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Vertex-pancyclic Multipartite Tournaments *

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Abstract : A c-partite tournament is an oriented graph obtained from a complete c-partite graph. A multipartite tournament is a c-partite tournament with $c \geq 2$. T being a multipartite tournament , we define $i_g(T) = \max |d^+(x) - d^-(y)|$ over all pairs of vertices $x, y \in V(T)$. We prove that if $V_1, V_2, \cdots, V_c$ are the partite sets of a c-partite $(c \geq 3)$ tournament T, with $|V_1| \leq |V_2| \leq \cdots \leq |V_1| + 1$ and $i_g(T) \leq 1$, then T is vertex-pancyclic.

Key words : multipartite tournaments , cycle ,vertex-pancyclicity

A c-partite tournament is an oriented graph obtained from a complete c-partite graph. A multipartite tournament is a c-partite tournament with $c \geq 2$. If T is a multipartite tournament and $x \in V(T)$, we denote $V(x)$ the partite set to which x belongs and denote $v_T^* = \min_i \{|V_i|\}$, where V_i are partite sets of T. A factor in a digraph is a spanning collection of vertex disjoint cycles. A digraph D is pancyclic if it contains cycles of lengths $3, 4, \cdots, |V(D)|$ and D is vertex-pancyclic if for each $w \in V(D)$ there are cycles of lengths $3, 4, \cdots, |V(D)|$ containing w. The local irregularity of a digraph D is defined as $i_l(D) = \max |d^+(x) - d^-(x)|$ over all vertices $x \in V(D)$ and the the global irregularity is defined as $i_g(D) = \max |d^+(x) - d^-(y)|$ over all pairs of vertices $x, y \in V(D)$. A digraph D is strong if for each $x, y \in V(D)$, there is a path from x to y. A digraph D is k-strong if D-X is strong for all sets of vertices X, $|X| < k$ and $X \subseteq V(D)$.

It is conjectured that all regular c-partite tournaments with $c \geq 4$ are pancyclic. In fact , Yeo [1] proves that when $c \geq 5$, all regular c-partite tournaments are vertex-pancyclic. Our main results are based on the technics of [1]. As for surveys on multipartite tournaments , see [2] and [1] , * *.

1　Terminology and notations

We shall assume that the reader is familiar with the standard terminology on graphs and

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digraphs and refer the reader to [3].

Let $D=(V,A)$ be a digraph. If $xy\in A(D)$, then we say that x dominates y and y is dominated by x. We also denote this by $x\to y$. If $X, Y\subseteq V(D)$ and there is no arc from Y to X, then we say that $X\Rightarrow Y$. For a vertex $x\in V(D)$, the in-degree $d^-(x)$ (out-degree $d^+(x)$) of x is the number of vertices dominating x (dominated by x) in D. Furthermore, we use $\triangle^+(D)$ and $\delta^+(D)$ ($\triangle^-(D)$ and $\delta^-(D)$) to denote the maximum and minimum out-degree (in-degree) in D respectively. If Q is a subgraph in D, then $D\langle Q\rangle$ is the subgraph induced by $V(Q)$.

Let D be a digraph and let $\{x,y\}\subseteq V(D)$. We will use the following definitions $\sigma(x,D)=d^+(x)-d^-(x)$ and $\sigma(x,y,D)=\sigma(x,D)-\sigma(y,D)$. Note that $i_l(D)=\max\{|\sigma(x,D)| : x\in V(D)\}$.

Let D be digraph and C be a cycle in D. If $u,v\in V(C)$, we denote $C[u,v]$ the directed path from u to v on C. If $x\in V(C)$, then x^+ denotes the successor of x on the cycle C. Analogously x^- denotes the predecessor of x on C. We define $x^+=x^{+1}$ and $x^{+\delta}=(x^{+(\delta-1)})^+$ for $\delta\geq 2$. Let $C_1, C_2, \cdots, C_l$ be cycles in D and let $F=C_1\cup C_2\cup\cdots\cup C_l$. If $C_1, C_2, \cdots, C_l$ are pairwise disjoint, then we call that F is a cycle subgraph in D. Let δ be an integer in $\{1,2,\cdots,|V(D)|\}$ and let $\{x,y\}\subseteq V(D)-V(C)$ be arbitrary. A δ-partner of (x,y) on F is a vertex $z\in V(F)$ such that $z\to x$ and $y\to z^{+\delta}$.

A cycle C_0 is k-reducible if there are cycles $C_1, C_2, \cdots C_k$ such that $C_{i+1}=C_i[w_i^+, w_i^-]w_i^+$ for all $i=0,1,\cdots,k-1$, where $w_i\in V(C_i)$. If $w\in V(D)$, then a cycle C_0 is (w,k)-reducible if it is k-reducible, and w belongs to all the cycles $C_1, C_2, \cdots, C_k$.

2 Lemmas and main results

Main Theorem Any c-partite ($c\geq 13$) tournament T with the partite sets $V_1, V_2, \cdots, V_c$ such that $|V_1|\leq|V_2|\leq\cdots\leq|V_c|\leq|V_1|+1$ and $i_g(T)\leq 1$ is vertex-pancyclic.

In order to prove this theorem, we need the following Lemmas:

Lemma 1 Let T be a c-partite tournament with partite sets $V_1, V_2, \cdots, V_c$ and $i_g(T)\leq 1$, then

(a) $\triangle^+(T)-\delta^+(T)\leq 2$.

(b) If $\triangle^+(T)-\delta^+(T)=2$, then $d^-(x)=\delta^+(T)+1$ for each $x\in V(T)$.

(c) $||V_i|-|V_j||\leq 2$, for all $i\neq j$.

(d) If $d^+(x)-d^+(y)=2$, then $d^+(x)=\triangle^+(T)$, $d^+(y)=\delta^+(T)$ and $|V(y)|=|V(x)|+2$.

Proof (a) Suppose there exist $u,v\in V(T)$ such that $d^+(u)-d^+(v)\geq 3$. Since $i_g(T)\leq$

1, we have $d^-(u) \geq d^+(u) - 1 \geq d^+(v) + 3 - 1 = d^+(v) + 2$, a contradiction.

(b) Suppose $\Delta^+(T) - \delta^+(T) = 2$. Let $u, v \in V(T)$ with $d^+(u) - d^+(v) = 2$, where $d^+(u) = \Delta^+(T)$ and $d^+(v) = \delta^+(T)$. Let $z \in V(T)$. If $d^-(z) \geq \delta^+(T) + 2$, then $d^-(z) - d^+(v) \geq 2$, a contradiction; If $d^-(z) \leq \delta^+(T)$, then $d^+(u) - d^-(z) \geq 2$, a contradiction too. So we have $d^-(x) = \delta^+(T) + 1$ for each $x \in V(T)$.

(c) Note that $d^+(x) + d^-(x) = |V(T)| - |V(x)|$ for each $x \in V(T)$. Let $x, y \in V(T)$ so that $V(x) = V_i$ and $V(y) = V_j$. Then $||V_i| - |V_j|| = ||V(x)| - |V(y)|| = |d^+(x) - d^+(y) + d^-(x) - d^-(y)| \leq |d^+(x) - d^-(y)| + |d^-(x) - d^+(y)| \leq 2$.

(d) By (a), it is easy to see that $d^+(x) = \Delta^+(T)$ and $d^+(y) = \delta^+(T)$. By (b), we know that $d^-(x) = d^-(y)$. Hence we have $|V(y)| - |V(x)| = (|V(T)| - d^+(y) - d^-(y)) - (|V(T)| - d^+(x) - d^-(x)) = d^+(x) - d^+(y) = 2$.

Corollary 2　Let T be a c-partite tournament with the partite sets $V_1, V_2, \cdots, V_c$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c| \leq |V_1| + 1$ and $i_g(T) \leq 1$, then $\Delta^+(T) - \delta^+(T) \leq 1$.

Lemma 3　Let T be a c-partite tournament of order p with the partite sets $V_1, V_2, \cdots, V_c$ and $i_g(T) \leq 1$. Let $r = \max_{1 \leq i \leq c}\{|V_i|\}$, then the connectivity κ of T satisfies: $\kappa(T) \geq (p - 2r)/3$.

Proof　Let S be any vertex set of T such that $T - S$ is not strong. Let $T_1, T_2, \cdots, T_l$ be the strong components of $T - S$, then there are T_i, T_j such that $N^+(V(T_i)) \subseteq S$ and $N^-(V(T_j)) \subseteq S$. Suppose, without loss of generality, that $|V(T_j)| \leq |V(T_i)|$ and $j = 1$, then $|V(T_1)| \leq (p - |S|)/2$. Since $i_g(T) \leq 1$, we have $\delta^-(T) \geq (p - r - 1)/2$. On the other hand, it is easy to verify that $\delta^-(T_1) \leq (|V(T_1)| - 1)/2$. Hence we have

$$(p - r - 1)/2 \leq \delta^-(T_1) + |S| \leq (|V(T_1)| - 1)/2 + |S| \leq (p + 3|S| - 2)/4.$$

Which yields $\kappa(T) = |S| \geq (p - 2r)/3$.

Lemma 4　([5]) Let T be a c-partite tournament with the partite sets $V_1, V_2, \cdots, V_c$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c| \leq |V_1| + 1$. If $i_l(T) \leq (|V(T)|) - |V_{c-1}| - 2|V_c| + 2)/2$, then T is Hamiltonian.

Lemma 5　Let T be a c-partite tournament with the partite sets $V_1, V_2, \cdots, V_c$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c| \leq |V_1| + 1$ and $i_g(T) \leq 1$. Then for any $\omega \in V(T)$, there exists a cycle C of length p in T containing ω, for all integers p with $|V(T)| \frac{2c-2}{3c-5} + \frac{4c}{3c-5} \leq p \leq |V(T)|$.

Proof　Let $n = |V(T)|$ and let T have partite sets $V_1, V_2, \cdots, V_c$ with $|V_1| = |V_2| = \cdots = |V_s| = |V_{s+1}| - 1 = \cdots = |V_c| - 1$, where $1 \leq s \leq c$.

Assume first, that $\omega \in V_1$. Let p be an integer with $n \frac{2c-2}{3c-5} + \frac{4c}{3c-5} \leq p \leq n$.

Let $k = [\frac{p}{c}]$ and $p = kc + r$, $0 \le r < c$. Let $V_i' \subseteq V_i$ such that $|V_i'| = k + 1$ for $i = c, c - 1, \cdots, c - r + 1$ and $|V_i'| = k$ for $i = 1, 2, \cdots, c - r$, and such that $\omega \in V_1'$.

Since

$$n\frac{2c-2}{3c-5} + \frac{4c}{3c-5} \le p$$

Substract $p\frac{2c-2}{3c-5}$ from both sides, we have

$$(n-p)\frac{2c-2}{3c-5} + \frac{4c}{3c-5} \le p\left(1 - \frac{2c-2}{3c-5}\right)$$

Multiply both sides with $\frac{3c-5}{2c}$, we have

$$(n-p)\frac{c-1}{c} + 2 \le p\frac{c-3}{2c}$$

As $p = kc + r$, we get that

$$n - p - \frac{n-p}{c} + 2 \le \frac{p-3k}{2} - \frac{3r}{2c}$$

Since $\frac{n-p}{c} - 1 < [\frac{n-p}{c}]$ and $\frac{3r}{2c} \ge 0$,

$$n - p - [\frac{n-p}{c}] + 1 < \frac{p-3k}{2}$$

Since n, p, k and $[\frac{n-p}{c}]$ are integers,

$$n - p - [\frac{n-p}{c}] + 1 \le \frac{p-3k-1}{2}$$

Let $T' = T\langle \cup_{i=1}^{c} V_i' \rangle$ and note that we have deleted at least $[(n-p)/c]$ vertices from each partite set. This implies that $i_l(T') \le n - p - [(n-p)/c] + 1 \le \frac{p-3(k+1)+2}{2}$. So, by Lemma 4, T' has a Hamiltonian cycle, which corresponds to a p-cycle in T containing ω.

The case that $\omega \in V_{s+1} \cup V_{s+2} \cup \cdots \cup V_c$ can be proved by the similar argument as above.

Lemma 6 Let T be a c-partite $(c \ge 5)$ tournament of order n with the partite sets V_1, V_2, $\cdots$, V_c such that $|V_1| \le |V_2| \le \cdots \le |V_c| \le |V_1| + 1$ and $i_g(T) \le 1$. Then for each pair x, y in $V(T)$, there is a path of length at most 3 form x to y.

Proof Suppose on the contrary that there are $x, y \in V(T)$ such that there is no (x, y)-path of length at most 3 in T. Thus $(N^-(y) \cup \{y\}) \Rightarrow (N^+(x) \cup \{x\})$ and $(N^-(y) \cup \{y\}) \cap (N^+(x) \cup \{x\}) = \emptyset$. This implies that $S = V(T) - N^+(x) - N^-(y) - \{x, y\}$ is a separating set in T. Thus, $|S| \le \frac{|V(x)| + |V(y)|}{2} - 2 \le |V_c| - 2 < \frac{n - 2|V_c|}{3}$ since $i_g(T) \le 1$ and $c \ge 5$. This contradicts Lemma 3.

Lemma 7 ([5]) Let T be a c-partite tournament. Let $F = C_1 \cup C_2 \cup \cdots \cup C_l$ be a factor in

T with the minimum number of cycles. Then there exists a partite set Q and an ordering of the cycles $C_1, C_2, \cdots, C_l$ in F such that $\{x^+, y^-\} \subseteq Q$ for each arc xy with $x \in V(C_j)$, $y \in V(C_1)$ $(j > 1)$.

Lemma 8 Let T be a c-partite tournament with the partite sets $V_1, V_2, \cdots, V_c$ such that $|V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c| \leqslant |V_1| + 1$ and $i_g(T) \leqslant 1$. Let $\omega \in V(T)$ be arbitrary. If $F = C_1 \cup C_2 \cup \cdots \cup C_l$ is a cycle subgraph in T with $\omega \in V(F)$, then there is a cycle C in T with $|V(C)| = |V(F)|$ and $\omega \in V(C)$.

Proof Let $F' = C_1' \cup C_2' \cup \cdots \cup C_m'$ be a cycle subgraph with $\omega \in V(F')$, and assume that m is as small as possible. If $m = 1$, we are done. So we assume that $m \geq 2$. By examining $T' = T\langle V(F')\rangle$ there exists a partite set Q such that the conditions of Lemma 7 hold. This implies that $\{x^+, y^-\} \subseteq Q$ for each arc xy with $x \in V(C_j')$, $y \in V(C_1')$ and $j > 1$. Assume, without loss of generality, that $\omega \in V(F') - V(C_1')$, otherwise we can consider the reverse digraph of T'.

Let $R = C_2' \cup C_3' \cup \cdots \cup C_m'$ and let $P = p_1 p_2 \cdots p_k$ be the shortest possible path from R to C_1'. Assume that $p_1 \in V(C_j')$, $j \in \{2, 3, \cdots, m\}$. By Lemma 6 we have that $2 \leq k \leq 4$. We now show Claim and three cases below.

Claim If $z \in V(C_1')$, $v \in V(R)$, $z \Rightarrow V(R)$, $V(C_1') \Rightarrow v$ and $V(z) = V(v)$, then there is a vertex u such that $v \to u \to z$. And if u is unique, then we have $(V(C_1') - \{z, z^+\}) \Rightarrow z$ and $v \Rightarrow (V(C_j') - \{v, v^-\})$.

In fact, suppose that $z \Rightarrow N^+(v)$, then we have $d^+(z) \geq d^+(v) + |\{z^+, v^-\}| = d^+(v) + 2$, which contradicts Corollary 2. And if u is unique, we can analogously check that $(V(C_1') - \{z, z^+\}) \Rightarrow z$ and $v \Rightarrow (V(C_j') - \{v, v^-\})$, otherwise we have $d^+(z) \geq d^+(v) + 2$, a contradiction.

Case 1 $k = 2$.

By applying Lemma 7 repeatedly, we have $\{p_1^+, p_2^-\} \subseteq Q$, $p_2^- \Rightarrow V(R)$ and $V(C_1') \Rightarrow p_1^+$. By Claim there is $z \in V(T) - V(F')$ such that $p_1^+ \to z \to p_2^-$.

If $p_2^{-3} \in Q$, then by Lemma 7 we have $p_2^{-3} \to p_1^{++}$. So the cycle subgraph $F'' = C_1'[p_2^-, p_2^{-3}] C_j'[p_1^{++}, p_1^+] z p_2^- \cup (F' - C_1' - C_j')$ has $|V(F'')| = |V(F')| = |V(F)|$ and as $p_2^{--} \neq w$, we have $\omega \in V(F'')$. Therefore F'' is a contradiction against the minimality of m. If $p_2^{-3} \notin Q$, then the cycle subgraph $F'' = C_1'[p_2, p_2^{-3}] p_1^+ z p_2^- C_j'[p_1^{++}, p_1] p_2 \cup (F' - C_1' - C_j')$ is also a contradiction against the minimality of m.

Case 2 $k = 3$.

By the minimality of k we must have $V(C_1') \Rightarrow V(R)$.

Subcase 2.1 $p_1^{++} = \omega$ and $V(p_1^{++}) \neq V(p_3^-)$.

The cycle subgraph $F'' = C_1'[p_3, p_3^-]C_j'[p_1^{++}, p_1]p_2p_3 \cup (F' - C_1' - C_j')$ has $\omega \in V(F'')$ since $p_1^+ \neq w$, which is a contradiction to the minimality of m.

Subcase 2.2 $p_1^{++} = w$, $V(p_1^{++}) = V(p_3^-)$ and $V(p_1^{+3}) \neq V(p_3^{-3})$.

By Claim there is a vertex $z \in V(T) - V(F')$ such that $p_1^{++} \to z \to p_3^-$. The cycle subgraph $F'' = C_1'[p_3^-, p_3^{-3}]C_j'[p_1^{+3}, p_1^{++}]zp_3^- \cup (F' - C_1' - C_j')$ has $w \in V(F'')$ since $p_3^{--} \neq w$ and $w \notin V(C_1')$.

Subcase 2.3 $p_1^{++} = w$, $V(p_1^{++}) = V(p_3^-)$ and $V(p_1^{+3}) = V(p_3^{-3})$.

If there is a vertex $z \in V(T) - V(F') - \{p_2\}$ such that $p_1^{++} \to z \to p_3^-$, then $F'' = C_1'[p_3, p_3^{-3}]p_1^{++}zp_3^- C_j'[p_1^{+3}, p1]p_2p_3 \cup (F' - C_1' - C_j')$ is a contradiction to the minimality of m. If there is no such z, then by claim we have $V(C_1^1) - \{p_3^-\} \Rightarrow p_3^-$, so $F'' = C_j'[p_1^+, p_1]p_2C_1'[p_3p_3^{-3}]p_3^- p_1^+ \cup (F' - C_1' - C_j')$ is a contradicfiom to the minimality of m.

Subcase 2.4 $p_1^{++} \neq w$ and $V(p_1^+) \neq V(p_3^{--})$.

The cycle subgraph $F'' = C_1'[p_3, p_3^{--}]C_j'[p_1^+, p_1]p_2p_3 \cup (F' - C_1' - C_j')$ is a contradiction to the minimality of m.

Subcase 2.5 $p_1^{++} \neq w$, $V(p_1^+) = V(p_3^{--})$ and $V(p_1^{+3}) \neq V(p_3^{-3})$.

By claim there is a vertex $z \in V(T) - V(F')$ with $p_1^+ \to z \to p_3^{--}$. So $F'' = C_1'[p_3^{--}, p_3^{-3}]C_j'[p_1^{+3}, p_1^+]zp_3^{--} \cup (F' - C_1' - C_j')$ is a contradiction to the minimality of m.

Subcase 2.6 $p_1^{++} \neq w$, $V(p_1^+) = V(p_3^{--})$ and $V(p_1^{+3}) = V(p_3^{-3})$.

If there is vertex $z \in V(T) - V(F') - \{p_2\}$ such that $p_1^+ \to z \to p_3^{--}$, then $F'' = C_1'[p_3, p_3^{-3}]p_1^+ zp_3^{--} C_j'[p_1^{+3}, p_1]p_2p_3 \cup (F' - C_1' - C_j')$ is a contradiction to the minimality of m. If no such vertex z exists, then we have $p_3^- \to p_1^+$ and $p_1^+ \to p_1^{++}$ by lemma 7, so $F'' = C_1'[p_3, p_3^-]C_j'[p_1^{+3}, p_1]p_2p_3 \cup (F' - C_1' - C_j')$ is contradiction to the minimality of m too.

Case 3. $k = 4$

By the minimality of k we have $V(C_1') \Rightarrow V(R)$. Furthermore, if $V(p_4^{-3}) = V(p_1^+)$, then Claim gives us a contradiction against the minimality of k. So we have $V(p_4^{-3}) \neq V(p_1^+)$. Now $F'' = C_1'[p_4, p_4^{-3}]C_j'[p_1^+, p_1]p_2p_3p_4 \cup (F' - C_1' - C_j')$ is a contradiction to the minimality of m.

Lemma 9 Let T be a c-partite ($c \geq 4$) tournament with the partite sets $V_1, V_2, \cdots, V_c$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c| \leq |V_1| + 1$ and $i_g(T) \leq 1$. Let $\delta \in \{1, 2, 3\}$ and let $F = C_1 \cup C_2 \cup \cdots \cup C_l$ be a cycle subgraph in T. Let $T' = T - V(F)$ and let $\{x, y\} \subseteq V(T')$.

Then there exists at least $(\sigma(x,y,T') - |V(x)\cap V(F)| - |V(y)\cap V(F)| - 1)/2$ distinct δ-partners of (x,y) in F.

Proof Define

$$A_1 = \{z \in V(F) \mid z \to x, y \to z^{(+\delta)}\}$$
$$A_2 = \{z \in V(F) \mid z \to x, z^{(+\delta)} \to y\}$$
$$A_3 = \{z \in V(F) \mid x \to z, y \to z^{(+\delta)}\}$$
$$A_4 = \{z \in V(F) \mid x \to z, z^{(+\delta)} \to y\}$$
$$A_5 = \{z \in V(F) \mid z \in V(x), y \to z^{(+\delta)}\}$$
$$A_6 = \{z \in V(F) \mid z \in V(x), z^{(+\delta)} \to y\}$$
$$A_7 = \{z \in V(F) \mid z \to x, z^{(+\delta)} \in V(y)\}$$
$$A_8 = \{z \in V(F) \mid x \to z, z^{(+\delta)} \in V(y)\}$$
$$A_9 = \{z \in V(F) \mid z \in V(x), y^{(+\delta)} \in V(y)\}.$$

Note that $\sigma(x,T) = \sigma(x,T') + |A_3| + |A_4| + |A_8| - |A_1| - |A_2| - |A_7|$ and $\sigma(y,T) = \sigma(y,T') + |A_1| + |A_3| + |A_5| - |A_2| - |A_4| - |A_6|$. Thus we have $\sigma(x,y,T) = \sigma(x,y,T') + 2|A_4| + |A_6| + |A_8| - 2|A_1| - |A_5| - |A_7| \leq 1$, which implies that $2|A_1| + 1 \geq \sigma(x,y,T') - |A_5| - |A_7| \geq \sigma(x,y,T') - |V(x)\cap V(F)| - |V(y)\cap V(F)|$.

Lemma 10 Let T be a c-partite $(c \geq 8)$ tournament with the partite sets $V_1, V_2, \cdots, V_c$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c| \leq |V_1| + 1$ and $i_g(T) \leq 1$. Then for each $\omega \in V(D)$ there exists a $(w,2)$-reducible 5-cycle in T.

Proof Let $A = N^+(\omega)$ and $B = N^-(\omega)$. Let $a \in A$ belong to the set A_l if and only if the longest path in $T\langle A\rangle$ which ends in a has length l. Analogously define the sets B_l such that $b \in B_l$ if and only if the longest path in $T\langle B\rangle$ which begins from b has length l. Let $A^* = \bigcup_{l=2}^{|A|} A_l$ and $B^* = \bigcup_{l=2}^{|B|} B_l$.

From the above definition it is obvious that A_0, A_1, B_0, B_1 are all independent sets. Furthermore, we have $A_0 \Rightarrow A_1 \cup A^*$, $A_1 \Rightarrow A^*$ and $B^* \cup B_1 \Rightarrow B_0$, $B^* \Rightarrow B_1$. We now consider the following cases:

Case 1 $A^* \neq \emptyset$

If $B \not\Rightarrow A^*$, then let $a_3 \in A^*$ and $b \in B$ be chosen such that $a_3 \to b$. Let $a_1 a_2 a_3$ be a path of length 2 in $T\langle A\rangle$ ending in a_3 and observe that $C = \omega a_1 a_2 a_3 b w$ is a cycle of length 5. Since $w a_2 a_3 b w$ and $w a_3 b w$ also are cycles, we have our desired cycles in this case.

Therefore assume that $B \Rightarrow A^*$. This implies that $S = V(w) - \{w\}$ is a separating set in T since $A \cup B \cup \{w\} - A^* \Rightarrow A^*$. However, $|S| \leq v_T^* < \frac{n - 2v_T^* - 2}{3} \leq \frac{n - 2r}{3}$ when $c \geq 8$, where $r = \max\{|V_i|\}$, which contradicts Lemma 3.

Case 2 $B^* \neq \emptyset$

This is analogous to case 1.

Case 3 $A^* = \emptyset$ and $B^* = \emptyset$.

In this case we have $c = 4$ since A_0, A_1, B_0, B_1 are all independent sets, which contradicts the initial assumption that $c \geq 7$.

Lemma 11 Let T be a c-partite ($c \geq 13$) tournament with the partite sets $V_1, V_2, \cdots V_c$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c| \leq |V_1| + 1$ and $i_g(T) \leq 1$. Let $w \in V(T)$ be arbitrary. Then for all integers p with $3 \leq p \leq (c-2)v_T^* - 1$, there exists a cycle C in T with $w \in V(C)$ and $|V(C)| = p$.

Proof Let $w \in V(T)$ and let F be a cycle subgraph in T such that $|V(F)| \leq p$, $w \in V(F)$ and $|V(F)| \equiv p \pmod 3$. Such a cycle must exist since by lemma 10 there exists a 3-cycle, a 4-cycle and a 5-cycle all including the vertex w. We choose F such that F contains the maximum number of vertices with the desired properties. If $|V(F)| = p$, then we are done by lemma 8. So we assume that $|V(F)| < p$, which implies that $|V(F)| \leq p - 3 \leq (c-2)v_T^* - 4$.

Let $T' = T - V(F)$ and note that if T' has a strong component with vertices from three of more partite sets, then there must exist a 3-cycle C in T and hence $F \cup C$ will be a contradiction against the maximality of $|V(F)|$, So there are vertices from at most two partite sets in any strong component of T', as $|V(F)| \leq (c-2)v_T^* - 4$, there are vertices from at least three partite sets in T', which implies that T' is not strong. Let $Q_1, Q_2, \cdots, Q_m$ be the strong components of T' such that $Q_i \Rightarrow Q_j$ for all $1 \leq i < j \leq m$.

Let $x \in Q_1$ be chosen such that $\sigma(x, T\langle Q_1\rangle) \geq 0$ and let $y \in Q_m$ be chosen such that $\sigma(y, T\langle Q_m\rangle) \leq 0$. Since $|V(T')| \geq 2v_T^* + 4 > 2(v_T^* + 1) + 1$, there is a vertex $z \in V(T') - V(x) - V(y)$. If $z \in Q_1$, then as Q_1 is strong and contains vertices from only two partite sets, there must be a vertex $z' \in V(z)$ such that $x \to z'$. This implies that $x \to z' \to y$ is a (x, y)-path of length 2 in T'. Analogously if $Z \in Q_m$, we can also obtain a (x, y)-path of length 2 in T' Finally if $z \notin Q_1 \cup Q_m$, then $x \to z \to y$ is a (x, y)-path of length 2 in T'. Hence there exists a (x, y)-path of length 2 in T', which we shall denote by R.

Since $\sigma(x, T') = \sigma(x, T\langle Q_1\rangle) + |V(T') - V(x) - Q_1|$ and $\sigma(y, T') = \sigma(y, T\langle Q_m\rangle) - |V(T') - V(y) - Q_m|$, we have that

$$\sigma(x, y, T') \geq |V(T') - V(x) - Q_1| + |V(T') - V(y) - Q_m| \geq$$

$$|V(T')| - |Q_1| - |V(x) \cap V(T')| + |V(T')| - |Q_m| - |V(y) \cap V(T')| \geq$$

$$|V(T')| - |V(x) \cap V(T')| - |V(y) \cap V(T')|.$$

By Lemma 9, $\{x, y\}$ has at least $(|V(T)| - 2(v_T^* + 1) - 1)/2 > 0$ 1-partners in F. Let z be any 1-partner in F and let $C \in F$ be the cycle with $z \in V(C)$. Now $(F - C) \cup C[z^+, z]R$ is a contradiction against the maximality of $|V(F)|$.

Proof of Main Theorem By lemma 5 and lemma 11, it is sufficient to show that $n(2c-2)/(3c-5) + 2c/(3c-5) + 1 \leq v_T^*(c-2)$. It holds when $c \geq 13$.

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多部竞赛图的点泛圈性

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摘　要: 把 c-部完全图的每条边任意加上一个方向后得到的定向图称为 c-部竞赛图,设 T 为 c-部竞赛图,定义 $i_g(T) = \max_{x,y \in V(T)} |d^+(x) - d^-(y)|$. 给出了 c-部竞赛图具有点泛圈性的一个充分条件,即:设 T 为 c-部竞赛图($c \geq 13$), $V_1, V_2, \cdots V_c$ 为 T 的各分部. 如果 $|V_1| \leq |V_2| \leq \cdots \leq |V_c| \leq |V_1| + 1$ 并且 $i_g(T) \leq 1$, 那么 T 具有点泛圈性.

关键词: 多部竞赛图,圈,点泛圈

中图分类号: O157.5