n$, $\exp_{D_1}(v_q, v_y) \leqslant \exp_{D_1}(v_q, v_n)$ for $q+1 \leqslant y \leqslant n$ and $\exp_{D_1}(v_q, v_n) = \exp_{D_0}(v_q, v_j) + n - q = (p-1)(q-1) + j + p - 1 \geqslant (p-1)(q-1) + (c-1)(p-1) + r + 1 + p - 1 = (p-1)(q-1) + q - p + 1$. Since $\exp_{D_1}(v_q, v_y) = \exp_{D_0}(v_q, v_y) \leqslant (p-1)(q-1) + q - p$ for $1 \leqslant y \leqslant q$, $\exp_{D_1}(v_q) = \exp_{D_1}(v_q, v_n) = (p-1)(q-1) + j + p - 1 = p(q-2) + (2p-q+j)$. Also we can get $\exp_{D_1}(v_j) = (p-1)(q-1) + q - 1 = p(q-2) + p$. By $j \leqslant q-p$, we have $2p-q+j \leqslant p$. Thus $\exp_{D_1}(v_q) \leqslant \exp_{D_1}(v_j)$. Since $\exp_{D_1}(v_q) < \exp_{D_1}(v_{q-1}) < \cdots < \exp_{D_1}(v_{q-p+1})$ and $\exp_{D_1}(v_j) < \exp_{D_1}(v_n) < \exp_{D_1}(v_{n-1}) < \cdots < \exp_{D_1}(v_{q+1})$,

$$\exp_D(k) \leqslant \exp_{D_1}(k) \leqslant F_1(p,q,k,j)$$

$$:= \begin{cases} p(q-2) + k + (2p-q+j-1) & \text{if } 1 \leqslant k \leqslant q-p-j+1, \\ p(q-2) + p + \lfloor \frac{1}{2}(k-q \\ \quad +p+j-1) \rfloor & \text{if } q-p-j+2 \leqslant k \\ & \quad \leqslant 3p-q+j+1, \\ p(q-2) + k - 1 & \text{if } 3p-q+j+2 \leqslant k \leqslant n. \end{cases}$$

Since $F_1(p,q,k,j) \leqslant F_1(p,q,k,j+1)$ and $j \leqslant q-p$, $\exp_D(k) \leqslant F_1(p,q,k,q-p) \leqslant M(p,q,k)$.

2. $q < 2p-1$. Let $p-1 = q-p+x$. Then $q-p+1 \leqslant q-x \leqslant q$.

2.1. $j \in \{q-p+1, q-p+2, \ldots, q-x\}$.

2.2. $j \in \{q-x+1, q-x+2, \ldots, q\}$.

2.3. $j \in \{1, 2, \ldots, q-p\}$.

The proofs of these subcases are analogous to the proof of (I) 1. So it is left to the readers.

(II) $i \neq j$.

If $\exp_D(v_q) \leqslant p(q-2) + p - l(D)$, then $\exp_D(k) \leqslant M(p,q,k)$ by Lemma 11.

If $\exp_D(v_q) > p(q-2) + p - l(D)$, then there exists a vertex $v \in V(D) \backslash V(D_0)$ such that $\exp_D(v_q) = \exp_D(v_q, v)$. Since D is primitive and $D_0 \subset D$, there exists a path $P := (v_j, v_{p+q-l(D)}, v_{p+q-l(D)+1}, \ldots, v_{p+q-l(D)+t-1}, v_i)$ such that $v_i, v_j \in V(D_0)$ and $v \in P$. Then $v = v_{p+q-l(D)+t-1}$ since $L(D) = \{p, q\}$. Let $D_1 = D_0 \cup P$ and $C_5 = P \cup (v_i \to v_j[D_0])$. Hence $\exp_{D_1}(v_q, v) \geqslant \exp_D(v_q, v) > p(q-2) + p - l(D)$. Thus $\exp_{D_1}(k) > p(q-2) + p - l(D)$. We claim that the positions of v_i and v_j have the following four possibilities.

1. $v_i, v_j \in V(C_1) \cap V(C_2)$ and $i < j$.
2. $v_i \in V(C_1) \cap V(C_2)$ and $v_j \in V(C_1) \backslash V(C_2)$.
3. $v_i \in V(C_1) \cap V(C_2)$ and $v_j \in V(C_2) \backslash V(C_1)$.
4. $v_i, v_j \in V(C_2) \backslash V(C_1)$ and $i < j$.

In fact, we can see it as follows:

If $v_i, v_j \in V(C_1) \cap V(C_2)$ and $i > j$, then $i - j = t + 1$, otherwise it is contradicts to $L(D) = \{p, q\}$. Thus there exists a vertex $v^* \in V(C_1) \cap V(C_2)$ such that $\exp_{D_1}(v_q, v) = \exp_{D_0}(v_q, v^*)$. This contradicts to the choice of P.

If $v_j \in V(C_1) \cap V(C_2)$ and $v_i \in V(C_1)\backslash V(C_2)$, let C_5 is a p-cycle (a q-cycle, resp.), then there exists $v^* \in V(C_1)$ ($v^* \in V(C_2)$, resp.) such that $\exp_{D_1}(v_q, v) = \exp_{D_0}(v_q, v^*)$. This contradicts to the choice of P.

If $v_j \in V(C_1) \cap V(C_2)$ and $v_i \in V(C_2)\backslash V(C_1)$, use an analogous proof of above, a contradiction too.

If $v_i, v_j \in V(C_1)\backslash V(C_2)$ and $i < j$, then the length of C_5 is not greater than $n - (p + q - l(D) - 1) + 1 + p - l(D) - 2 = n - q < p$. This contradicts $L(D) = \{p, q\}$ with $p < q$.

If $v_i, v_j \in V(C_1)\backslash V(C_2)$ and $i > j$, let C_5 be a q-cycle, then $|E(C_5) \cap E(C_1)| \geqslant l(D) + 2$. This contradicts the choice of C_1 and C_2. Let C_5 be a p-cycle, there exists $v^* \in V(C_1)\backslash V(C_2)$ such that $\exp_{D_1}(v_q, v) = \exp_{D_0}(v_q, v^*)$. This contradicts the choice of P.

If $v_i, v_j \in V(C_2) \cap V(C_1)$ and $i > j$. By the choice of C_1 and C_2, C_5 is a q-cycle. Then there exists $v^* \in V(C_2)$ such that $\exp_{D_1}(v_q, v) = \exp_{D_0}(v_q, v^*)$. This contradicts the choice of P.

Now we divide into the following cases:

1. $v_i, v_j \in V(C_1) \cap V(C_2)$ and $i < j$ or $v_i \in V(C_1) \cap V(C_2)$ and $v_j \in V(C_1)\backslash V(C_2)$ or $v_i \in V(C_1) \cap V(C_2)$ and $v_j \in V(C_2)\backslash V(C_1)$. Let $s := \exp_D(v_q) - [p(q - 2) + p - l(D)]$.

We claim that $q - i \leqslant l(D) - s$. In fact, if $v_j \in V(C_1)$, then $p(q - 2) + p - l(D) + s = \exp_D(v_q) = \exp_D(v_q, v) \leqslant \exp_{D_1}(v_q, v) \leqslant \exp_{D_0}(v_j) + t = (p - 1)(q - 1) + j - q + t$. Thus $l(D) - s \geqslant 2q - j - t - 1$. Also $j - i + t + 1 \leqslant q$ since C_5 is a p-cycle or q-cycle, then $l(D) - s \geqslant q - i$. If $v_j \in V(C_2)\backslash V(C_1)$, then $p(q - 2) + p - l(D) + s = \exp_D(v_q) = \exp_D(v_q, v) \leqslant \exp_{D_1}(v_q, v) \leqslant \exp_{D_0}(v_j) + t = (p - 1)(q - 1) + j + t$. Thus $l(D) - s \geqslant q - j - t - 1$. Also $j + q - i + t + 1 \leqslant q$ since C_5 is a p-cycle or q-cycle, then $l(D) - s \geqslant q - i$.

Since $D_1 \subset D$, $R_1(v_y) \supset \{v_{y+1}\}$ for $1 \leqslant y \leqslant q - 1$ or $q + 1 \leqslant y \leqslant p + q - l(D) - 2$ or $p + q - l(D) \leqslant y \leqslant p + q - l(D) + t - 1$, $R_1(v_{p+q-l(D)-1}) \supset \{v_{q-l(D)}\}$ and $R_1(v_{p+q-l(D)+t-1}) \supset \{v_i\}$, $\exp_D(v_y) \leqslant \exp_D(v_{y+1}) + 1$ for $1 \leqslant y \leqslant q - 1$ or $q + 1 \leqslant y \leqslant p + q - l(D) - 2$ or $p + q - l(D) \leqslant y \leqslant p + q - l(D) + t - 1$, $\exp_D(v_{p+q-l(D)-1}) \leqslant \exp_D(v_{q-l(D)}) + 1$ and $\exp_D(v_{p+q-l(D)+t-1}) \leqslant \exp_D(v_i) + 1$. Also since $\exp_D(1) \leqslant \exp_D(v_q) = p(q - 2) + p - l(D) + s$,

$$
\begin{aligned}
&\exp_D(k) \leqslant G_0(p, q, k, l(D), s, i) \\
&\quad := \begin{cases} p(q-2) + k + (p - l(D) + s - 1) & \text{if } 1 \leqslant k \leqslant q - i + 2, \\ p(q-2) + (p + q - l(D) & \text{if } q - i + 3 \leqslant k \leqslant 2p - q + i, \\ \quad + s - i + 1) + \lfloor \frac{1}{2}(k - q + i - 2) \rfloor & \\ p(q-2) + k & \text{if } 2p - q + i + 1 \leqslant k \leqslant n. \end{cases}
\end{aligned}
$$

Since $q-i \leqslant l(D)-s$, $q-i+2 \leqslant l(D)-s+2$ and $2p-q+i \geqslant 2p-l(D)+s$. Thus

$$\exp_D(k) \leqslant G_0(p,q,k,l(D),s,i) \leqslant G(p,q,k,l(D),s)$$
$$:= \begin{cases} p(q-2)+k+(p-l(D)+s-1) & \text{if } 1 \leqslant k \leqslant l(D)-s+2, \\ p(q-2)+(p+1) & \text{if } l(D)-s+3 \leqslant k \\ \quad +\lfloor \frac{1}{2}(k-l(D)+s-2)\rfloor & \quad \leqslant 2p-l(D)+s, \\ p(q-2)+k & \text{if } 2p-l(D)+s+1 \leqslant k \leqslant n. \end{cases}$$

Since $G(p,q,k,l(D),s) \leqslant G(p,q,k,l(D),s+1)$ and $s \leqslant t \leqslant n-p-q+l(D)+1$, $\exp_D(k) \leqslant G(p,q,k,l(D),n-p-q+l(D)+1) \leqslant M(p,q,k)$.

2. $v_i, v_j \in V(C_2)\backslash V(C_1)$ and $i<j$. It is not difficult to check that $\exp_D(v_q) \leqslant p(q-2)+(p-q+j+t+1)$ and $\exp_D(v_j) \leqslant p(q-2)+\max\{n-p-q+l(D)+2, n+p-q-t-j+1\}$.

2.1. $n-p-q+l(D)+2 \geqslant n+p-q-t-j+1$.

2.1.1. $n-p-q+l(D)+2 \geqslant p-q+j+t+1$.

2.1.2. $n-p-q+l(D)+2 < p-q+j+t+1$.

2.2. $n-p-q+l(D)+2 < n+p-q-t-j+1$.

2.2.1. $p-q+j+t+1 \leqslant n+p-q-t-j+1$.

2.2.2. $p-q+j+t+1 > n+p-q-t-j+1$.

The proofs of these subcases are analogous to the proof of (II) 1. So it is left to the readers.

To sum up, the bound is obtained. In the following, we show that this bound is sharp in a sense. Let D^* consist of C_q and the walks $(v_q, v_{q+1}, \ldots, v_n, v_{n-p+1})$. By Lemma 10, $\exp_{D^*}(v_q, v_y) = (p-1)(q-1)+y-q$ for $n-p+1 \leqslant y \leqslant n$ and $\exp_{D^*}(v_q, v_y) = (p-1)(q-1)+y$ for $1 \leqslant y \leqslant n-p$. So we have $\exp_{D^*}(k) = M(p,q,k)$. □

3. Gap system between $\lfloor \frac{1}{2}(n-2)^2 \rfloor + k + 1$ and $n^2-3n+2+k$

By [1], it is known that $\max\{\exp_D(k) \mid D \in PD_n\} = n^2-3n+2+k$. So $E_n(k) \subset \{1, 2, \ldots, n^2-3n+2+k\}$. For determining $E_n(k)$, in this section, we determine the gap systems between $\lfloor \frac{1}{2}(n-2)^2 \rfloor + k + 1$ and $n^2-3n+2+k$.

Theorem 4. *Let positive integers* p, q, n, k *and* m *be given such that* $(p,q) \in L(n)$, $2 \leqslant k \leqslant n-1$ *and* $m(p,q,k) \leqslant m \leqslant M(p,q,k)$. *Then there exists* $D \in PD_n$ *with* $L(D) = \{p,q\}$ *such that* $\exp_D(k) = m$.

Proof. We denote by C_q the cycle of the form $(v_1, v_2, \ldots, v_q, v_1)$.

Case 1. $p=2$. Then q is odd.

Subcase 1.1. $q=n$. Then $m(p,q,k) = m(2,n,k) = n-1+\max\{k-1,0\} = n-2+k$ and $M(p,q,k) = M(2,n,k) = 2(n-2)+k$. For $0 \leqslant a \leqslant n-2$, let $D =$

$C_n \cup \{(v_i, v_{i-1}) \mid a+2 \leqslant i \leqslant n\}$. It is easy to check that $\exp_D(k) = (n-2)+a+k$. Thus $\{m(2,n,k), m(2,n,k)+1, \ldots, M(2,n,k)\} \subset E_n(k)$.

Subcase 1.2. $q = n-1$. Then $m(p,q,k) = m(2,n-1,k) = n-2+\max\{k-2,0\} = n-4+k$, $M(p,q,2) = M(2,n-1,2) = 2n-3$ and $M(p,q,k) = M(2,n-1,k) = 2(n-3)+k$ for $3 \leqslant k \leqslant n-1$. For $0 \leqslant a \leqslant n-3$. Let $V(D) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ and let the subdigraph D_1 of D induced by $V(D)\backslash v_n$ be like the digraph in Subcase 1.1. If v_n is a copy of v_1, then it follows from Subcase 1.1 that $\exp_D(k) = n-3+a+k$ for $2 \leqslant k \leqslant n-1$. If v_n is a copy of v_{n-1}, then it follows from Subcase 1.1 that $\exp_D(k) = n-4+a+k$ for $2 \leqslant k \leqslant n-1$. If $D = C_{n-1} \cup \{(v_1, v_n), (v_n, v_1)\}$, then $\exp_D(2) = 2n-3$. Thus $\{m(2,n-1,k), m(2,n-1,k)+1, \ldots, M(2,n-1,k)\} \subset E_n(k)$.

Case 2. $p \geqslant 3$.

Subcase 2.1. $m(p,q,k) \leqslant m \leqslant p(q-2)+k-1$. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$. Consider C_q with additional arcs $\{(v_{p+i}, v_{1+i}) \mid 0 \leqslant i \leqslant q-p-a\}$, where $0 \leqslant a \leqslant q-p-1$. The walk $(v_{q-a-t}, v_{q+1}, \ldots, v_{q+t}, v_{q-a+1})$ $(0 \leqslant t \leqslant n-q)$ is added. Further let v_j be a copy of v_{q-a} for $q+t < j \leqslant n$. We can check that $\exp_D(k) = (p-1)(q-1)+a+\max\{k+q-n+t-1, 0\}$. In fact, the proof of it is analogous to the proof of Theorem 2. Thus $\{m(p,q,k)\ , m(p,q,k)+1, \ldots, p(q-2)+k-1\} \subset E_n(k)$.

Subcase 2.2. $p(q-2)+k \leqslant m \leqslant M(p,q,k)$. Let D consist of C_q and the walks $(v_q, v_{q+1}, \ldots, v_{a+p}, v_{a+1})$ and $(v_{q+p+a-n}, v_{a+p+1}, v_{a+p+2}, \ldots, v_n, v_1)$, where $q-p \leqslant a \leqslant n-p$. Since $p+q > n$, $v_{a+1}, v_{q+p+a-n} \in C_q$.

Let D_0 consist of C_q and the walk $(v_q, v_{q+1}, \ldots, v_{a+p}, v_{a+1})$. Then $\exp_{D_0}(v_q, v_y) = (p-1)(q-1)+y-q$ for $a+1 \leqslant y \leqslant a+p$ and $\exp_{D_0}(v_q, v_y) = (p-1)(q-1)+y$ for $1 \leqslant y \leqslant a$ by Lemma 10. Thus $\exp_D(v_q, v_y) \leqslant \exp_{D_0}(v_q, v_{q+p+a-n}) + y-(a+p) \leqslant (p-1)(q-1)+(q+p+a-n-q)+n-(a+p) = (p-1)(q-1)$ for $a+p+1 \leqslant y \leqslant n$ and $\exp_D(v_q, v_y) = \exp_{D_0}(v_q, v_y) \leqslant \exp_{D_0}(v_q, v_a) = (p-1)(q-1)+a$ for $a+1 \leqslant y \leqslant a+p$. If $v_q \to v_a[(p-1)(q-1)+a-1]$, then there exist nonnegative integers k_1, k_2 such that $(p-1)(q-1)+a-1 = a+k_1p+k_2q$, i.e. $k_1p+k_2q = (p-1)(q-1)-1$, a contradiction. Hence $\exp_D(v_q) = (p-1)(q-1)+a$. Since $R_{n-a-p}(v_{a+p+q-n}) \supset \{v_q\}$, $\exp_D(v_{a+p+q-n}) \leqslant \exp_D(v_q)+n-a-p = (p-1)(q-1)+n-p$. If $v_{a+p+q-n} \to v_a[(p-1)(q-1)+n-p-1]$, then there exist nonnegative integers k_1, k_2 such that $(p-1)(q-1)+n-p-1 = n-p+k_1p+k_2q$. Thus $k_1p+k_2q = (p-1)(q-1)-1$, a contradiction. Hence $\exp_D(v_{a+p+q-n}) = (p-1)(q-1)+n-p$. Also it is easy to see that $R_{q-y}(v_y) = \{v_q\}$ for $a+p+q-n+1 \leqslant y \leqslant q$, $R_{a+p+q-n-y}(v_y) = \{v_{n+p+q-n}\}$ for $1 \leqslant y \leqslant a+p+q-n-1$, $R_{a+p-y+1}(v_y) = \{v_{a+1}\}$ for $q+1 \leqslant y \leqslant a+p$ and $R_{n-y+1}(v_y) = \{v_1\}$ for $a+p+1 \leqslant y \leqslant n$. So we have $\exp_D(v_q) < \exp_D(v_{q-1}) < \cdots < \exp_D(v_{a+1}) < \exp_D(v_a) < \cdots < \exp_D(v_1) < \exp_D(v_n) < \exp_D(v_{n-1}) < \cdots < \exp_D(v_{a+p+1})$ and $\exp_D(v_{a+1}) < \exp_D(v_{a+p}) < \exp_D(v_{a+p-1}) < \cdots < \exp_D(v_{q+1})$. Thus

$$\exp_D(k) = \begin{cases} (p-1)(q-1)+a+k-1 & \text{if } 1 \leqslant k \leqslant q-a+1, \\ (p-1)(q-1)+q & \text{if } q-a+2 \leqslant k \leqslant a-q+2p, \\ \quad +\lfloor \frac{1}{2}(k-(q-a+1)) \rfloor & \\ p(q-2)+k & \text{if } a-q+2p+1 \leqslant k \leqslant n. \end{cases}$$

Let a take over all numbers in $\{q-p, q-p+1, \ldots, n-p\}$. Then we get $\{p(q-2)+k, p(q-2)+k+1, \ldots, M(p,q,k)\} \subset E_n(k)$.

By Cases 1 and 2, the proof of the theorem is completed. □

Corollary 3. *Let* $m \in \{\lfloor \frac{1}{2}(n-2)^2 \rfloor + k + 1, \ldots, n^2 - 3n + 2 + k\} \setminus \bigcup_{(p,q)\in L(n)} \{m(p,q,k), \ldots, M(p,q,k)\}$. *Then there is no* $D \in PD_n$ *such that* $\exp_D(k) = m$ *for* $2 \leqslant k \leqslant n-1$ *and* $n \geqslant 4$.

Proof. Let $D \in PD_n$. If $1 \in L(D)$, then $\exp_D(k) \leqslant n - 2 + k \leqslant \lfloor \frac{1}{2}(n-2)^2 \rfloor + k$ by Lemma 3 and $n \geqslant 4$. So it follows from Lemma 6 and Theorems 1–4. □

4. $\{1, 2, \ldots, \lfloor \frac{1}{2}(n-2)^2 \rfloor + k\} \subset E_n(k)$ for all n, k with $2 \leqslant k \leqslant n-1$ except $n = 11$ and $9 \leqslant k \leqslant 10$

Lemma 12. *For any* n, k *with* $2 \leqslant k \leqslant n-1$, *we have* $E_{n-1}(k-1) \cup E_{n-1}(k) \subset E_n(k)$.

Proof. Suppose $m \in E_{n-1}(k-1)$. Then there exists $D_1 \in PD_{n-1}$ such that $\exp_{D_1}(k-1) = m$. Let $V(D) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$. Let the subdigraph of D induced by $V(D)\backslash v_n$ be like the digraph D_1. Further, let v_n be a copy of vertex u which has exponent $\exp_{D_1}(1)$. i.e. $\exp_{D_1}(u) = \exp_{D_1}(1)$ for $k \geqslant 2$. Then $\exp_D(k) = \exp_{D_1}(k-1) = m$. Thus $m \in E_n(k)$.

Suppose $m \in E_{n-1}(k)$. Then there exists $D_1 \in PD_{n-1}$ such that $\exp_{D_1}(k) = m$. Let $V(D) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$. Let the subdigraph of D induced by $V(D)\backslash v_n$ be like the digraph D_1. Further let v_n be a copy of vertex u which has exponent $\gamma(D_1)$. i.e. $\exp_{D_1}(u) = \gamma(D_1)$ for $k \leqslant n-1$. Then $\exp_D(k) = \exp_{D_1}(k) = m$. Thus $m \in E_n(k)$. □

Lemma 13 [7]. $E_n(1) = \{1, 2, \ldots, \frac{1}{2}(n^2 - 3n + 4)\} \cup \bigcup_{(p,q)\in L(n)} \{(p-1)(q-1), \ldots, p(q-2) + n - q + 1\}$.

Lemma 14. *Let* n, k *be integers with* $2 \leqslant k \leqslant n-1$. *Then* $\{1, 2, \ldots, \lfloor \frac{1}{2}(n-2)^2 \rfloor + k\} \subset E_n(k)$ *if and only if* $E_n^{(1)}(k) := \{\lfloor \frac{1}{2}(n-3)^2 \rfloor + k, \ldots, \lfloor \frac{1}{2}(n-2)^2 \rfloor + k\} \subset E_n(k)$.

Proof. The necessity is trivial. Now suppose $E^{(1)}_{n-i}(k-i) \subset E_{n-i}(k-i)$ is true for all $i = 0, 1, 2, \ldots, \; k-2$. By Lemma 12, we get $\{\lfloor\frac{1}{2}(n-i-3)^2\rfloor + k - i, \lfloor\frac{1}{2}(n-i-3)^2\rfloor + k - i + 1, \ldots, \lfloor\frac{1}{2}(n-i-2)^2\rfloor + k - i\} \subset E_n(k)$ for all $i = 0, 1, 2, \ldots, \; k-2$. Thus $\{\lfloor\frac{1}{2}(n-k-1)^2\rfloor + 2, \lfloor\frac{1}{2}(n-k-1)^2\rfloor + 3, \ldots, \lfloor\frac{1}{2}(n-2)^2\rfloor + k\} \subset E_n(k)$. Also $E_{n-k+1}(1) \subset E_n(k)$ for all n, k with $2 \leqslant k \leqslant n-1$ by Lemma 12, and $\{1, 2, \ldots, \lfloor\frac{1}{2}(n-k+1-2)^2\rfloor + 1\} \subset \{1, 2, \ldots, \frac{1}{2}[(n-k+1)^2 - 3(n-k+1)+4]\} \subset E_{n-k+1}(1)$ by Lemma 13. Thus $\{1, 2, \ldots, \lfloor\frac{1}{2}(n-2)^2\rfloor + k\} \subset E_n(k)$. □

Lemma 15. *If n is even and $n \geqslant 8$, then $E^{(1)}_n(k) \subset E_n(k)$ for $2 \leqslant k \leqslant n-1$.*

Proof. If $(p, q) \in L(n)$, then $\{m(p,q,k), \; m(p,q,k)+1, \ldots, \; M(p,q,k)\} \subset E_n(k)$ by Theorem 4. Let $p = \frac{1}{2}(n-2)$, $q = n-1$. Then $m(p,q,k) = (p-1)(q-1) + \max\{k+q-n-1, 0\} = \frac{1}{2}(n^2-6n+8)+k-2$ and $M(p,q,k) \geqslant p(q-2)+k = \frac{1}{2}(n^2-5n+6)+k$. Thus $\{\frac{1}{2}(n^2-6n+4)+k, \ldots, \frac{1}{2}(n^2-5n+6)+k\} \subset E_n(k)$.

Case 1. $n \equiv 0 \pmod 4$. Let $p = \frac{1}{2}(n-2)$, $q = n$. Then $m(p,q,k) = (p-1)(q-1) + \max\{k+q-n-1, 0\} = \frac{1}{2}(n^2-5n+4)+k-1$ and $M(p,q,k) \geqslant p(q-2)+k = \frac{1}{2}(n^2-4n+4)+k$. Thus $\{\frac{1}{2}(n^2-5n+2)+k, \ldots, \frac{1}{2}(n^2-4n+4)+k\} \subset E_n(k)$.

Case 2. $n \equiv 2 \pmod 4$. Let $p = \frac{1}{2}n$, $q = n-2$. Then $m(p,q,k) = (p-1)(q-1) + \max\{k+q-n-1, 0\} \leqslant \frac{1}{2}(n^2-5n+6)+k$ and $M(p,q,k) \geqslant p(q-2)+k = \frac{1}{2}(n^2-4n)+k$. Thus $\{\frac{1}{2}(n^2-5n+2)+k, \ldots, \frac{1}{2}(n^2-4n)+k\} \subset E_n(k)$. Let $p = \frac{1}{2}n$, $q = n-1$. Then $m(p,q,k) = (p-1)(q-1) + \max\{k+q-n-1, 0\} = \frac{1}{2}(n^2-4n)+k$ and $M(p,q,k) \geqslant p(q-2)+k = \frac{1}{2}(n^2-3n)+k$. Thus $\{\frac{1}{2}(n^2-4n)+k, \ldots, \frac{1}{2}(n^2-3n)+k\} \subset E_n(k)$.

To sum up, the proof of the lemma is completed. □

Lemma 16. *Let n, k be integers with $2 \leqslant k \leqslant n-1$. If n is odd and $n \geqslant 5$, then $E^{(1)}_n(k) \subset E_n(k)$ if and only if $\{\frac{1}{2}(n^2-5n+6)+k, \ldots, \frac{1}{2}(n^2-4n-1)+k\} \subset E_n(k)$.*

Proof. Let $p = \frac{1}{2}(n-1)$, $q = n-2$. Then $m(p,q,k) = (p-1)(q-1) + \max\{k+q-n-1, 0\} = \frac{1}{2}(n^2-6n+9) + \max\{k-3, 0\} \leqslant \frac{1}{2}(n^2-6n+9) + k$ and $M(p,q,k) \geqslant p(q-2)+k = \frac{1}{2}(n^2-5n+4)+k$. Thus $\{\frac{1}{2}(n^2-6n+9)+k, \ldots, \frac{1}{2}(n^2-5n+4)+k\} \subset E_n(k)$. Let $p = \frac{1}{2}(n-1)$, $q = n$. Then $m(p,q,k) = (p-1)(q-1) + \max\{k+q-n-1, 0\} = \frac{1}{2}(n^2-4n+3) + k - 1 = \frac{1}{2}(n^2-4n+1)+k$ and $M(p,q,k) \geqslant p(q-2)+k = \frac{1}{2}(n^2-3n+2)+k$. Thus $\{\frac{1}{2}(n^2-4n+1)+k, \ldots, \frac{1}{2}(n^2-3n+2)+k\} \subset E_n(k)$. So the proof of this lemma is completed. □

Lemma 17. *Let* $n \equiv 1 \pmod 4$ *and* $n \geqslant 9$.
1. *If* $2 \leqslant k \leqslant \frac{1}{2}(n+1)$, *then* $E_n^{(1)}(k) \subset E_n(k)$.
2. *If* $\frac{1}{2}(n+3) \leqslant k \leqslant \frac{1}{2}(n+5)$, *then* $E_n^{(1)}(k) \subset E_n(k)$ *if and only if* $\frac{1}{2}(n^2-4n-1)+k \in E_n(k)$.
3. *If* $\frac{1}{2}(n+7) \leqslant k \leqslant n-1$, *then* $E_n^{(1)}(k) \subset E_n(k)$ *if and only if* $\{\frac{1}{2}(n^2-4n-3)+k, \frac{1}{2}(n^2-4n-1)+k\} \subset E_n(k)$.

Proof. Let $p = \frac{1}{2}(n+1)$, $q = n-3$. Then $2p-q=4$. Thus $\gcd(p,q)=4$, 2 or 1. Since $n \equiv 1 \pmod 4$ and $n \geqslant 9$, then $(p,q) \in L(n)$. Since $m(p,q,k) = (p-1)(q-1) + \max\{k+q-n-1, 0\} = \frac{1}{2}(n^2-5n+4) + \max\{k-4, 0\} \leqslant \frac{1}{2}(n^2-5n+4)+k$ and

$$M(p,q,k) \geqslant \begin{cases} \frac{1}{2}(n^2-4n-5)+k+2 & \text{if } 1 \leqslant k \leqslant \frac{1}{2}(n+1), \\ \frac{1}{2}(n^2-4n-5)+k+1 & \text{if } \frac{1}{2}(n+3) \leqslant k \leqslant \frac{1}{2}(n+5), \\ \frac{1}{2}(n^2-4n-5)+k & \text{if } \frac{1}{2}(n+7) \leqslant k \leqslant n. \end{cases}$$

The proof of this lemma is completed by Lemma 16 and Theorem 4. □

Lemma 18. *Let* n, k *be integers with* $2 \leqslant k \leqslant n-1$. *If* $n \equiv 3 \pmod 4$ *and* $n \geqslant 15$, *then* $E_n^{(1)}(k) \subset E_n(k)$ *if and only if* $\{\frac{1}{2}(n^2-4n-19)+k, \ldots, \frac{1}{2}(n^2-4n-1)+k\} \subset E_n(k)$.

Proof. Let $p = \frac{1}{2}(n+3)$, $q = n-5$. Then $2p-q=8$. Thus $\gcd(p,q)=8$, 4, 2 or 1. Since $n \equiv 3 \pmod 4$ and $n \geqslant 15$, $(p,q) \in L(n)$. Then $m(p,q,k) = (p-1)(q-1) + \max\{k+q-n-1, 0\} = \frac{1}{2}(n^2-5n-6) + \max\{k-6, 0\} < \frac{1}{2}(n^2-5n+6)+k$ and $M(p,q,k) \geqslant p(q-2)+k = \frac{1}{2}(n^2-4n-21)+k$. Thus $\{\frac{1}{2}(n^2-5n+6)+k, \ldots, \frac{1}{2}(n^2-4n-21)+k\} \subset E_n(k)$. Thus the proof of this lemma is completed by Lemma 16. □

Let $r_1=3, r_2=7, r_3=11, r_4=19, r_5=23, r_6=31, r_7, \ldots, r_i, \ldots$ be the infinite sequence of all prime numbers of the form $4k+3$, and $\mathbf{P_3} = \{r_1, r_2, \ldots, r_i, \ldots\}$.

Lemma 19. *If* $n \geqslant 43$ *and* $n \neq 50, 61, 72, 83, 94, 105$, *then there exists a prime* r (*depending on* n) *satisfying the following properties*:
(B1) $r \in \mathbf{P_3}$;
(B2) $\frac{1}{4}(r-5)(r+3) \geqslant 21$;
(B3) $\frac{1}{4}(r-1)(r+5) \leqslant n-3$;
(B4) $n \not\equiv \frac{1}{2}(r+1) \pmod r$.

Proof. The proof of this lemma is analogous to the proof of Lemma 6 in [9]. □

Lemma 20. *If* n *is odd*, $n \geqslant 43$ *and* $n \neq 61, 83, 105$, *then* $E_n^{(1)}(k) \subset E_n(k)$.

Proof. Let $p = \frac{1}{2}[n + \frac{1}{2}(r-1)]$ and $q = n - \frac{1}{2}(r+1)$, where r is the prime number satisfying properties (B1)–(B4) in Lemma 19. By (B1), $r \equiv 3 \pmod 4$, p is a integer since n is odd. By (B3), we have $r < \frac{1}{3}(2n-1)$, and then $p < q$. Also we have that either $\gcd(p,q) = r$ or $\gcd(p,q) = 1$ since $2p - q = r$. But if $\gcd(p,q) = r$, then $q = n - \frac{1}{2}(r+1) \equiv 0 \pmod r$. So $n \equiv \frac{1}{2}(r+1) \pmod r$, this contradicts (B4). Hence $\gcd(p,q) = 1$. Thus $(p,q) \in L(n)$. Now we use Theorem 4 to get that $m(p,q,k) = (p-1)(q-1) + \max\{k + q - n - 1, 0\} = \frac{1}{2}[n^2 - 4n - \frac{1}{4}(r-5)(r+3)] + \max\{k - \frac{1}{2}(r+3), 0\} \leqslant \frac{1}{2}(n^2 - 4n - 21) + k$, $M(p,q,k) \geqslant p(q-2) + k = \frac{1}{2}\{n^2 - 4n + 3 - [n - 3 - \frac{1}{4}(r-1)(r+5)]\} + k \geqslant \frac{1}{2}(n^2 - 4n + 3) + k$ and $\{\frac{1}{2}(n^2 - 4n - 19) + k, \ldots, \frac{1}{2}(n^2 - 4n + 3) + k\} \subset E_n(k)$. Hence, by Lemmas 17 and 18, this lemma follows. □

Lemma 21. *Let n be odd with $n \geqslant 5$ and $\frac{1}{2}(n+1) \leqslant i \leqslant n-2$. If D is the digraph with a Hamiltonian cycle $(v_1, v_2, \ldots, v_n, v_1)$ and two additional arcs (v_n, v_2), (v_i, v_{i+2}), then $\exp_D(k) = \frac{1}{2}(n^2 - 3n + 4) - i + k$ for $1 \leqslant k \leqslant 2i - n + 1$.*

Proof. It is easy to see that $\exp_D(v_{i+1}) > \exp_D(v_{i+2}) > \cdots > \exp_D(v_{n-1}) > \exp_D(v_n)$ and $\exp_D(v_1) > \exp_D(v_2) > \cdots > \exp_D(v_i)$. So $\exp_D(1) = \min\{\exp_D(v_n), \exp_D(v_i)\}$. Since $|R_{(n-2)j+i}(v_n)| = \min\{n, 3 + 2j\}$ and $|R_{(n-2)j+i-1}(v_n)| = \min\{n, 2 + 2j\}$, we have $\exp_D(v_n) = \frac{1}{2}(n-2)(n-3) + i$. Similarly it can be proved that $\exp_D(v_i) = \frac{1}{2}(n-2)(n-3) + n - i$. Thus $\exp_D(1) = \exp_D(v_i) = \frac{1}{2}(n-2)(n-3) + n - i$. Hence $\exp_D(k) = \frac{1}{2}(n^2 - 3n + 4) - i + k$ for $1 \leqslant k \leqslant 2i - n + 1$. □

Corollary 4. *Let n be odd and $n \geqslant 5$. Then $\{\frac{1}{2}(n^2 - 5n + 12), \ldots, \frac{1}{2}(n^2 - 4n + 7)\} \subset E_n(2)$.*

Proof. Let i take over all integers in $\{\frac{1}{2}(n+1), \ldots, n-2\}$ in Lemma 21. □

Theorem 5. $\{1, 2, \ldots, \lfloor \frac{1}{2}(n-2)^2 \rfloor + 2\} \subset E_n(2)$ *for all $n \geqslant 3$.*

Proof. Since $E_{n-1}(1) \subset E_n(2)$ and $\{1, 2, \ldots, \frac{1}{2}(n^2 - 3n + 6)\} \subset E_n(1)$, we have $\{1, 2, \ldots, \frac{1}{2}(n^2 - 5n + 10)\} \subset E_n(2)$.

Case 1. n is odd. Let $p = \frac{1}{2}(n-1)$, $q = n$. Then $m(p,q,2) = \frac{1}{2}(n^2 - 4n + 5)$ and $M(p,q,2) \geqslant \frac{1}{2}(n^2 - 3n + 2) + 2 \geqslant \lfloor \frac{1}{2}(n-2)^2 \rfloor + 2$. Thus $\{\frac{1}{2}(n^2 - 4n + 5), \ldots, \lfloor \frac{1}{2}(n-2)^2 \rfloor + 2\} \subset E_n(2)$. By Corollary 4, $\{\frac{1}{2}(n^2 - 5n + 12), \ldots, \frac{1}{2}(n^2 - 4n + 7)\} \subset E_n(2)$.

Case 2. n is even.

Subcase 2.1. $n \equiv 0 \pmod 4$. Let $p = \frac{1}{2}(n-2)$, $q = n$. Then $m(p,q,2) = \frac{1}{2}(n^2 - 5n + 6)$ and $M(p,q,2) \geqslant \frac{1}{2}(n^2 - 4n + 4) + 2$. Thus $\{\frac{1}{2}(n^2 - 5n + 6), \ldots, \lfloor \frac{1}{2}(n-2)^2 \rfloor + 2\} \subset E_n(2)$.

Table 1

n	k	$E_n^{(2)}(k)$	n	k	$E_n^{(2)}(k)$
7	$3 \leqslant k \leqslant 6$	$10+k$	29	$16 \leqslant k \leqslant 17$	$362+k$
9	$6 \leqslant k \leqslant 7$	$22+k$	29	$18 \leqslant k \leqslant 28$	$361+k, 362+k$
9	$k=8$	$21+k, 22+k$	31	$3 \leqslant k \leqslant 30$	$409+k, \ldots, 418+k$
11	$3 \leqslant k \leqslant 10$	$36+k, 37+k, 38+k$	33	$18 \leqslant k \leqslant 19$	$478+k$
13	$8 \leqslant k \leqslant 9$	$58+k$	33	$20 \leqslant k \leqslant 32$	$477+k, 478+k$
13	$10 \leqslant k \leqslant 12$	$57+k, 58+k$	35	$3 \leqslant k \leqslant 34$	$533+k, \ldots, 542+k$
15	$3 \leqslant k \leqslant 14$	$78+k, \ldots, 82+k$	37	$20 \leqslant k \leqslant 21$	$610+k$
17	$10 \leqslant k \leqslant 11$	$110+k$	37	$22 \leqslant k \leqslant 36$	$609+k, 610+k$
17	$12 \leqslant k \leqslant 16$	$109+k, 110+k$	39	$3 \leqslant k \leqslant 38$	$673+k, \ldots, 682+k$
19	$3 \leqslant k \leqslant 18$	$136+k, \ldots, 142+k$	41	$22 \leqslant k \leqslant 23$	$758+k$
21	$12 \leqslant k \leqslant 13$	$178+k$	41	$24 \leqslant k \leqslant 40$	$757+k, 758+k$
21	$14 \leqslant k \leqslant 20$	$177+k, 178+k$	61	$32 \leqslant k \leqslant 33$	$1738+k$
23	$3 \leqslant k \leqslant 22$	$210+k, \ldots, 218+k$	61	$34 \leqslant k \leqslant 60$	$1737+k, 1738+k$
25	$14 \leqslant k \leqslant 15$	$262+k$	83	$3 \leqslant k \leqslant 82$	$3269+k, \ldots, 3278+k$
25	$16 \leqslant k \leqslant 24$	$261+k, 262+k$	105	$54 \leqslant k \leqslant 55$	$5302+k$
27	$3 \leqslant k \leqslant 26$	$301+k, \ldots, 310+k$	105	$56 \leqslant k \leqslant 104$	$5301+k, 5302+k$

Subcase 2.2. $n \equiv 2 \pmod 4$. Let $p = \frac{1}{2}n, q = n-2$. Then $m(p,q,2) = \frac{1}{2}(n^2 - 5n+6)$ and $M(p,q,2) \geqslant \frac{1}{2}(n^2-4n)+2$. Thus $\{\frac{1}{2}(n^2-5n+6), \ldots, \frac{1}{2}(n^2-4n)+2\} \subset E_n(2)$. Let $p = \frac{1}{2}n, q = n-1$. Then $m(p,q,2) = (p-)(q-1) + \max\{2+q-n-1, 0\} = \frac{1}{2}(n-2)^2)$ and $M(p,q,2) \geqslant \frac{1}{2}(n^2-3n+2)+2 \geqslant \frac{1}{2}(n-2)^2+2$. Thus $\{\frac{1}{2}(n^2-5n+6), \ldots, \lfloor\frac{1}{2}(n-2)^2\rfloor+2\} \subset E_n(2)$.

To sum up, $\{1, 2, \ldots, \lfloor\frac{1}{2}(n-2)^2\rfloor+2\} \subset E_n(2)$ for $n \geqslant 3$. $\square$

Theorem 6. $\{1, 2, \cdots, \lfloor\frac{1}{2}(n-2)^2\rfloor+k\} \subset E_n(k)$ *for all integers* n, k *with* $3 \leqslant k \leqslant n-1$ *except* $n = 11$ *and* $9 \leqslant k \leqslant 10$.

Proof. *Case* 1. $4 \leqslant n \leqslant 6$. Since the proof of this case is not difficult, it is left to the reader.

Case 2. $n \geqslant 7$. We divide the proof into six steps.

(1) By Lemmas 14–18 and 20, for proving this theorem, it is enough to prove that $E_n^{(2)}(k) \subset E_n(k)$, where $E_n^{(2)}(k)$ is as shown in Table 1.

(2) Let $p = \frac{1}{2}(n+1)$, $q = n-2$. Then $2p-q = 3$. Thus $\gcd(p,q) = 3$ or 1. If $n-2 \not\equiv 0 \pmod 3$ and $n \geqslant 7$, then $(p,q) \in L(n)$. Since $m(p,q,k) = \frac{1}{2}(n-1)(n-3) + \max\{k-3, 0\} = \frac{1}{2}(n^2-4n-3)+k$, $M(p,q,k) \geqslant p(q-2)+k = \frac{1}{2}(n+1)(n-4)+k \geqslant \frac{1}{2}(n^2-4n-1)+k$, we have $\{\frac{1}{2}(n^2-4n-3)+k, \frac{1}{2}(n^2-4n-1)+k\} \subset E_n(k)$ by Theorem 4. Also, for proving $E_n^{(2)}(k) \subset E_n(k)$, it is enough to prove that $E_n^{(3)}(k) \subset E_n(k)$, where $E_n^{(3)}(k)$ is as shown in Table 2.

(3) Let $p = \frac{1}{2}(n+3)$, $q = n-4$. Then $2p-q = 7$. Thus $\gcd(p,q) = 7$ or 1. If $n-4 \not\equiv 0 \pmod 7$ and $n \geqslant 13$, then $(p,q) \in L(n)$. Since $m(p,q,k) = \frac{1}{2}(n+$

Table 2

n	k	$E_n^{(3)}(k)$	n	k	$E_n^{(3)}(k)$
11	$3 \leqslant k \leqslant 10$	$36+k, 37+k, 38+k$	29	$18 \leqslant k \leqslant 28$	$361+k, 362+k$
15	$3 \leqslant k \leqslant 14$	$78+k, \ldots, 80+k$	31	$3 \leqslant k \leqslant 30$	$409+k, \ldots, 416+k$
17	$10 \leqslant k \leqslant 11$	$110+k$	35	$3 \leqslant k \leqslant 34$	$533+k, \ldots, 542+k$
17	$12 \leqslant k \leqslant 16$	$109+k, 110+k$	39	$3 \leqslant k \leqslant 38$	$673+k, \ldots, 680+k$
19	$3 \leqslant k \leqslant 18$	$136+k, \ldots, 140+k$	41	$22 \leqslant k \leqslant 23$	$758+k$
23	$3 \leqslant k \leqslant 22$	$210+k, \ldots, 218+k$	41	$24 \leqslant k \leqslant 40$	$757+k, 758+k$
27	$3 \leqslant k \leqslant 26$	$301+k, \ldots, 308+k$	83	$3 \leqslant k \leqslant 82$	$3269+k, \ldots, 3278+k$
29	$16 \leqslant k \leqslant 17$	$362+k$			

Table 3

n	k	$E_n^{(4)}(k)$	n	k	$E_n^{(4)}(k)$
11	$3 \leqslant k \leqslant 10$	$36+k, 37+k, 38+k$	31	4	413, 414, 415
19	3	139	31	$5 \leqslant k \leqslant 30$	$409+k, 410+k$
23	3	213, 214, 215	35	3	$536, \ldots, 539$
23	4	214, 215	35	4	537, 538, 539
23	$5 \leqslant k \leqslant 22$	$210+k$	35	$5 \leqslant k \leqslant 34$	$533+k, 534+k$
27	3	$304, \ldots, 307$	39	$3 \leqslant k \leqslant 38$	$673+k, \ldots, 680+k$
27	4	$305, \ldots, 307$	83	3	$3272, \ldots, 3275$
27	$5 \leqslant k \leqslant 26$	$301+k, 302+k$	83	4	3273, 3274, 3275
31	3	$412, \ldots, 415$	83	$5 \leqslant k \leqslant 82$	$3269+k, 3270+k$

$1)(n-5)+\max\{k-5,0\}=\frac{1}{2}(n^2-4n-5)+\max\{k-5,0\}$, $M(p,q,k)\geqslant p(q-2)+k=\frac{1}{2}(n+3)(n-6)+k=\frac{1}{2}(n^2-3n-18)+k$, we have $\{\frac{1}{2}(n^2-4n-5)+\max\{k-5,0\},\ldots,\frac{1}{2}(n^2-3n-18)+k\}\subset E_n(k)$ by Theorem 4. Also, for proving $E_n^{(3)}(k)\subset E_n(k)$, it is enough to prove that $E_n^{(4)}(k)\subset E_n(k)$, where $E_n^{(4)}(k)$ is as shown in Table 3.

(4) Let $p=\frac{1}{2}(n+5)$, $q=n-6$. Then $2p-q=11$. Thus $\gcd(p,q)=11$ or 1. If $n-6\not\equiv 0 \pmod{11}$ and $n\geqslant 19$, then $(p,q)\in L(n)$. Since $m(p,q,k)=\frac{1}{2}(n+3)(n-7)+\max\{k-8,0\}\leqslant\frac{1}{2}(n^2-4n-21)+k$, $M(p,q,k)\geqslant p(q-2)+k=\frac{1}{2}(n+5)(n-8)+k=\frac{1}{2}(n^2-3n-40)+k$, we have $\{\frac{1}{2}(n^2-4n-21)+k,\ldots,\frac{1}{2}(n^2-3n-40)+k\}\subset E_n(k)$ by Theorem 4. Also, for proving $E_n^{(4)}(k)\subset E_n(k)$, it is enough to prove that $E_n^{(5)}(k)\subset E_n(k)$, where $E_n^{(5)}(k)$ is as shown in Table 4.

(5) For convenience, we use the symbol $(n,k;\ p,q;\ m(p,q,k),\ M(p,q,k))$. From $(11,k;\ 4,11;\ 36+k,\ 36+k)$, $(11,k;\ 7,8;\ 42+\max\{k-4,0\},\ M(7,8,k))$ with $M(7,8,k)\geqslant 42+k$, $(39,k;\ 25,29;\ 672+\max\{k-11,0\},M(25,29,k))$ with $M(25,29,k)\geqslant 675+k$, $(39,k;\ 23,32;\ 682+\max\{k-8,0\},\ M(23,32,k))$ with $M(23,32,k)\geqslant 690+k$ and $(83,k;\ 47,72;\ 3266+\max\{k-12,0\},\ M(47,72,k))$ with $M(47,72,k))\geqslant 3290+k$, we have: for proving $E_n^{(5)}(k)\subset E_n(k)$, it is enough to prove that $E_n^{(6)}(k)\subset E_n(k)$, where $E_n^{(6)}(k)$ is as shown in Table 5.

Table 4

n	k	$E_n^{(5)}(k)$	n	k	$E_n^{(5)}(k)$
11	$3 \leqslant k \leqslant 10$	$36+k, 37+k, 38+k$	83	4	3273, 3274, 3275
39	$3 \leqslant k \leqslant 38$	$673+k, \ldots, 680+k$	83	$5 \leqslant k \leqslant 82$	$3269+k, 3270+k$
83	3	$3272, \ldots, 3275$			

Table 5

n	k	$E_n^{(6)}(k)$	n	k	$E_n^{(6)}(k)$
11	3	40, 41	39	4	680, 681
11	$4 \leqslant k \leqslant 10$	$37+k$	39	5	681
39	3	679, 680, 681			

(6) Let $n = 11$ and $i = 9$ in Lemma 21, then we get $\exp_D(k) = 37 + k$ for $1 \leqslant k \leqslant 8$. Let $n = 11$ and $i = 8$ in Lemma 21, then we get $\exp_D(3) = 41$. Let $n = 39$ and $i = 29$ in Lemma 21, then we get $\exp_D(k) = 676 + k$ for $3 \leqslant k \leqslant 5$. Let $n = 39$ and $i = 28$ in Lemma 21, then we get $\exp_D(k) = 677 + k$ for $3 \leqslant k \leqslant 4$. Let $n = 39$ and $i = 27$ in Lemma 21, then we get $\exp_D(3) = 681$.

Combining Cases 1 and 2, the proof of this theorem is completed. □

Corollary 5. $\{1, 2, \ldots, 36 + k, 38 + k, 39 + k, 40 + k\} \subset E_{11}(k)$ *for* $9 \leqslant k \leqslant 10$.

5. Proof of Main Theorem

First of all, we list $E_{11}(k)$ for $9 \leqslant k \leqslant 10$ without proof.

Theorem 7. *For* $9 \leqslant k \leqslant 10$, $E_{11}(k) = \{1, 2, \ldots, 36 + k, 38 + k, 39 + k, 40 + k\} \cup \bigcup_{(p,q)\in L(11)} \{m \mid m(p, q, k) \leqslant m \leqslant M(p, q, k)\}$.

Proof of Main Theorem. By Theorem 7, it is enough to prove that Main Theorem holds for all integers n, k with $2 \leqslant k \leqslant n-1$ except for $n = 11$ and $9 \leqslant k \leqslant 10$. By Lemma 6 and Theorems 1–6, we get $\{1, 2, \ldots, \lfloor\frac{1}{2}(n-2)^2\rfloor + k\} \cup \bigcup_{(p,q)\in L(n)} \{m \mid m(p, q, k) \leqslant m \leqslant M(p, q, k)\} \subset E_n(k) \subset \{1, 2, \ldots, \lfloor\frac{1}{2}(n-2)^2\rfloor + k\} \cup \bigcup_{(p,q)\in L'(n)} \{m \mid m(p, q, k) \leqslant m \leqslant M(p, q, k)\}$. Let $D \in PD_n$. If $1 \in L(D)$, then $\exp_D(k) \leqslant n - 2 + k \leqslant \lfloor\frac{1}{2}(n-2)^2\rfloor + k$ for $n \geqslant 4$. Thus $\{m(1, q, k), m(1, q, k) + 1, \ldots, M(1, q, k)\} \subset \{1, 2, \ldots, \lfloor\frac{1}{2}(n-2)^2\rfloor + k\}$. So $E_n(k) = \{1, 2, \ldots, \lfloor\frac{1}{2}(n-2)^2\rfloor + k\} \cup \bigcup_{(p,q)\in L(n)} \{m \mid m(p, q, k) \leqslant m \leqslant M(p, q, k)\}$

for $n \geqslant 4$. Further, it is easy to check that $E_3(2) = \{1, 2, 3, 4\}$. So the proof is completed. □

Remark. Combining [7,9] and the Main Theorem, we have

$$E_n(k) = \left\{1, 2, \ldots, \lfloor \tfrac{1}{2}(n-2)^2 \rfloor + k\right\} \cup \bigcup_{(p,q)\in L(n)} \{m \mid m(p,q,k) \leqslant m \leqslant M(p,q,k)\}$$

for all n, k with $1 \leqslant k \leqslant n$ except $n = 11$ and $9 \leqslant k \leqslant 11$. Also

$$E_{11}(k) = (\{1, 2, \ldots, 40 + k\} \backslash \{37 + k\}) \cup \bigcup_{(p,q)\in L(11)} \{m \mid m(p,q,k) \leqslant m \leqslant M(p,q,k)\}$$

for $9 \leqslant k \leqslant 11$. So the problem of local exponent sets of primitive digraphs is completely solved.

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