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A Note on List Improper Coloring Planar Graphs

KO-WEI LIH
Institute of Mathematics, Academia Sinica, Nankang, Taipei 115, Taiwan, R.O.C.
makwlih@sinica.edu.tw

ZENGMIN SONG
Department of Mathematics, Hong Kong Baptist University
Kowloon, Hong Kong

WEIFAN WANG
Department of Mathematics, Liaoning University
Shenyang 110036, P.R. China

KEMIN ZHANG
Department of Mathematics, Nanjing University
Nanjing 210093, P.R. China

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Abstract—A graph G is called $(k,d)^*$-choosable if, for every list assignment L satisfying $|L(v)| = k$ for all $v \in V(G)$, there is an L-coloring of G such that each vertex of G has at most d neighbors colored with the same color as itself. In this note, we prove that every planar graph without 4-cycles and l-cycles for some $l \in \{5,6,7\}$ is $(3,1)^*$-choosable. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

All graphs considered in this paper are finite, loopless, and without multiple edges unless stated otherwise. For a plane graph G, we denote its vertex set, edge set, face set, and minimum degree by $V(G)$, $E(G)$, $F(G)$, and $\delta(G)$, respectively. For $x \in V(G) \cup F(G)$, let $d_G(x)$ (or simply $d(x)$) denote the degree of x in G. A vertex (or face) of degree k is called a k-vertex (or k-face). Let $N_G(u)$ (or simply $N(u)$) denote the set of neighbors of a vertex u in G. Two faces of a plane graph are said to be adjacent if they have at least one common boundary edge. A vertex v and a face f are said to be incident if v lies on the boundary of f. For $x \in V(G) \cup F(G)$, we use $F_k(x)$ to denote the set of all k-faces that are incident or adjacent to x, and $V_k(x)$ to denote the set of all k-vertices that are incident or adjacent to x. For $f \in F(G)$, we write $f = [u_1u_2\cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in the clockwise order. A 3-face $[u_1u_2u_3]$ is called an (m_1, m_2, m_3)-face if $d(u_i) = m_i$ for $i = 1,2,3$.

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Let the integer $m > 1$. A graph G is m-colorable with impropriety d, or simply $(m,d)^*$-colorable, if the vertices of G can be colored with m colors so that each vertex has at most d of the same color as itself. An $(m,0)^*$-coloring is an ordinary proper m-coloring. A list assignment of G is a function L that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An L-coloring with impropriety d, or simply $(L,d)^*$-coloring, is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that v has at most d neighbors colored with $\phi(v)$. A graph is m-choosable with impropriety d, or simply $(m,d)^*$-choosable, if there exists an $(L,d)^*$-coloring for every list assignment L with $|L(v)| = m$ for all $v \in V(G)$. Obviously, $(m,0)^*$-choosability is the ordinary m-choosability introduced by Erdős, Rubin and Taylor [1], and independently by Vizing [2].

The notion of list improper coloring was introduced independently by Škrekovski [3] and Eaton and Hull [4]. They proved that every planar graph is $(3,2)^*$-choosable and every outerplanar graph is $(2,2)^*$-choosable. Let $g(G)$ denote the girth of a graph G, i.e, the length of a shortest cycle. Recently, Škrekovski [5] proved that every planar graph G is $(2,1)^*$-choosable if $g(G) \geq 9$, $(2,2)^*$-choosable if $g(G) \geq 7$, $(2,3)^*$-choosable if $g(G) \geq 6$, and $(2,d)^*$-choosable if $g(G) \geq 5$ and $d \geq 4$. Thomassen [6] proved that every planar graph with girth at least five is 3-choosable. Voigt [7] found an example of a planar graph without 3-cycles that is not 3-choosable. However, Škrekovski [8] proved that every planar graph without 3-cycles is $(3,1)^*$-choosable. Steinberg [9, p. 42] conjectured that every planar graph without 4-cycles and 5-cycles is 3-colorable. This conjecture still remains open. The best partial result, due to Borodin [10] and independently to Sanders and Zhao [11], shows that every planar graph without k-cycles for all $4 \leq k \leq 9$ is 3-colorable. We do not know if there exists a planar graph without 4-cycles and 5-cycles that is 3-colorable, yet non-3-choosable. In view of the result of [8], we would like to know if every planar graph without 4-cycles and 5-cycles is $(3,1)^*$-choosable. In this note, we will present a positive solution to this problem. Our result may be regarded as a solution to a weakened form of Steinberg's three-color conjecture.

2. MAIN THEOREM

Given a list improper coloring of the graph G and a vertex v, let $\mathrm{Im}(v)$ denote the number of neighbors of v that are colored with the same color as v. We call $\mathrm{Im}(v)$ the impropriety of v with respect to the coloring.

LEMMA 1. *Let G be a graph and $d \geq 1$ an integer. If G is not $(k,d)^*$-choosable but every subgraph of G with fewer vertices is, then the following facts hold.*

(1) $\delta(G) \geq k$.

(2) *If $u \in V(G)$ is a k-vertex and v is a neighbor of u, then $d(v) \geq k + d$.*

PROOF. Let x be an arbitrary vertex of G and L an arbitrary list assignment satisfying $|L(v)| = k$ for all $v \in V(G)$. By the assumption, there is an $(L,d)^*$-coloring of $G - x$. Then the neighbors of x must use up all k colors, for otherwise the $(L,d)^*$-coloring can be extended to G. Statement (1) thus follows. To prove (2), we assume that $d(v) \leq k+d-1$ for some $v \in N(u)$. By the assumption, there is an $(L,d)^*$-coloring ϕ of $G - u$. If $\mathrm{Im}(v) = d$ with respect to ϕ, then the degree constraint on v implies that the number of distinct colors used by the neighbors of v in $G - u$ is at most $k - 1$. Then we recolor v to make $\mathrm{Im}(v) = 0$ in the modified $(L,d)^*$-coloring. Therefore, the chosen coloring ϕ may be assumed to satisfy $\mathrm{Im}(v) \leq d - 1$. Since there are at most $k - 1$ vertices in $G - u - v$ that were adjacent to u, we may color u so that, among all neighbors of u in G, only v may have the same color as u. Thus, ϕ is extended to an $(L,d)^*$-coloring of G. This contradicts the choice of G. ∎

LEMMA 2. *Let G be a graph such that G is not $(k,d)^*$-choosable but every subgraph of G with fewer vertices is. If $d(u) \leq k + d$ for a given $u \in V(G)$, then $d(v) \geq k + d$ for some $v \in N(u)$.*

PROOF. Suppose that $d(v) \leq k + d - 1$ for every $v \in N(u)$. Let L be an arbitrary list assignment satisfying $|L(x)| = k$ for all $x \in V(G)$. By the assumption, there is an $(L,d)^*$-coloring ϕ of $G - u$.

We may argue similarly to the proof of Lemma 1 that $\text{Im}(v) \leq d-1$ for every neighbor v of u with respect to ϕ. If every color in $L(u)$ is used by at least $d+1$ neighbors of u, then there exist at least $k(d+1) > k+d \geq d(u)$ neighbors of u which is absurd. Hence, there is a color in $L(u)$ that is used by at most d neighbors of u. We extend ϕ to color u with that color. The extended ϕ is an $(L,d)^*$-coloring of G. This contradicts the choice of G. ∎

To prove our main theorem, we only need the case for $k=3$ and $d=1$ in Lemmas 1 and 2. However, we have arrived at a much shorter proof for the results in [5] by means of these lemmas.

THEOREM 3. *Let G be a plane graph without 4-cycles and l-cycles for some $l \in \{5,6,7\}$. Then G is $(3,1)^*$-choosable.*

PROOF. Suppose that this theorem is false. Let G be a counterexample with the fewest vertices. We first assume that G is 2-connected. Thus, the boundary of every face of G forms a cycle, and every vertex v of G is incident to exactly $d(v)$ distinct faces. Every subgraph H of G with fewer vertices is still a plane graph without 4-cycles and l-cycles for some $l \in \{5,6,7\}$, hence, H is $(3,1)^*$-choosable. Let L denote an arbitrary list assignment of G satisfying $|L(v)| = 3$ for all $v \in V(G)$. The following facts hold for G.

(a) $\delta(G) \geq 3$.
(b) G does not contain two adjacent 3-vertices.
(c) G contains neither a 4-face nor two adjacent 3-faces.
(d) G does not contain a (3,4,4)-face.

Facts (a) and (b) follow from Lemma 1, and (b) implies that $|V_3(f)| \leq \lfloor d(f)/2 \rfloor$ for all $f \in F(G)$. Fact (c) holds since G does not contain any 4-cycle. It also implies that $|F_3(v)| \leq \lfloor d(v)/2 \rfloor$ for all $v \in V(G)$. The proof of Fact (d) goes as follows. Suppose to the contrary that G contains a $(3,4,4)$-face $[uvw]$ such that $d(u)=3$ and $d(v)=d(w)=4$. By the minimality of G, $G-\{u,v,w\}$ has an $(L,1)$-list coloring ϕ. Define $L'(x) = L(x) - A(x)$ for every $x \in \{u,v,w\}$, where $A(x)$ denotes the set of colors that ϕ assigns to the neighbors of x in $G-\{u,v,w\}$. Thus $|L'(u)| \geq 2$, $|L'(v)| \geq 1$, and $|L'(w)| \geq 1$. An $(L',1)$-coloring of the 3-cycle $uvwu$ can be constructed easily. Hence, G is $(L,1)$-colorable, this contradicts the choice of G.

Graph G satisfies Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ which can be rewritten in the following form:

$$\sum\{d(v)-4 \mid v \in V(G)\} + \sum\{d(f)-4 \mid f \in F(G)\} = -8.$$

Let w denote the weight function defined on $V(G) \cup F(G)$ by $w(v) = d(v)-4$ if $v \in V(G)$ and $w(f) = d(f)-4$ if $f \in F(G)$. Thus, the total sum of weights is the negative number -8. We are going to introduce discharging rules so that the total sum of weights is kept fixed while the discharging is in progress. However, once the discharging is finished, we can show that the resulting weight function w' is nowhere negative. Thus, the following contradiction is arrived and the existence of G is absurd.

$$0 \leq \sum\{w'(x) \mid x \in V(G) \cup F(G)\} = \sum\{w(x) \mid x \in V(G) \cup F(G)\} = -8.$$

Now we list our discharging rules.

(R1) For every vertex v with $d(v) \geq 5$, we transfer $1/3$ from v to each incident 3-face.
(R2) For every 3-face f, we transfer $1/3$ from f to each incident 3-vertex.
(R3) For every face f with $d(f) \geq 5$, we transfer $1/3$ from f to each incident 3-vertex and $1/3$ from f to each adjacent 3-face.

It remains to show that the resulting weight function w' satisfies $w'(x) \geq 0$ for all $x \in V(G) \cup F(G)$. It is evident that $w'(x) = w(x) = 0$ for all $x \in V(G) \cup F(G)$ with $d(x) = 4$. Let $v \in V(G)$. By (a), $d(v) \geq 3$. If $d(v) = 3$, then, by (R2) and (R3), $w'(v) = w(v) + 3 \cdot (1/3) = 0$ since G does not contain any 4-faces and v is incident to three distinct faces. If $d(v) \geq 5$, then by (c) and (R1), $w'(v) \geq w(v) - (1/3)|F_3(v)| \geq w(v) - (1/3)\lfloor d(v)/2 \rfloor \geq 0$. Now let $f \in F(G)$. First suppose that $d(f) = 3$ and $f = [x_1x_2x_3]$. If e is one of the boundary edges of f, we use f_e to denote the face in G adjacent to f and sharing the same boundary edge e with f. It follows from (c) that $d(f_e) \geq 5$, where e equals to x_1x_2, x_2x_3, or x_3x_1. We claim that $f_{x_1x_2}$, $f_{x_2x_3}$, and $f_{x_3x_1}$ are pairwise distinct. If two of them, say $f_{x_1x_2}$ and $f_{x_2x_3}$, are identical, then either $d(x_2) \leq 2$ or x_2 is a cut vertex. Yet both are impossible. If f is incident to at least one 3-vertex, then it follows from (d) that the boundary of f contains a vertex of degree at least five. Thus, $w'(f) \geq w(f) + 4 \cdot (1/3) - (1/3) = 0$ by (R1) to (R3). If f is not incident to any 3-vertex, then $w'(f) \geq w(f) + 3 \cdot (1/3) = 0$. If $d(f) \geq 6$, then $w'(f) \geq w(f) - (1/3)(|V_3(f)| + |F_3(f)|) \geq w(f) - (1/3) \cdot d(f) \geq 0$. If G does not contain any 5-cycle, then we are done. Otherwise, we finally assume $d(f) = 5$. Since G does not contain any l-cycle for $l = 6$ or 7, we have $|F_3(f)| \leq 1$. By (b), $|V_3(f)| \leq 2$. It follows from (R3) that $w'(f) \geq w(f) - 3 \cdot (1/3) \geq 0$. The proof of the 2-connected case is complete.

We next suppose that G contains cut vertices. We may choose a block B of G that contains a unique cut vertex t^* of G. Let f_0 denote the exterior face of the plane graph B. Thus, B contains no 4-cycles and l-cycles for some $l \in \{5, 6, 7\}$. Moreover B satisfies the following properties.

(a′) $d_B(t^*) \geq 2$ and $d_B(v) = d_G(v) \geq 3$ for all $v \in V(B) - \{t^*\}$.
(b′) B does not contain two adjacent 3-vertices u and v such that $u, v \in V(B) - \{t^*\}$.
(c′) B contains neither a 4-face nor two adjacent 3-faces in $F(B) - \{f_0\}$.
(d′) B does not contain a 3-face $[v_1v_2v_3]$ such that $v_1, v_2, v_3 \in V(B) - \{t^*\}$, $d_B(v_1) = 3$, and $d_B(v_2) = d_B(v_3) = 4$.

Euler's formula applied to B implies the following:

$$\sum \{d_B(v) - 4 \mid v \in V(B)\} + \sum \{d_B(f) - 4 \mid f \in F(B)\} = -8.$$

Define the weight function w on $V(B) \cup F(B)$ by $w(x) = d_B(x) - 4$ for all $x \in V(B) \cup F(B)$. Thus, $\sum\{w(x) \mid x \in V(B) \cup F(B)\} = -8$. We redistribute the weight $w(x)$ for every $x \in V(B) \cup F(B)$ according to the following discharging rules.

(r1) For every vertex $v \in V(B) - \{t^*\}$ with $d_B(v) \geq 5$, we transfer 1/3 to each incident 3-face except f_0.
(r2) For every 3-face $f \in F(B) - \{f_0\}$, we transfer 1/3 to each incident 3-vertex except t^*.
(r3) For every face $f \in F(B) - \{f_0\}$ with $d_B(f) \geq 5$, we transfer 1/3 to each incident 3-vertex except t^* and each adjacent 3-face except f_0.
(r4) We transfer 1/3 from t^* to each incident face except f_0.
(r5) We transfer 1/3 from f_0 to each adjacent face and each incident vertex except t^*.

Let w^* denote the final weight function when the discharging is complete. The total sum of the new weights $w^*(x)$ is kept fixed. We can show that $w^*(x) \geq 0$ for all $x \in (V(B) \cup F(B)) - \{t^*, f_0\}$ by the same argument for the 2-connected case. Since $d_B(t^*) \geq 2$ and t^* is incident to at most $d_B(t^*) - 1$ faces except f_0, we have $w^*(t^*) \geq w(t^*) - (1/3)(d_B(t^*) - 1) = d_B(t^*) - 4 - (1/3)d_B(t^*) + (1/3) = (2/3)d_B(t^*) - (11/3) \geq -7/3$ by (r4). Note that f_0 is adjacent to at most $d_B(f_0)$ faces and incident to exactly $d_B(f_0) - 1$ vertices except t^*. Since $d_B(f_0) \geq 3$, we have $w^*(f_0) \geq d_B(f_0) - 4 - (1/3)(2d_B(f_0) - 1) = (1/3)d_B(f_0) - (11/3) \geq -8/3$ by (r5). It follows that $w^*(t^*) + w^*(f_0) \geq -5$. Consequently, we obtain the following contradiction and the proof is complete.

$$0 \leq \sum \{w^*(x) \mid x \in (V(B) \cup F(B)) - \{t^*, f_0\}\} = -8 - w^*(t^*) - w^*(f_0) \leq -3. \qquad \blacksquare$$

COROLLARY 4. *Every plane graph without 4-cycles and 5-cycles is* $(3,1)^*$*-choosable.*

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