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c-Pancyclic Partial Ordering and $(c-1)$-Pan-Outpath Partial Ordering in Semicomplete Multipartite Digraphs

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Abstract An outpath of a vertex v in a digraph is a path starting at v such that v dominates the end vertex of the path only if the end vertex also dominates v. First we show that letting D be a strongly connected semicomplete c-partite digraph ($c \geq 3$), and one of the partite sets of it consists of a single vertex, say v, then D has a c-pancyclic partial ordering from v, which generalizes a result about pancyclicity of multipartite tournaments obtained by Gutin in 1993. Then we prove that letting D be a strongly connected semicomplete c-partite digraph with $c \geq 3$ and letting v be a vertex of D, then D has a $(c-1)$-pan-outpath partly ordering from v. This result improves a theorem about outpaths in semicomplete multipartite digraphs obtained by Guo in 1999.

Keywords Semicomplete multipartite digraphs, Outpaths, Cycles
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1 Introduction

We use the terminology and notation of [1]. A *digraph* $D = (V(D), A(D))$ is determined by its set of vertices $V(D)$, and its set of arcs $A(D)$. If xy is an arc of a digraph D, then we say that x *dominates* y, denoted by $x \to y$. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \to B$. By a *cycle* (*path,* resp.) we mean a directed cycle (directed path, resp.). A cycle of length k is called a k*-cycle.* A digraph D is *strongly connected*, if for any two vertices x and y, there are a path from x to y and a path from y to x in D. If S is a set of vertices in a digraph D, then $D[S]$ is the subdigraph induced by S.

A *semicomplete n-partite digraph* is a digraph obtained from a complete n-partite graph by replacing each edge with an arc, or a pair of mutually opposite arcs with the same end vertices.

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An *n-partite tournament* is a semicomplete n-partite digraph with no cycles of length 2 and a *tournament* is an n-partite tournament having exactly n vertices.

An *outpath* of a vertex x (an arc xy, resp.) in D is a path starting at x (xy, resp.) such that x dominates the end vertex of the path only if the end vertex also dominates x. An outpath of length k is called a k-outpath.

The concept of the so-called outpath was introduced by Guo [2], which is an extension of the notation of a cycle in tournaments, i.e., a vertex v of a tournament T is in a k-cycle if and only if v has a $(k-1)$-outpath.

In [3], [4] Hendry introduced the concept of pancyclic ordering. In [5] Tewes considered pancyclic ordering in strongly connected in-tournaments. For semicomplete multipartite digraphs, we introduce slightly weak concepts of c-pancyclic partial ordering and $(c-1)$-pan-outpath partial ordering, which are all generalizations of the concept of vertex pancyclicity in tournaments.

A semicomplete c-partite digraph D with $c \geq 3$ has *c-pancyclic partial ordering*, if there are c vertices in D which can be labelled $x_1, x_2, \ldots, x_c$ such that $D[\{x_1, x_2, \ldots, x_t\}]$ is Hamiltonian for every t $(3 \leq t \leq c)$. The ordering $x_1, x_2, \ldots, x_c$ is called a *c-pancyclic partial ordering from* x_1, denoted by $\langle x_1, x_2, \ldots, x_c \rangle$.

A semicomplete c-partite digraph D with $c \geq 3$ has a $(c-1)$*-pan-outpath partial ordering from* v, if there are c vertices in D which can be labelled $x_1(= v), x_2, \ldots, x_c$ such that $D[\{x_1, x_2, \ldots, x_t\}]$ has a $(t-1)$-outpath from x_1 for every t $(3 \leq t \leq c)$. The ordering $x_1, x_2, \ldots, x_c$ is also denoted by $\langle x_1, x_2, \ldots, x_c \rangle$.

The well-known theorem of Moon [6] says that if T is a strong tournament on n vertices, then every vertex of T is in a k-cycle for all $k \in \{3, 4, \ldots, n\}$.

Guo [2] proves that letting D be a strongly connected semicomplete c-partite digraphs with $c \geq 3$ and letting v be a vertex of D, then v has a $(k-1)$-outpath for all $k \in \{3, 4, \ldots, c\}$, which generalizes the theorem on tournaments due to Moon.

As another generalization of the theorem on tournaments due to Moon, Gutin [7] proves that letting D be a strongly connected c-partite $(c \geq 3)$ tournament, and one of the partite sets of it consists of a single vertex, say v, then for each $p \in \{3, 4, \ldots, c\}$ there is a p-cycle of D containing v.

In this paper, we improve both the result of Guo and that of Gutin.

2 Main Results

Theorem 1 *Let D be a strongly connected semicomplete c-partite digraph with $c \geq 3$, and one of the partite sets of it consists of a single vertex, say v. Then D has a c-pancyclic partial ordering from v.*

Proof Let $V_1 = \{v\}$, V_2, $\ldots$, V_c be the partite sets of D. First we show that v lies on a 3-cycle. Since D is strongly connected and $c \geq 3$, it is easy to show that there exists a cycle of length at least 3 which contains v. Let $C = v_1 v_2 \cdots v_k v_1$ with $v_1 = v$ be such a shortest cycle. Suppose that $k \geq 4$. Since v is adjacent with every vertex of $V(D) - \{v\}$, we have $v_1 v_3 \in A(D)$ or $v_3 v_1 \in A(D)$. If $v_1 v_3 \in A(D)$, then v is in a $(k-1)$-cycle $v_1 v_3 \cdots v_k v_1$, which contradicts the choice of the cycle C. If $v_3v_1 \in A(D)$, then v is in a 3-cycle $v_1v_2v_3v_1$, which also contradicts the choice of the cycle C. Therefore, $k = 3$ and v lies on a cycle of length 3.

Suppose now that D contains m vertices $v_1 = v,\, v_2, \ldots, v_m$ such that $D[\{v_1, v_2, \ldots, v_t\}]$ is Hamiltonian for every t $(3 \le t \le m)$, where m satisfies $3 \le m < c$. We shall show that D contains $m+1$ vertices $x_1 = v, x_2, \ldots, x_{m+1}$ such that $D[\{x_1, x_2, \ldots, x_t\}]$ is Hamiltonian for every t $(3 \le t \le m+1)$.

Without loss of generality, we assume the Hamiltonian cycle of $D[\{v_1, v_2, \ldots, v_m\}]$ is $C_m = v_1v_2\cdots v_mv_1$ with $v_1 = v$. Let $S = \{x|x \in V_i, V_i \cap \{v_1, v_2, \ldots, v_m\} = \emptyset, 2 \le i \le c\}$. It is clear that $S \ne \emptyset$ and every vertex of S is adjacent with all vertices of $\{v_1, v_2, \ldots, v_m\}$. If there is a vertex x in S such that $N^+(x) \cap \{v_1, v_2, \ldots, v_m\} \ne \emptyset$ and $N^-(x) \cap \{v_1, v_2, \ldots, v_m\} \ne \emptyset$, then it is easy to check that x can be inserted into C_m to form an $(m+1)$-cycle and $\langle v_1, v_2, \ldots, v_m, x\rangle$ has the desired property. So we assume that S can be decomposed into two subsets S_1 and S_2 such that $S_2 \to \{v_1, v_2, \ldots, v_m\} \to S_1$. Without loss of generality, we assume that S_1 is not empty. Since D is strongly connected, there is a path from S_1 to $\{v_1, v_2, \ldots, v_m\}$. Let $P = y_1y_2\ldots y_q$ be such a shortest path. It is obvious that $q \ge 3$ and $y_q = v_k$ for some $1 \le k \le m$. We consider the following two cases:

Case 1 $V(P) \cap S_2 \ne \emptyset$.

Since S_2 dominates $\{v_1, v_2, \ldots, v_m\}$ and P is a shortest path, we have $v_1 \to y_{q-2}$ and the vertex y_{q-1} must be in S_2, otherwise there is $y_i \in S_2$, $(1 \le i \le q-2)$ and then $y_1y_2\ldots y_iv_1$ is a shorter path from S_1 to $\{v_1, v_2, \ldots, v_m\}$, a contradiction. Hence $\langle v_1, y_{q-2}, y_{q-1}, v_m, v_{m-1}, \ldots, v_3\rangle$ has the desired property.

Case 2 $V(P) \cap S_2 = \emptyset$.

Subcase 2.1 $y_q = v_1$.

Clearly, $y_q \to y_i$ for all $1 \le i \le q-2$. Hence D contains $m+1$ vertices such that $\langle y_q = v_1, y_{q-1}, y_{q-2}, \ldots, y_1, v_2, v_3, \ldots, v_{m-q+2}\rangle$ has the desired property.

Subcase 2.2 $y_q \ne v_1$.

By Subcase 2.1 we may assume that $v_1 \notin N^+(y_{q-1})$. Recall that $y_q = v_k$, $2 \le k \le m$. For convenience, let $y_{q+i} = v_{k+i}$ for all i satisfying $0 \le i \le m-k$ and denote $P' = y_1y_2\cdots y_{q+m-k}$. Let $\alpha = \max\{j|v_1 \to P'[y_1, y_j]\}$. Clearly $q-1 \le \alpha \le q+m-k$. If $\alpha \ge q+m-k-1$, then D contains $m+q-1$ vertices such that $\langle v_1, y_{q+m-k}, y_{q+m-k-1}, \ldots, y_1, v_2, v_3, \ldots, v_{k-1}\rangle$ has the desired property.

So we may assume that $q-1 \le \alpha \le q+m-k-2$, $2 \le k \le m$. Note that $y_{\alpha+1} \to v_1$. Let $\beta = \max\{i|P'[y_{\alpha+1}, y_i] \to v_1\}$. Clearly $\alpha+1 \le \beta \le q+m-k$. Since $y_{q+m-k} \to v_1$, we have either $\beta = q+m-k$ or $\beta \le (q+m-k)-1$.

If $\beta = q+m-k$, then D contains $m+q-1$ vertices such that $\langle v_1, y_{\alpha+1}, y_\alpha, \ldots, y_1, y_{\alpha+2}, y_{\alpha+3}, \ldots, y_{q+m-k}, v_2, v_3, \ldots, v_{k-1}\rangle$ has the desired property.

If $\beta \le (q+m-k)-1$, then $v_1 \to y_{\beta+1}$, D contains $m+q-1$ vertices such that $\langle v_1, y_{\alpha+1}, y_\alpha, \ldots, y_1, y_{\alpha+2}, y_{\alpha+3}, \ldots, y_\beta, y_{\beta+1}, \ldots, y_{q+m-k}, v_2, v_3, \ldots, v_{k-1}\rangle$ has the desired property.

This completes the proof of Theorem 1.

Corollary 2 [7] *Let T be a strongly connected c-partite $(c \ge 3)$ tournament, and one of the* *partite sets of it consists of a single vertex, say v. Then for each $p \in \{3, 4, \ldots, c\}$ there is a p-cycle of T containing v.*

The method for the proof of the following Theorem 3 comes from [2], this proof is shorter than the original one in the manuscript.

Theorem 3 *Let D be a strongly connected semicomplete c-partite digraph with $c \geq 3$ and let v be a vertex of D. Then D has a $(c-1)$-pan-outpath partial ordering from v.*

Proof Let V_1, V_2,..., V_c be the partite sets of D and assume, without loss of generality, that $v \in V_1$. If $V_1 = \{v\}$, then by Theorem 3.1, D has a c-pancyclic partial ordering from v. Hence D has a $(c-1)$-pan-outpath partial ordering from v.

Suppose now that $|V_1| \geq 2$. By adding arcs from $V_1 \backslash \{v\}$ to v, we obtain a semicomplete $(c+1)$-partite digraph D' which is also strongly connected. Note that the vertex v forms a partite set by itself in D'. By the same argument as above, D' has a $(c+1)$-pancyclic partial ordering from v, say $\langle v_1 = v, v_2, \ldots, v_{c+1} \rangle$. If the Hamiltonian cycle C_k of $D'[\{v_1, v_2, \ldots, v_k\}]$ contains no arc from $V_1 \backslash \{v\}$ to v, then clearly the path $C_k[v_1, v_1^-]$ is a $(k-1)$-outpath of v in D, where $3 \leq k \leq c+1$. If the Hamiltonian cycle C_k of $D'[\{v_1, v_2, \ldots, v_k\}]$ contains an arc from $V_1 \backslash \{v\}$ to v, we delete it and obtain a $(k-1)$-outpath of v in D, where $3 \leq k \leq c+1$. So $\langle v_1, v_2, \ldots, v_{c+1} \rangle$ is a c-pan-outpath partial ordering of D.

Corollary 4 [2] *Let D be a strongly connected semicomplete c-partite digraph with $c \geq 3$ and let v be a vertex of D. Then v has a $(k-1)$-outpath for all $k \in \{3, 4, \ldots, c\}$.*

Corollary 5 [7] *Let T be a strong tournament. Then T has a pancyclic ordering from v for every $v \in V(T)$.*

Corollary 6 [6] *Every strong tournament is vertex pancyclic.*

From the proof of Theorem 3, we can obtain the following theorem about long outpaths:

Theorem 7 *Let D be a strongly connected semicomplete c-partite digraph $(c \geq 3)$ with partite sets V_1, V_2, ..., V_c. If $|V_i| \geq 2$ for all $i = 1, 2, \ldots, c$, then D has a c-pan-outpath partial ordering from v for every $v \in V(D)$.*

Lastly we give a problem about long outpaths in a strongly connected semicomplete n-partite digraph.

Problem 8 *Can we give conditions to ensure that for every vertex v, v has a k-outpath in a strongly connected semicomplete c-partite $(c \geq 3)$ digraph D with $k > c$?*

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