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A SEVEN-COLOR THEOREM ON EDGE-FACE COLORING OF PLANE GRAPHS [1]

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Abstract Melnikov(1975) conjectured that the edges and faces of a plane graph G can be colored with $\Delta(G)+3$ colors so that any two adjacent or incident elements receive distinct colors, where $\Delta(G)$ denotes the maximum degree of G. This paper proves the conjecture for the case $\Delta(G) \le 4$.

Key words Plane graph, chromatic number, coloring

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1 Introduction

Let G be a plane graph with the vertex set $V(G)$, the edge set $E(G)$, the face set $F(G)$, and the maximum degree $\Delta(G)$. The edge-face chromatic number $\chi_{ef}(G)$ of G is the minimum number of colors assigned to $E(G) \cup F(G)$ such that any two adjacent or incident elements have different colors. By the definition, $\chi_{ef}(G) \ge \Delta(G)$ is trivial. In 1975, Melnikov[4] raised the following conjecture.

Conjecture 1.1 For every plane graph G, $\chi_{ef}(G) \le \Delta(G)+3$.

The conjecture has been confirmed for the case $\Delta(G) \le 3$[3,5] and for the case $\Delta(G) \ge 8$[2]. In particular, Borodin[2] proved that every plane graph G with $\Delta(G) \ge 10$ is $(\Delta(G)+1)$-edge-face colorable. The purpose of this paper is to settle the case $\Delta(G)=4$.

We shall use the standard terms and symbols in [1] except the following notations defined. Let $p(G)$ and $q(G)$ denote the vertex number and the edge number of a plane graph G, respectively. Let $V_k(G)$ denote a set of all the vertices in G of degree k, $k=0,1,\cdots,\Delta(G)$. For $f \in F(G)$, we denote the boundary of f by $b(f)$. The unique outer face of G is written as $f_{\text{out}}(G)$. Let $\sigma(y)$ denote the color assigned to $y \in E(G) \cup F(G)$ in a given edge-face coloring σ. For $u \in V(G)$, let $C_\sigma(u)$ denote the set of colors which are assigned to the edges incident to u in σ.

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2 Preliminary

Let G be a plane graph and u a cut vertex of G. Then u is said to be splitable if there are plane graphs G_1 and G_2 such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{u\}$, and $\min\{d_{G_1}(u), d_{G_2}(u)\} \le 2$.

Lemma 2.1 Let G be a plane graph and u a splitable cut vertex of G. Then

$$\chi_{ef}(G) \le \max\{\chi_{ef}(G_1), \chi_{ef}(G_2), d_G(u)+1\}.$$

Proof Since u is a splitable cut vertex of G, there are plane graphs G_1 and G_2 such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{u\}$, and $\min\{d_{G_1}(u), d_{G_2}(u)\} \le 2$. Suppose that $d_{G_1}(u) \le 2$. Obviously, $d_G(u) = d_{G_1}(u) + d_{G_2}(u)$. Moreover, we may suppose that u lies on the outer face of G, and hence on the outer faces of G_1 and G_2 at the same time. Let $k = \max\{\chi_{ef}(G_1), \chi_{ef}(G_2), d_G(u)+1\}$. Thus G_i has a k-edge-face coloring σ_i with a color set C_i for $i = 1, 2$. By $d_G(u)+1 \le k$ and $d_{G_1}(u) \le 2$, we can choose σ_1 and σ_2 so that (a) $C_1 = C_2$; (b) $\sigma_1(f_{\rm out}(G_1)) = \sigma_2(f_{\rm out}(G_2))$; and (c) $C_{\sigma_1}(u) \cap C_{\sigma_2}(u) = \emptyset$. Therefore, combining σ_1 with σ_2, we obtain a k-edge-face coloring σ of G with the color set C_1. It follows that $\chi_{ef}(G) \le k$.

Lemma 2.2 If G is a plane graph with $\Delta(G) \le 5$, then each cut vertex of G is splitable.

Lemma 2.3 Let G be a connected plane graph with cut edges and $p(G) \ge 3$. Then G contains at least one splitable cut vertex.

Lemmas 2.2 and 2.3 are obvious. Let G be a 2-connected plane graph and $\{e_1, e_2\}$ be a 2-edge cut of G. Then both e_1 and e_2 must simultaneously lie on the boundary of some face f_0. We say that $\{e_1, e_2\} \subseteq b(f_0)$ is a special 2-edge cut of G if $G - e_1 - e_2$ is disconnected, and there exists an edge $e_0 \in b(f_0) - \{e_1, e_2\}$ such that e_0 is adjacent to both e_1 and e_2. Let $e_0 = x_1x_2$ with $e_i \cap e_0 = \{x_i\}$, $i = 1, 2$. In fact, $\{x_1, x_2\}$ is a 2-vertex cut of G. Set $G - e_1 - e_2 = H_1 \cup H_2$, where $e_0 \in E(H_2)$, $G_1 = H_1 \cup \{e_0, e_1, e_2\}$, and $G_2 = H_2$. Hence $G = G_1 \cup G_2$, $G_1 \cap G_2 = \{e_0\}$, and $f_0 \in F(G_1)$.

Lemma 2.4 If $\Delta(G) \le 4$ and $\chi_{ef}(G_i) \le 7$ for $i = 1, 2$, then $\chi_{ef}(G) \le 7$.

Proof Let f_1 denote the face of G separated by e_1 and e_2 and $f_1 \ne f_0$, and f_2 the face of G with e_0 as a boundary edge and $f_2 \ne f_0$. First, by $\chi_{ef}(G_2) \le 7$, G_2 has a 7-edge-face coloring σ_2 with a color set C. It is easy to see that, for any 7-edge-face coloring σ_1 of G_1, three colors $\sigma_1(f_0)$, $\sigma_1(e_0)$ and $\sigma_1(f_1)$ must be pairwise distinct, and they can be prescribed any values. We now fix $\sigma_1(e_0) = \sigma_2(e_0)$, and $\sigma_1(f_1) = \sigma_2(f_1)$. If e_1 and e_2 are required to color different colors in σ_1, we put $\sigma_1(e_1) \in C - (C_{\sigma_2}(x_1) \cup \{\sigma_1(f_1)\})$, and $\sigma_1(e_2) \in C - (C_{\sigma_2}(x_2) \cup \{\sigma_1(f_1), \sigma_1(e_1)\})$. Otherwise we let

$$\sigma_1(e_1) = \sigma_1(e_2) \in C - (C_{\sigma_2}(x_1) \cup C_{\sigma_2}(x_2) \cup \{\sigma_1(f_1)\}).$$

Afterwards, $\sigma_1(f_0) \in C - \{\sigma_1(e_1), \sigma_1(e_2), \sigma_1(f_1), \sigma_1(e_0), \sigma_2(f_2)\}$. Noting that $d_{G_1}(x_i) = 2$ and $d_{G_2}(x_i) = d_G(x_i) - 1 \le 3$, we have $|C_{\sigma_2}(x_i)| \le 3, i = 1, 2$, $|C_{\sigma_2}(x_1) \cup C_{\sigma_2}(x_2)| \le 5$. Since $|C| = 7$, the above coloring is available. Combining σ_1 with σ_2, we obtain a 7-edge-face coloring of G.

Suppose that $\{e_1, e_2\} \subseteq b(f_0)$ is a non-special 2-edge cut of a 2-connected plane graph G. Thus $d_G(f_0) \ge 6$. Let $e_1 = x_1y_1$, $e_2 = x_2y_2$, and $G - e_1 - e_2 = H_1 \cup H_2$ with $x_1, x_2 \in V(H_1)$, $y_1, y_2 \in V(H_2)$. Then clearly $x_1x_2, y_1y_2 \notin E(G)$. Set $\overline{H}_1 = H_1 + x_1x_2$ and $\overline{H}_2 = H_2 + y_1y_2$.

So $\overline{H}_i$ is a 2-connected plane graph with $q(\overline{H}_i) < q(G)$, $i = 1, 2$. The following two lemmas are straightforward.

Lemma 2.5 If $\Delta(G) \leq 4$ and $\chi_{ef}(\overline{H}_i) \leq 7$, $i = 1, 2$, then $\chi_{ef}(G) \leq 7$.

Lemma 2.6 Let G be a plane multigraph with $\Delta(G) \geq 3$, and $\widetilde{G}$ a subdivision of G. If G is $(\Delta(G) + 3)$-edge-face colorable, then so is $\widetilde{G}$.

Let G be a connected plane graph and C_k denote a cycle in G of length k. Then the rest of G is partitioned into two edge-disjoint parts, where the part which contains the outer face of G is called the exterior of C_k and the other the interior. Let $V_{\text{int}}(C_k)$ and $V_{\text{ext}}(C_k)$ denote the sets of vertices which are contained in the interior and the exterior of C_k, respectively. If $V_{\text{int}}(C_k) \neq \emptyset$ and $V_{\text{ext}}(C_k) \neq \emptyset$, we say that C_k is a k-separating cycle of G. In particular, C_3 is called a separating triangle. We claim that each 2-separating cycle $C_2 = x_1e_1x_2e_2x_1$ will yield a 2-edge cut of G when G is 2-connected and $\Delta(G) = 4$. Note that e_1 and e_2 are, in fact, two multiple edges between x_1 and x_2. Let $G_1 = G[V_{\text{int}}(C_2) \cup \{x_1, x_2\}]$. Thus $d_{G_1}(x_1) = d_{G_1}(x_2) = 3$ and $d_G(x_1) = d_G(x_2) = 4$. Let $e_i^0 \in E(G) - E(G_1)$ be incident to x_i in G, $i = 1, 2$. Then $\{e_1^0, e_2^0\}$ is a 2-edge cut of G.

For $u \in V(G)$ with $d_G(u) \geq 3$, we set $\overline{N}_G(u) = N_G(u) \cup \{u\}$. If $G[\overline{N}_G(u)]$ contains no a separating triangle passing through u, then u is said to be a regular vertex of G. Clearly, if u is a regular vertex of G, then, for any $v, w \in N_G(u)$, $vw \in E(G)$ if and only if $uvw \in F(G)$. Similarly, let f be a face of G and $f_1, f_2, \cdots, f_m$ be its neighbour faces in the clockwise order, here $m = d_G(f) \geq 3$. If, for any $i \neq j$, f_i is adjacent to f_j if and only if $|i - j| = 1 (\text{mod m})$ and the common boundary vertex of f_i, f_j, and f is of degree 3, then f is called a regular face of G.

Lemma 2.7 If G is a 2-connected simple plane graph with $\delta(G) \geq 3$, then G contains a regular vertex of degree no more than 5.

Proof If G contains no separating triangle, then the theorem is obvious. Thus suppose that G contains separating triangles. Choose a separating triangle $T = xyz$ with as few internal vertices as possible. Consider the graph $H = G[V_{\text{int}}(T) \cup \{x, y, z\}]$. Since G is 2-connected, so is H. This implies that $\delta(G) \geq 2$. Moreover, $V_2(H) \subseteq \{x, y, z\}$ and $|V_2(H)| \leq 1$ by $\delta(G) \geq 3$. Now let us estimate the number of vertices in H of degree at most 5. Let $p_i = |V_i(H)|$. By

$$\sum_{u \in V(H)} d_H(u) = 2q(H) \leq 6p(H) - 12,$$

we have

$$\sum_{i=2}^{\Delta(H)} ip_i \leq 6p(H) - 12 = 6 \sum_{i=2}^{\Delta(H)} p_i - 12.$$

Equivalently,

$$4p_2 + 3p_3 + 2p_4 + p_5 \geq 12.$$

If $p_2 = 0$, then $3p_3 + 2p_4 + p_5 \geq 12$, and it follows easily that $p_3 + p_4 + p_5 \geq 4$. If $p_2 = 1$, then $3p_3 + 2p_4 + p_5 \geq 8$ and further we have $p_3 + p_4 + p_5 \geq 3$. Thus we obtain in two cases that $p_2 + p_3 + p_4 + p_5 \geq 4$. This implies that there is at least one vertex $u \in V(H) - \{x, y, z\}$ such that $3 \leq d_H(u) = d_G(u) \leq 5$. We claim that u is just a desired vertex to the theorem. In fact, if there exists a separating triangle $T' = uvw$ in $G[\overline{N}_G(u)]$, then obviously $\emptyset \neq V_{\text{int}}(T') \subseteq V_{\text{int}}(T)$.

But it follows from $u \in V_{\text{int}}(T) - V_{\text{int}}(T')$ that $1 \le |V_{\text{int}}(T')| < |V_{\text{int}}(T)|$, which contradicts the choice of T.

Lemma 2.8 If G is a 2-connected and 3-edge connected simple plane graph with $\delta(G) \ge 3$, then G contains a regular face of degree no more than 5.

Proof First note that G^*, the dual of G, is a 2-connected simple plane graph with $\delta(G^*) \ge 3$. Thus G^* contains a regular vertex u^* of degree at most 5 by Lemma 2.7. From the one-to-one relation between $F(G)$ and $V(G^*)$, it follows that f, the image of u^*, is a regular face of G with degree no more than 5.

In order to prove the following lemma, we say that a color α is forbidden at the vertex v if any edge incident to v can not get α. Moreover, we write $J_n = \{1, 2, \cdots, n\}$.

Lemma 2.9 Let $C = u_1u_2\cdots u_nu_1$ be a cycle of length $n \ge 3$ and B a set of six colors. Let $B_i \subset B$ denote a subset of all forbidden colors at the vertex u_i and $|B_i| = 2$, $i = 1, 2, \cdots, n$. Then $E(C)$ can be properly colored with the colors in B whatever $B_1, B_2, \cdots, B_n$ are prescribed.

Proof We declare that all suffixes here are taken modulo n. For $i \in J_n$, let $A_{i,i+1} = B - B_i - B_{i+1}$, which is a set of admissible colors of the edge u_iu_{i+1}. In view of $|B| = 6$ and $|B_i| = 2$, we have

$$|A_{i,i+1}| = |B| - |B_i \cup B_{i+1}| \ge |B| - |B_i| - |B_{i+1}| = 2,$$

and $|A_{i,i+1}| = 2$ if and only if $B_i \cap B_{i+1} = \emptyset$.

If there is $j \in J_n$ such that $B_j \cap B_{j+1} \ne \emptyset$, i.e., $|A_{j,j+1}| \ge 3$, we color $u_{j+1}u_{j+2}$, $u_{j+2}u_{j+3}$, $\cdots$, $u_{j-1}u_j$, and u_ju_{j+1}, successively. Otherwise, we have

$$B_i \cap B_{i+1} = \emptyset \quad \text{for all } i \in J_n. \tag{2.1}$$

First suppose $n = 2k + 1 \ge 3$. Let us prove the following inequality

$$|\bigcup_{i=1}^{2k+1} A_{i,i+1}| \ge 3.$$

Suppose that this is not true, we immediately have $A_{1,2} = A_{2,3} = \cdots = A_{2k+1,1}$ and $|A_{i,i+1}| = 2$. Furthermore,

$$B_1 \cup B_2 = B_2 \cup B_3 = \cdots = B_{2k} \cup B_{2k+1} = B_{2k+1} \cup B_1$$

by $B_i \subset B$ for all $i \in J_n$. Since $2k + 1$ is odd, it follows

$$B_1 = B_3 = \cdots = B_{2k-1} = B_{2k+1}, \tag{2.2}$$

or

$$B_2 = B_4 = \cdots = B_{2k} = B_{2k+1}. \tag{2.3}$$

But $B_1 = B_{2k+1}$ in (2.2) and $B_{2k} = B_{2k+1}$ in (2.3) contradict (2.1) because $u_1u_{2k+1}, u_{2k}u_{2k+1} \in E(C)$. Thus there is $j \in J_n$ such that $A_{j-1,j} \ne A_{j,j+1}$. We color u_ju_{j+1} with some color from $A_{j,j+1} - A_{j-1,j}$, then color $u_{j+1}u_{j+2}, u_{j+2}u_{j+3}, \cdots, u_{j-1}u_j$, successively.

Next let $n = 2k \ge 4$. If there is $j \in J_n$ such that $A_{j,j+1} \ne A_{j-1,j}$, we shall have a coloring similarly to the previous case. If $A_{1,2} = A_{2,3} = \cdots = A_{2k-1,2k} = A_{2k,1}$, a proper coloring can be easily given.

3 Seven-Color Theorem

Theorem 3.1 If G is a multiple plane graph with $\Delta(G) = 4$, then $\chi_{ef}(G) \leq 7$.

Proof We proceed by induction on $q(G)$. The theorem holds trivially for $q(G) \leq 4$. Let G be a multiple plane graph with $\Delta(G) = 4$ and with $m(\geq 5)$ edges. By Lemmas 2.2, 2.5 and 2.6, we may suppose that G is 2-connected, without 2-edge cut, without separating 2-cycle, and $\delta(G) \geq 3$.

If G contains two multiple edges e_1 and e_2 between two vertices u and v, then the 2-cycle $C = ue_1ve_2u$ forms a 2-face f_1 of G. By the induction assumption, $G - e_1$ has a 7-edge-face coloring λ. Based on λ, we can color e_1 and f_1, successively.

Now assume that G has no multiple edges. By Lemma 2.8, G contains a regular face f with $3 \leq d_G(f) \leq 5$. We only give a proof for the case $d_G(f) = 5$ since other cases can be similarly handled. Suppose that $b(f) = u_1e_1u_2e_2u_3e_3u_4e_4u_5e_5u_1$, where $e_i = u_iu_{i+1}$ for each $i \in J_5$. Let $F(f) = \{f_1, f_2, f_3, f_4, f_5\}$ denote a set of faces in G each of which is adjacent to f, satisfying $e_i \in b(f_i)$ for $i \in J_5$. Since $3 \leq d_G(u_i) \leq 4$, we only consider the following four cases to form a 7-edge-face coloring σ of G.

Case 0 $|V_3(G) \cap V(b(f))| = 0$.

Since $d_G(u_i) = 4$ for all $i \in J_5$, Lemma 2.8 implies that any two faces of $F(f)$ are nonadjacent. Let $H = G - \{e_1, e_2, e_3, e_4, e_5\}$. Let f_0 denote the face of H with u_1, u_2, u_3, u_4, u_5 as boundary vertices. By the induction assumption, H has a 7-edge-face coloring λ with a color set C. Let $B_i = C_\lambda(u_i)$ for $i \in J_5$, and $B = C - \{\lambda(f_0)\}$. It is clear that $|B| = |C| - 1 = 6$ and $|B_i| = 2$ for all $i \in J_5$. By Lemma 2.9, $e_1, e_2, \cdots, e_5$ can be properly colored with the colors in B. Then we put

$$\sigma(f_1) = \sigma(f_2) = \cdots = \sigma(f_5) = \lambda(f_0),$$

$$\sigma(f) \in C - \{\lambda(f_0), \sigma(e_1), \sigma(e_2), \sigma(e_3), \sigma(e_4), \sigma(e_5)\}.$$

Case 1 $|V_3(G) \cap V(b(f))| = 1$.

Let $d_G(u_1) = 3$, $d_G(u_i) = 4$, $i = 2, 3, 4, 5$. By Lemma 2.8, no two faces of $F(f) - \{f_5\}$ are adjacent. Define the graph $H = G - \{e_1, e_2, e_3, e_4\}$. Let f_0 denote the face of H with e_5 as a boundary edge and $f_0 \neq f_5$. By the induction assumption, H has a 7-edge-face coloring λ with a color set C. Restore G and discolor the edge e_5. Now we put

$$\sigma(f_5) = \lambda(f_5), \sigma(f_1) = \sigma(f_2) = \sigma(f_3) = \sigma(f_4) = \lambda(f_0).$$

Then let $B_i = C_\lambda(u_i)$, $i = 2, 3, 4$, $B_j = C_\lambda(u_j) - \{\lambda(e_5)\}$, $j = 1, 5$. We claim that there must exist some color $\alpha \in C - \{\lambda(f_0), \lambda(f_5)\}$ such that α can be assigned to two edges in $b(f)$. In fact, if this were absurd, then each color of $C - \{\lambda(f_0), \lambda(f_5)\}$ must occur on $\bigcup_{i=1}^{5} B_i$ at least twice, and so the total number of times is at least 10. But, by $|B_1| = 1$ and $|B_i| = 2$, $i \neq 1$, we have $|\bigcup_{i=1}^{5} B_i| \leq \sum_{i=1}^{5} |B_i| = 9$, a contradiction. If α can be used to color both e_1 and e_3, we put $\sigma(e_1) = \sigma(e_3) = \alpha$, and then color e_2, e_4, e_5, and f, successively. If α can be used to color both e_2 and e_4, we put $\sigma(e_2) = \sigma(e_4) = \alpha$ and then color e_3, e_1, e_5, and f successively.

Case 2 $|V_3(G) \cap V(b(f))| = 2$.

If there is $i \in J_5$ such that $d_G(u_i) = d_G(u_{i+1}) = 3$, the proof is simple. Thus we suppose $d_G(u_1) = d_G(u_4) = 3$, $d_G(u_i) = 4$, $i = 2, 3, 5$. So f_4 is not adjacent to f_5, and f_1, f_2, and

f_3 are pairwise nonadjacent. Set $H = G - \{e_1, e_2, e_3, e_4, e_5\} + u_1u_4$. Let λ be a 7-edge-face coloring of H with a color set C. Let f' and f'' stand for the faces of H separated by u_1u_4 with $u_5 \in b(f')$ and $u_2, u_3 \in b(f'')$. In G we first color f_4 and f_5 with $\lambda(f')$, and f_1, f_2, f_3 with $\lambda(f'')$. Then let $B_i = C_\lambda(u_i)$, $i = 2,3,5$, and $B_j = C_\lambda(u_j) - \{\lambda(u_1u_4)\}$, $j = 1,4$. Since $|B_1| = |B_4| = 1$, $|B_2| = |B_3| = |B_5| = 2$, and $|\bigcup_{i=1}^5 B_i| \le \sum_{i=1}^5 |B_i| = 8$, it follows that there is $\alpha \in C - \{\lambda(f'), \lambda(f'')\}$ such that α can be properly assigned to two edges in $b(f)$. If $\sigma(e_1) = \sigma(e_3) = \alpha$, we color e_2, e_4, e_5, and f successively. If $\sigma(e_3) = \sigma(e_5) = \alpha$, we color e_2, e_1, e_4, and f successively. For the other cases, we shall give a similar coloring.

Case 3 $|V_3(G) \cap V(b(f))| \ge 3$.

Without loss of generality, we suppose that $d_G(u_1) = d_G(u_2) = 3$. Form a 7-edge-face coloring λ of $G - e_1$. Restore G and remove the colors from e_2, e_3, e_4, e_5. If there is $e_i \ne e_1$ such that $d_G(u_i) = d_G(u_{i+1}) = 4$, we color $e_i, f, e_{i+1}, e_{i+2}, \cdots, e_5, e_{i-1}, e_{i-2}, \cdots, e_2$, and e_1 successively. Otherwise we color f, e_2, e_3, e_4, e_5, and e_1 successively.

We now exhaust all cases and therefore the proof is complete.

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