Acta Mathematicae Applicatae Sinica, English Series
Vol. 18, No. 4 (2002) 681–684

Acta Mathematicae Applicatae Sinica, English Series

Cycles Containing a Given Arc in Regular Multipartite Tournaments

Guo-fei Zhou, Ke-min Zhang

Department of Mathematics, Nanjing University, Nanjing 210093, China (E-mail: gfzhou@nju.edu.cn)

Abstract In this paper we prove that if T is a regular n-partite tournament with $n \geq 6$, then each arc of T lies on a k-cycle for $k=4,5,\cdots,n$. Our result generalizes theorems due to Alspach[1] and Guo[3] respectively.

Keywords Multitepartite tournaments, cycle
2000 MR Subject Classification 05C20

1 Introduction

We follow the terminologies and notations of [2]. Let $D = \big(V(D), A(D)\big)$ be a digraph. If xy is an arc of a digraph D, then we say that x dominates y, denoted by $x \to y$. More generally, if A and B are two disjoint vertex sets of D such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \Rightarrow B$. The outset $N^+(x)$ of a vertex x is the set of vertices dominated by x in D, and the inset $N^-(x)$ is the set of vertices dominating x in D. The irregularity $i(D)$ is max $|d^+(x) - d^-(y)|$ over all vertices x and y of D ($x = y$ is admissible). If $i(D) = 0$, we say D is regular. A k-cycle is a cycle of length k. Let $U \subseteq V(D)$, we use $D\langle U\rangle$ to denote the subdigraph induced by U. Let T be a multipartite tournament and $x \in V(T)$, we use $V(x)$ to denote the partite set of T to which x belongs. A k-outpath of an arc xy in a multipartite tournament is a directed path with length k starting from xy such that x does not dominate the end vertex of the directed path.

Guo and Volkmann[4] proved that every partite set of a strongly connected n-partite tournament has at least one vertex which lies on a k-cycle for each k, $3 \leq k \leq n$. Yeo[6] proves that if T is a regular n-partite tournament of order p with $n \geq 5$, then each vertex of T lies on a cycle of length k, for $k = 3, 4, \cdots, p$. Furthermore, Guo[3] proved that if T is a regular n-partite ($n \geq 3$) tournament, then every arc of T has an outpath of length $k-1$ for all k satisfying $3 \leq k \leq n$.

In this paper we show that, if T is a regular n-partite tournament with $n \geq 6$, then each arc of T lies on a k-cycle for $k = 4, 5, \cdots, n$.

2 Main results

Lemma. *Let T be a regular n-partite tournament and let $e = (u, v)$ be an arbitrary arc in T, if e is not contained in any 4-cycle, then*

(1) $N^+(v) \cap N^-(u) = V(w)$ for some $w \in V(T)$.

(2) $V(u) \Rightarrow v$ and $u \Rightarrow V(v)$.

Proof. (1) If there are two vertices $x, y \in N^+(v) \cap N^-(u)$ such that $x \to y$, then $uvxyu$ is a 4-cycle containing e, which is a contradiction to the initial hypothesis. Hence $N^+(v) \cap N^-(u)$

Manuscript received October 8, 1999. Revised August 12, 2002.

is either an empty set or an independent set. So there is $w \in V(T)$ such that $N^+(v) \cap N^-(u) \subseteq V(w)$. Assume that $N^+(v) \cap N^-(u) \neq V(w)$. Then since T is regular, each partite set has the same cardinality (say s), hence we have $\left|N^+(v) \cap N^-(u)\right| < \left|V(w)\right| = s$. On the one hand,

$$\left|N^+(v)\right| + \left|N^-(u)\right| = \frac{n-1}{2}s + \frac{n-1}{2}s = (n-1)s;$$

On the other hand, since

$$\left|N^+(v)\right| + \left|N^-(u)\right| = \left|N^+(v) \cap N^-(u)\right| + \left|N^+(v) \cup N^-(u)\right| < s + (n-2)s = (n-1)s,$$

which implies that $(n-1)s < (n-1)s$, a contradiction. Therefore we have $N^+(v) \cap N^-(u) = V(w)$.

(2) Suppose that there is $x \in V(u)$ such that $v \to x$. If $u \Rightarrow N^+(x)$, then $d^+(u) \geq d^+(x) + \left|\{v\}\right|$, which contradicts the regularity of T. So there is $y \in N^+(x)$ such that $y \to u$, and then $uvxyu$ is a 4-cycle containing e, again a contradiction. Hence we have $V(u) \Rightarrow v$.

Using arguments similar to that of the Lemma, we can also prove that $u \Rightarrow V(v)$.

Theorem. *Let T be a regular n-partite tournament with $n \geq 6$, then each arc of T lies on a k-cycle for $k = 4, 5, \cdots, n$.*

Proof. Since T is regular, it is not difficult to check that all partite sets of T have the same cardinality, say s. So it is clear that $\left|N^+(x)\right| = \left|N^-(x)\right| = \frac{(n-1)s}{2}$ for each $x \in V(T)$.

Let $e = (u, v)$ be any arc in T, we shall first show that e lies on a 4-cycle.

Let $A = N^+(v)$ and $B = N^-(u)$. If there exist a, b in $A \cap B$ such that $a \to b$, then $uvabu$ is a 4-cycle containing e. So we may assume that $A \cap B$ is either an empty set or an independent set. Let $t = |A \cap B|$, then $t \leq s$. By the Lemma, we know that $V(u) \Rightarrow v$, it follows that $V(u) \cap A = \emptyset$. Let k be the number of partite sets in $T\langle A - B\rangle$, then $k \leq n-2$. Again by the Lemma we know that there is x in A such that $d^+_{T[A]}(x) \leq \frac{(n-3)(|A|-t)}{2(n-2)}$, it follows that

$$\left|N^+(x) - A\right| \geq \frac{(n-1)s}{2} - \frac{(n-3)(|A|-t)}{2(n-2)}.$$

If there is $y \in N^+(x) \cap B$, then $uvxyu$ will be a 4-cycle containing e. So we assume that $N^+(x) \cap B = \emptyset$. It follows that

$$\left|V(T)\right| \geq |A| + |B| - |A \cap B| + \left|N^+(x) - A\right| + \left|\{u, v\}\right|,$$

that is

$$ns \geq \frac{(n-1)s}{2} + \frac{(n-1)s}{2} - t + \frac{(n-1)s}{2} - \frac{(n-3)(|A|-t)}{2(n-2)} + 2,$$

it follows that $t \geq \frac{[(n-3)s+4](n-2)}{3n-7}$, since $t \leq s$, we have $s \geq \frac{[(n-3)s+4](n-2)}{3n-7}$, which implies that $(n^2 - 8n + 13)s + 4(n-2) \leq 0$, this is impossible since $n \geq 6$. Therefore we have that e is contained in a 4-cycle.

Let $C = v_1 v_2 \cdots v_m v_1$ be an m-cycle containing e, where $e = (v_m, v_1)$ and $4 \leq m \leq n-1$. It suffices to show that e is contained in an $(m+1)$-cycle. In the following, we always assume that $s \geq 2$, since otherwise by [1] the theorem is valid.

Note that for each $x \in V(C)$, $N^+(x) - V(C) \neq \emptyset$ and $N^-(x) - V(C) \neq \emptyset$, otherwise we have $m = |V(C)| \geq d^+(x) + 1 \geq (n-1)s/2 + 1 \geq n$, a contradiction.

Let

$S = \{$all vertices that beong to partite sets that are not represented on $C\}$;

$A = \{x | x \in S, V(C) \Rightarrow x\}$; $B = \{y | y \in S, y \Rightarrow V(C)\}$; $X = S - A - B$.

We consider the following cases:

Case 1. $A \neq \emptyset$ or $B \neq \emptyset$.

Without loses of generality, we may assume that $A \neq \emptyset$; for the case that $B \neq \emptyset$, we only need to consider the converse of T.

Let $a \in A$, by the definition of A, we have $V(C) \Rightarrow a$. Note that $N^-(v_m) - V(C) \neq \emptyset$.

If there is $b \in N^-(v_m) - V(C)$ such that $a \to b$, then $v_1 v_2 \cdots v_{m-2} a b v_m v_1$ will be an $(m+1)$-cycle containing e.

If $N^-(v_m) - V(C) \Rightarrow a$, then we have $d^-(a) \geq d^-(v_m) + \big|\{v_m, v_1\}\big|$, which contradicts the regularity of T.

So we assume that there is $b \in N^-(v_m) - V(C)$ such that $V(a) = V(b)$. Since $s \geq 2$, $N^-(b) - V(C) \neq \emptyset$. Hence if there is $c \in N^-(b) - V(C)$ such that $a \to c$, then $v_1 v_2 \cdots v_{m-3} a c b v_m v_1$ will be an $(m+1)$-cycle containing e; and if $N^-(b) - V(C) \Rightarrow a$, then we have $d^-(a) \geq d^-(b) + \big|\{v_m\}\big|$, which contradicts the regularity of T.

Case 2. $A = \emptyset$ and $B = \emptyset$.

In this case we have $X \neq \emptyset$. If there is $x \in X$ such that $v_1 \to x$, then since x is adjacent to each vertex of $V(C)$, $V(C) \Rightarrow x$, which is a contradiction that $x \in X$. So we have $X \Rightarrow v_1$. Similarly, we have $v_m \Rightarrow X$.

Subcase 2.1. There is $x \in X$ such that $x \to v_{m-1}$.

Then we have $x \Rightarrow \{v_1, v_2, \cdots, v_{m-1}\}$. Since $X \Rightarrow v_1$, each vertex in $N^+(v_1) - V(C)$ is adjacent to x.

If there is $y \in N^+(v_1) - V(C)$ such that $y \to x$, then $v_1 y x v_3 \cdots v_m v_1$ is an $(m+1)$-cycle containing e.

If $x \Rightarrow N^+(v_1) - V(C)$, then $d^+(x) \geq d^+(v_1) + \big|\{v_1\}\big|$, a contradiction to the regularity of T.

Subcase 2.2. $v_{m-1} \Rightarrow X$.

By considering the converse of T of subcase 2.1, we may also assume that $X \Rightarrow v_2$.

Subcase 2.2.1. $m = 4$.

If $n \geq 7$, then we have $d^+(v_4) \geq (n-4)s + 1 \geq 3s + 1$; but $d^-(v_4) \leq 3s - 1$. Thus we have $d^+(v_4) \neq d^-(v_4)$, a contradiction. So we consider the case that $n = 6$.

Let t be number of partite sets in $T\langle X\rangle$. Since $n = 6$ and $m = 4$, $t \geq 2$. If $t \geq 3$, then $d^+(v_4) \geq |X| + \big|\{v_1\}\big| \geq 3s + 1$, which contradicts the regularity of T. So we assume that $t = 2$. Let $x, y \in X$ such that $x \to y$. Since $n = 6$, $T\langle V(C)\rangle$ is a tournament. If $v_2 \to v_4$, then we have $x \Rightarrow N^+(v_1) - V(C)$, since otherwise let $z \in N^+(v_1) - V(C)$ such that $z \to x$, then $v_1 z x v_2 v_4 v_1$ is a 5-cycle containing e. Hence $d^+(x) \geq d^+(v_1) + \big|\{v_1, y\}\big| - \big|\{v_3\}\big|$, a contradiction. So we have $v_4 \to v_2$. Similarly we have $v_3 \to v_1$.

Since $\big|V(T)\big| \geq \big|N^+(v_1) - V(C)\big| + \big|N^-(v_4) - V(C)\big| - \big|N^+(v_1) \cap N^-(v_4) - V(C)\big| + |X| + \big|V(C)\big|$, that is $6s \geq \frac{5}{2}s - 1 + \frac{5}{2}s - 1 - \big|N^+(v_1) \cap N^-(v_4) - V(C)\big| + |2s| + 4$, which follows that $\big|N^+(v_1) \cap N^-(v_4) - V(C)\big| \geq s + 2$. Hence there exist $c, d \in N^+(v_1) \cap N^-(v_4) - V(C)$ such that $c \to d$. Note that $v_1 c d v_4 v_1$ is another 4-cycle containing e, from the same argument as above, we have $v_4 \to c$ and $d \to v_1$, this contradicts that $c \to v_4$ and $v_1 \to d$.

Subcase 2.2.2. $m \geq 5$.

Suppose first that there is $x \in X$ such that $x \to v_{m-2}$.

Then $x \Rightarrow \{v_1, v_2, \cdots, v_{m-2}\}$. Since $x \to v_1$, x is adjacent to each vertex of $N^+(v_1) - V(C)$. If there is $y \in N^+(v_1) - V(C)$ such that $y \to x$, then $v_1 y x v_3 \cdots v_m v_1$ is an $(m+1)$-cycle containing e. So we may assume that $x \Rightarrow N^+(v_1) - V(C)$, it follows that $v_1 \Rightarrow \{v_2, v_3, \cdots, v_{m-1}\}$, since otherwise let $v_i \in \{v_2, v_3, \cdots, v_{m-1}\}$ such that $v_i \to v_1$, we must have $d^+(x) \geq d^+(v_1) + \big|\{v_1\}\big|$, a contradiction to regularity of T. Hence we have $v_m \Rightarrow N^+(v_1) - V(C) - V(v_m)$; otherwise let $y \in N^+(v_1) - V(C) - V(v_m)$ such that $y \to v_m$, then $v_1 v_3 \cdots v_{m-1} x y v_m v_1$ will be an $(m+1)$-cycle containing e, a contradiction. On the other hand, if $v_1 \to v_i$, then we must have $v_m \to v_{i-1}$, otherwise $v_1 v_i \cdots v_{m-1} x v_2 \cdots v_{i-1} v_m v_1$ will be an $(m+1)$-cycle containing e. It follows that

$$\big|N^+(v_m) \cap V(C)\big| \geq \big|N^+(v_1) \cap V(C)\big| - \big|V(v_m) \cap V(C)\big| + 1,$$

hence we have

$$d^+(v_m) \geq d^+(v_1) + |X| - \left|N^+(v_1) \cap V(v_m) - V(C)\right| - \left(|V(v_m) \cap V(C)| - 1\right),$$

which yields $d^+(v_m) \geq d^+(v_1) + (n-m)s - s + 1 \geq d^+(v_1) + 1$, a contradiction to the regularity of T.

Therefore we assume that $v_{m-2} \Rightarrow X$. Consider the converse of T, we can also assume that $X \Rightarrow v_3$.

Now let $x \in X$. If there exists $y \in N^+(v_1) - V(C)$ such that $y \to x$, then $v_1 y x v_3 \cdots v_m v_1$ is an $(m+1)$-cycle containing e. So in the following we assume that $x \Rightarrow N^+(v_1) - V(C)$.

If there is $z \in N^+(v_1) - V(C) - V(v_m)$ such that $z \to v_m$, then $v_1 v_2 \cdots v_{m-2} x z v_m v_1$ will be an $(m+1)$-cycle containing e, a contradiction, so we have $v_m \Rightarrow N^+(v_1) - V(C) - V(v_m)$; On the other hand, if $v_1 \to v_i$, then we must have $v_m \to v_{i-1}$, otherwise $v_1 v_i \cdots v_{m-1} x v_2 \cdots v_{i-1} v_m v_1$ will be an $(m+1)$-cycle containing e. It follows that

$$\left|N^+(v_m) \cap V(C)\right| \geq \left|N^+(v_1) \cap V(C)\right| - \left|V(v_m) \cap V(C)\right| + 1.$$

hence we have

$$d^+(v_m) \geq d^+(v_1) + |X| - \left|N^+(v_1) \cap V(v_m) - V(C)\right| - \left(|V(v_m) \cap V(C)| - 1\right),$$

which yields $d^+(v_m) \geq d^+(v_1) + (n-m)s - s + 1 \geq d^+(v_1) + 1$, a contradiction to the regularity of T.

This completes the proof of the Theorem.

3 Remarks

For $n \leq 5$, the theorem does not hold any more.

Let $V_1 = A_1,\ V_2 = A_2 \cup A_3,\ V_3 = \{u\} \cup A_4,\ V_4 = \{v\} \cup A_5,\ V_5 = A_6 \cup A_7$, where $|V_i| = 8$ and $|A_2| = |A_3| = |A_6| = |A_7| = 4$. We construct a 5-partite tournament as follows:

(1) Add arcs between A_2 and A_7 such that $T\langle A_2 \cup A_7\rangle$ is regular.

(2) Add arcs between A_3 and A_6 such that $T\langle A_3 \cup A_6\rangle$ is regular.

(3) Let $A_1 \Rightarrow A_2 \cup V_4 \cup A_7,\ A_2 \Rightarrow A_4 \cup A_5,\ A_3 \Rightarrow A_1 \cup A_7 \cup \{u, v\},\ u \Rightarrow A_2 \cup A_7,\ A_4 \Rightarrow A_3 \cup A_6,\ V_3 \Rightarrow V_4,\ A_5 \Rightarrow A_1 \cup A_3 \cup a_6,\ v \Rightarrow A_1 \cup A_2 \cup A_7,\ A_6 \Rightarrow A_1 \cup A_2 \cup \{u, v\},\ A_7 \Rightarrow A_4 \cup A_5$.

We can check that this is a regular 5-partite tournament but uv is not contained in any 4-cycle.

Moreover, we construct a 6-partite tournament T as follows: Let C_1, C_2, C_3 be three disjoint 4-cycles, let $V(T) = V(C_1) \cup V(C_2) \cup V(C_3)$, and let $V(C_1) \Rightarrow V(C_2) \Rightarrow V(C_3) \Rightarrow V(C_1)$. In this tournament, we can see that each arc of any C_i is not contained in a 3-cycle. So our result is the best possible in a certain sense.

Finally, we raise the following conjecture:

Conjecture. Let T be a regular n-partite $(n \geq 6)$ tournament of order p, then each arc of T lies on a k-cycle, for $k = 4, 5, \cdots, p$.

References

[1] Alspach, B. Cycles of each length in regular tournaments. *Canad. Math. Bull.*, 10: 283–286 (1967)
[2] Bondy, J.A., Murty, U.S.R. Graph theory with applications. MacMillan Press, New York (1976)
[3] Guo, Y. Outpaths in semicomplete multipartite digraphs. *Discrete Applied Math.*, 95: 273–277 (1999)
[4] Guo, Y., Volkmann, L. Cycles in multipartite tournaments. *J. Combin. Theory* (Seirs B), 62: 363–366 (1994)
[5] Volkmann L. Cycles in multipartite tournaments: results and problems. *Discrete Math.*, 245: 19–53 (2002)
[6] Yeo, A. Diregular c-partite tournaments are vertex-pancyclic when $c \geq 5$. *J. Graph Theory*, 32: 137–152 (1999)