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Discrete Mathematics 226 (2001) 419–421

DISCRETE MATHEMATICS

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Note

A bound for multicolor Ramsey numbers ☆

Ling-sheng Shi *, Ke-min Zhang

Department of mathematics, Nanjing University, Nanjing, 210093, People's Republic of China

Received 25 November 1998; revised 24 September 1999; accepted 7 February 2000

Abstract

The Ramsey number $R(G_1, G_2, \ldots, G_n)$ is the smallest integer p such that for any n-edge coloring $(E_1, E_2, \ldots, E_n)$ of K_p, $K_p[E_i]$ contains G_i for some i, G_i as a subgraph in $K_p[E_i]$. Let $R(m_1, m_2, \ldots, m_n) := R(K_{m_1}, K_{m_2}, \ldots, K_{m_n})$, $R(m; n) := R(m_1, m_2, \ldots, m_n)$ if $m_i = m$ for $i = 1, 2, \ldots, n$. A formula is obtained for $R(G_1, G_2, \ldots, G_n)$.

Keywords: Formula; Ramsey number

An n-edge coloring $(E_1, E_2, \ldots, E_n)$ of K_p is called a $(G_1, G_2, \ldots, G_n; p)((m_1, m_2, \ldots, m_n; p)$, resp.)-decomposition if for all i, $G_i(K_{m_i}$, resp.) is not contained in $K_p[E_i]$. It is clear that $R(G_1, G_2, \ldots, G_n) = p_0 + 1$ iff $p_0 = max\{p$:there exists a $(G_1, G_2, \ldots, G_n; p)$-decomposition$\}$. We denote by $N(G_i)$ the number of subgraphs of $K_p[E_i]$ which are isomorphic to G_i. The $(G_1, G_2, \ldots, G_n; p)$-decomposition is called a $(G_1, G_2, \ldots, G_n; p)$-Ramsey decomposition if $p = R(G_1, G_2, \ldots, G_n) - 1$. If G and H are two graphs, $G \circ H$ denotes either the disjoint union or the join (see [1]) of G and H. Let G_i^k be a graph of order k and let $G_i = G_i^{m_i - n_i} \circ G_i^{n_i}$. Taking any vertex v_i, let $G_i^{n_i+1} = \{v_i\} \circ G_i^{n_i}$. The number of subgraphs of $G_i^{n_i}$ which are isomorphic to $G_i^{n_i+1}$ is denoted by $n(i)$. Thus we have:

Theorem 1. *For any $(G_1, G_2, \ldots, G_n; p)$-decomposition and if $G_i = G_i^{m_i - n_i} \circ G_i^{n_i}$, $i \in \{1, 2, \ldots, n\}$, we have*

$$n(i)N(G_i^{n_i+1}) \leqslant N(G_i^{n_i})[R(G_1, \ldots, G_{i-1}, G_i^{m_i - n_i}, G_{i+1}, \ldots, G_n) - 1]. \tag{1}$$

Proof. In a $(G_1, G_2, \ldots, G_n; p)$-decomposition $(E_1, E_2, \ldots, E_n)$, by the definition of $R(G_1, \ldots, G_{i-1}, G_i^{m_i - n_i}, G_{i+1}, \ldots, G_n)$ and for any $G_i^{n_i} \subset K_p[E_i]$, there are at most

☆ The project supported by NSFC and NSFJS.

* Corresponding author.

E-mail address: zkmfl@nju.edu.cn (K.-m. Zhang).

0012-365X/01/$ - see front matter
PII: S0012-365X(00)00175-8

$R(G_1,\dots,G_{i-1},G_i^{m_i-n_i},G_{i+1},\dots,G_n)-1$ vertices v in $K_p[E_i]-V(G_i^{n_i})$ such that $\{v\}\circ G_i^{n_i}=G_i^{n_i+1}$, otherwise there exists a subgraph G of $K_p[E_i]-V(G_i^{n_i})$ of order $R(G_1,\dots,G_{i-1},G_i^{m_i-n_i},G_{i+1},\dots,G_n)$, either there exists a $G_i^{m_i-n_i}\subset G$ such that $G_i^{m_i-n_i}\circ G_i^{n_i}=G_i\subset K_p[E_i]$, or there exists a $G_j\subset K_p[E_j], j\neq i$; a contradiction. Hence by the definition of $N(G_i^{n_i})$ and $n(i)$, (1) follows. □

Theorem 1 generalizes the Theorem 1 in [4].

Corollary 1. *If* $G_i=K_{m_i}$ or $K_{m_i}-e$, $i\in\{1,2,\dots,n\}$, *then for any* $(G_1,G_2,\dots,G_n;p)$-*decomposition, we have*

$$(n_i+1)N(K_{n_i+1})\leqslant N(K_{n_i})\,[R(G_1,\dots,G_{i-1},G_i^{m_i-n_i},G_{i+1},\dots,G_n)-1], \tag{2}$$

where $G_i^{m_i-n_i}=K_{m_i-n_i}$ or $K_{m_i-n_i}-e, 0<n_i<m_i-1$ *and* $m_i>2$.

In particular if $G_i=K_i^m{:=}K_m$, $i=1,2,\dots,n$ *and* $m>2$, *we have*

$$(m-1)\sum_{i=1}^{n}N(K_i^{m-1})\leqslant[R(m;n-1)-1]\sum_{i=1}^{n}N(K_i^{m-2}). \tag{3}$$

Proof. Note that for any K_{k+1}, it contains exactly $k+1$ K_k. Hence, by (1), (2) follows. Furthermore, since $R(K_1^2,K_2^m,\dots,K_n^m)=R(m;n-1)$ and (2), we obtain (3). □

Note that (3) and the following facts:

when $n=2$ (see [3]),

$$N(K_1^2)+N(K_2^2)=(1/2)p(p-1)>2p=N(K_1^1)+N(K_2^1)\quad \text{if } p\geqslant R(3,3)=6,$$

$$N(K_1^2)+N(K_2^2)\leqslant N(K_1^1)+N(K_2^1)\quad \text{if } p<R(3,3)=6,$$

$$N(K_1^3)+N(K_2^3)=\tbinom{p}{3}-(1/2)\sum_{i=1}^{n}d_{1i}d_{2i}\geqslant(1/6)p(p-1)(p-2)$$

$$-(1/8)p(p-1)^2>(1/2)p(p-1)=N(K_1^2)+N(K_2^2)\quad \text{if } p\geqslant R(4,4)=18,$$

$$N(K_1^3)+N(K_2^3)\leqslant N(K_1^2)+N(K_2^2)\quad \text{if } p<R(4,4)=18,$$

where d_{ij} is the degree of vertex v_j in $K_p[E_i]$.

when $n=3$,

$$2\sum_{i=1}^{3}N(K_i^2)=p(p-1)>15p=5\sum_{i=1}^{3}N(K_i^1)\quad \text{if } p\geqslant R(3,3,3)=17,$$

(see [2]).

$$2\sum_{i=1}^{3}N(K_i^2)\leqslant 5\sum_{i=1}^{3}N(K_i^1)\quad \text{if } p<R(3,3,3)=17.$$

So we raise a conjecture which generalizes Conjecture 2 in [3] as follows:

Conjecture. *If* $m(>3)$, $n(>1)$ *and* p *are natural numbers,* $p \geqslant R(m;n)$, *then*

$$(m-1)\sum_{i=1}^{n} N(K_i^{m-1}) > [R(m;n-1)-1]\sum_{i=1}^{n} N(K_i^{m-2}).$$

Theorem 2. *For any graph* G_i *of order* $m_i(>1)$ *and* $G_i = \{v_i\} \circ G_i^{m_i-1}$ *for* $i=1,2,\ldots,n$

$$R(G_1,G_2,\ldots,G_n) \leqslant \sum_{i=1}^{n} R(G_1,\ldots,G_{i-1},G_i^{m_i-1},G_{i+1},\ldots,G_n) - n + 2. \qquad (4)$$

Proof. Using Theorem 1 for $n_i = 1$, $i = 1,2,\ldots,n$ and $p = R(G_1,G_2,\ldots,G_n) - 1$, we have

$$2N(K_i^2) \leqslant p[R(G_1,\ldots,G_{i-1},G_i^{m_i-1},G_{i+1},\ldots,G_n) - 1] \text{ for } i = 1,2,\ldots,n.$$

Then $p(p-1) = 2\binom{p}{2} = 2\sum_{i=1}^{n} N(K_i^2) \leqslant p[\sum_{i=1}^{n} R(G_1,\ldots,G_{i-1},G_i^{m_i-1},G_{i+1},\ldots,G_n) - n]$.

Thus we obtain (4). □

Theorem 2 is a generalization of Theorem 2 in [4] and of the classical inquality $R(m_1,m_2,\ldots,m_n) \leqslant \sum_{i=1}^{n} R(m_1,\ldots,m_{i-1},m_i-1,m_{i+1},\ldots,m_n) - n + 2$.

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