Journal of Shanghai University (English Edition), 2003, **7**(1): 46～48
Article ID: 1007-6417(2003)01-0046-03

Upper Bounds for Ramsey Numbers $R(m,n,l)$ and $R(m,n,l,s)$ with Parameter

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Abstract In the paper some new upper bounds with parameters were obtained for the classical Ramsey numbers $R(m,n,l)$ and $R(m,n,l,s)$. By using the upper bounds, it was proved that $R(4,4,4) \le 236$.

Key words Ramsey number, (m,n,l,p)-graph, Ramsey graph.
MSC 2000 05D10

In 1935, P. Erdös and G Szekeres obtained the classical inequality $R(m,n) \le R(m-1,n) + R(m,n-1)$.

In 1968, K. Walker proved that $R(n,n) \le 4R(n-2,n) + 2$.

In 1998, Huang Y. R. and Zhang K. M.[1,2] proved that

$$R(m,n) \le \frac{1}{2}(\beta + 3\gamma + 5) + \frac{1}{2}\sqrt{\gamma(4\alpha + 2\beta - 3\gamma + 6) + (\beta + 1)^2}$$

In this paper, we obtain some new upper bounds for $R(m,n,l)$ and $R(m,n,l,s)$ with parameters.

1 The Upper Bound for $R(m,n,l)$

Let $r(m,n,l) = R(m,n,l) - 1$ and $p = r(m,n,l)$, an (m,n,l,p)-graph is a graph of the order p which does not contains the subgraph K_m with color a, and nor subgraph K_n with color b, nor subgraph K_l with color c for any three edge-coloring a, b, c.

Let G be a graph with three edge-coloring a, b, c and i be a vertex of G. Denote the number of K_3 containing v_i with color a (resp. color b, color c) by Δ_i^a (resp. Δ_i^b, Δ_i^c), the number of edges with an end vertex i of color a (resp. color b, color c) by d_i^a (resp. d_i^b, d_i^c), and the number of K_3 in G with color a (resp. color b, color c) by Δ^a (resp. Δ^b, Δ^c)

Received Jul. 3, 2002; Revised Oct. 16, 2002
Project supported by the National Natural Science Foundation of China (Grant No. 10171062)
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Theorem 1 Let $3 \le m \le n \le l$. Assume that

$r(m-2,n,l) \le \alpha_a$, $r(m,n-2,l) \le \alpha_b$, $r(m,n,l-2) \le \alpha_c$;
$r(m,n-1,l-1) \le \beta_a$, $r(m-1,n,l-1) \le \beta_b$, $r(m-1,n-1,l) \le \beta_c$;
$r(m-1,n,l) \le \gamma_a$, $r(m,n-1,l) \le \gamma_b$, $r(m,n,l-1) \le \gamma_c$.

When $m = n = l$, let $\alpha_a = \alpha, \beta_a = \beta, \gamma_a = \gamma$, $\delta = \beta_a + \beta_b + \beta_c + 1$, $A = \max(\delta, 2\gamma_a + r(m,n-1,l-1) - \alpha_a, 2\gamma_b + r(m-1,n,l-1) - \alpha_b, 2\gamma_c + r(m-1,n-1,l) - \alpha_c)$, Then the following inequalities must hold:

$$R(m,n,l) \le 2 + \min_{t>\delta} \frac{1}{4(t-\delta)}\{(t + \alpha_a - \beta_a)^2 + (t + \alpha_b - \beta_b)^2 + (t + \alpha_c - \beta_c)^2\} \tag{1}$$

$$R(n,n,n) \le 2 + \min_{t>\delta} \frac{3(t + \alpha - \beta)^2}{4(t - 3\beta - 1)} \tag{2}$$

$$R(m,n,l) \le 2 + \min_{t>A} \frac{1}{t-\delta}\{\gamma_a(t + \alpha_a - r(m,n-1,l-1) - \gamma_a) + \gamma_b(t + \alpha_b - r(m-1,n,l-1) - \gamma_b) + \gamma_c(t + \alpha_c - r(m-1,n-1,l) - \gamma_c)\} \tag{3}$$

$$R(n,n,n) \le 2 + \min_{t>A} \frac{3(t + \alpha - \beta - \gamma)\gamma}{t - 3\beta - 1} \tag{4}$$

Proof Let G be an (m,n,l,p)-Ramsey graph and i be a vertex of G, then we have the following inequalities:

$$\Delta_i^a \ge \binom{d_i^a}{2} - \frac{1}{2}[r(m-1,n-1,l) + r(m-1,n,l-1)]d_i^a$$

$$\frac{1}{2}\sum_{i=1}^{p}\alpha_a d_i^a \geq 3\Delta_a \geq \frac{1}{2}\sum_{i=1}^{p}[(d_i^a)^2 - (\delta - r(m, n-1, l-1))d_i^a]$$

$$\sum_{i=1}^{p}(\alpha_a d_i^a + \alpha_b d_i^b + \alpha_c d_i^c) \geq \sum_{i=1}^{p}[(d_i^a)^2 + (d_i^b)^2 + (d_i^c)^2 - \delta(p-1) + r(m, n-1, l-1)d_i^a + r(m-1, n, l-1)d_i^b + r(m-1, n-1, l)d_i^c]$$

It is clear that for any t we have

$$\sum_{i=1}^{p}[d_i^a(t + \alpha_a - r(m, n-1, l-1) - d_i^a) + d_i^b(t + \alpha_b - r(m-1, n, l-1) - d_i^b) + d_i^c(t + \alpha_c - r(m-1, n-1, l) - d_i^c)] \geq p(p-1)(t-\delta) \quad (*)$$

From (*) we have

$$(p-1)(t-\delta) \leq [t + \alpha_a - r(m, n-1, l-1)]^2 + [t + \alpha_b - r(m-1, n, l-1)]^2 + [t + \alpha_c - r(m-1, n-1, l)]^2$$

From this, it follows that (1) and (2) are true.

Since $d_i^a \leq \gamma_a$, $d_i^b \leq \gamma_b$, $d_i^c \leq \gamma_c$, using the $t > A$, we have the inequality

$$\gamma_a[t + \alpha_a - r(m, n-1, l-1) - \gamma_a] + \gamma_b[t + \alpha_b - r(m-1, n, l-1) - \gamma_b] + \gamma_c[t + \alpha_c - r(m-1, n-1, l) - \gamma_c] \geq (p-1)(t-\delta)$$

Now we obtain (3) and (4).

2 The Upper Bound for $R(m, n, l, s)$

Let $r(m, n, l, s) = R(m, n, l, s) - 1$ and $p = r(m, n, l, s)$. An (m, n, l, s, p)-graph is a graph with order p which does not contains the subgraph K_m with color a, and nor subgraph K_n with color b, nor subgraph K_l with color c, nor subgraph K_s with color d for any four edge-coloring a, b, c, d.

Let G be a graph with four edge-coloring a, b, c, d, and i be a vertex of G. Denote the number of K_3 containing v_i with color a (resp. color b, color c, color d) by Δ_i^a (resp. $\Delta_i^b, \Delta_i^c, \Delta_i^d$), the number of edges with an end vertex i of color a (resp. color b, color c, color d) by d_i^a (resp. d_i^b, d_i^c, d_i^d), and the number of K_3 in G with color a (resp. color b, color c, color d) by Δ^a (resp. Δ^b, Δ^c, Δ^d)

Theorem 2 Let $3 \leq m \leq n \leq l \leq s$. Assume that

$r(m-2, n, l, s) \leq \alpha_a$, $r(m, n-2, l, s) \leq \alpha_b$,
$r(m, n, l-2, s) \leq \alpha_c$, $r(m, n, l, s-2) \leq \alpha_d$,
$r(m-1, n, l, s) \leq \gamma_a$, $r(m, n-1, l, s) \leq \gamma_b$,
$r(m, n, l-1, s) \leq \gamma_c$, $r(m, n, l, s-1) \leq \gamma_d$,

$$B = \max(2\gamma_a - 1 - \alpha_a - r(m-1, n-1, l, s) - r(m-1, n, l-1, s) - r(m-1, n, l, s-1), 2\gamma_b - 1 - \alpha_b - r(m-1, n-1, l, s) - r(m, n-1, l-1, s) - \gamma(m, n-1, l, s-1), 2\gamma_c - 1 - \alpha_c - r(m-1, n, l-1, s) - r(m, n-1, l-1, s) - r(m, n, l-1, s-1), 2\gamma_d - 1 - \alpha_d - r(m-1, n, l, s-1) - r(m, n-1, l, s-1) - r(m, n, l-1, s-1))$$

When $m = n = l$, let $\alpha_a = \alpha, \beta_a = \beta, \gamma_a = \gamma$, Then the following inequalities must hold:

$$R(m, n, l, s) \leq 2 + \min_{t>0}\frac{1}{4t}\{[t + R(m-2, n, l, s) + R(m-1, n-1, l, s) + R(m-1, n, l-1, s) + R(m-1, n, l, s-1) - 3]^2 + [t + R(m, n-2, l, s) + R(m-1, n-1, l, s) + R(m, n-1, l-1, s) + R(m, n-1, l, s-1) - 3]^2 + [t + R(m, n, l-2, s) + R(m-1, n, l-1, s) + R(m, n-1, l-1, s) + R(m, n, l-1, s-1) - 3]^2 + [t + R(m, n, l, s-2) + R(m-1, n, l, s-1) + R(m, n-1, l, s-1) + R(m, n, l-1, s-1) - 3]^2\} \quad (5)$$

$$R(n, n, n, n) \leq 2 + \min_{t>0}\frac{3(t + \alpha + 3R(n-1, n-1, n, n) - 2)^2}{4t} \quad (6)$$

$$R(m, n, l, s) \leq 2 + \min_{t>B}\frac{1}{t}\{\gamma_a(t + 1 + \alpha_a + r(m-1, n-1, l, s) + r(m-1, n, l-1, s) + r(m-1, n, l, s-1) - \gamma_a) + \gamma_b(t + 1 + \alpha_b + r(m-1, n-1, l, s) + r(m, n, l-1, s) + r(m, n-1, l, s-1) - \gamma_b) + \gamma_c(t + 1 + \alpha_c + r(m-1, n, l-1, s) + r(m, n-1, l-1, s) + r(m, n, l-1, s-1) - \gamma_c) + \gamma_d(t + 1 + \alpha_d + r(m-1, n, l, s-1) + r(m, n-1, l, s-1) + r(m, n, l-1, s-1) - \gamma_d)\} \quad (7)$$

$$R(n, n, n, n) \leq 2 + \min_{t>B}\frac{4}{t}[R(n-1, n, n, n) - 1] \times [t + R(n-2, n, n, n) + 3R(n-1, n-1, n, n) - R(n-1, n, n, n) - 2] \quad (8)$$

Proof Let G be an (m, n, l, p)-Ramsey graph, and i be a vertex of G, then we have the following inequalities:

$$\Delta_i^a \geq \binom{d_i^a}{2} - \frac{1}{2}[r(m-1,n-1,l,s) + r(m-1,n,l-1,s) + r(m-1,n,l,s-1)]d_i^a$$

$$\frac{1}{2}\sum_{i=1}^{p}\alpha_a d_i^a \geq 3\Delta_a \geq \frac{1}{2}\sum_{i=1}^{p}\{(d_i^a)^2 - [1 + r(m-1,n-1,l,s) + r(m-1,n,l-1,s) + r(m-1,n,l,s-1)]\}d_i^a$$

Since $d_i^a + d_i^b + d_i^c + d_i^d = p - 1$, for any t we have:

$$\begin{aligned} tp(p-1) \leq \sum_{i=1}^{p}\{ & d_i^a(t+1+\alpha_a + r(m-1,n-1,l,s) + r(m-1,n,l-1,s) + r(m-1,n,l,s-1) - d_i^a) + \\ & d_i^b(t+1+\alpha_b + r(m-1,n-1,l,s) + r(m,n,l-1,s) + r(m,n-1,l,s-1) - d_i^b) + \\ & d_i^c(t+1+\alpha_c + r(m-1,n,l-1,s) + r(m,n-1,l-1,s) + r(m,n,l-1,s-1) - d_i^c) + \\ & d_i^d(t+1+\alpha_d + r(m-1,n,l,s-1) + r(m,n-1,l,s-1) + r(m,n,l-1,s-1) - d_i^d)\} \end{aligned} \quad (**)$$

By (**) we have:

$$\begin{aligned} 4t(p-1) \leq & [t + \alpha_a + r(m-1,n-1,l,s) + r(m-1,n,l-1,s) + r(m-1,n,l,s-1) + 1]^2 + \\ & [t + \alpha_b + r(m-1,n-1,l,s) + r(m,n-1,l-1,s) + r(m,n-1,l,s-1) + 1]^2 + \\ & [t + \alpha_c + r(m-1,n,l-1,s) + r(m,n-1,l-1,s) + r(m,n,l-1,s-1) + 1]^2 + \\ & [t + \alpha_d + r(m-1,n,l,s-1) + r(m,n-1,l,s-1) + r(m,n,l-1,s-1) + 1]^2 \end{aligned}$$

From this, (5) and (6) are true.

Since $d_i^a \leq \gamma_a$, $d_i^b \leq \gamma_b$, $d_i^c \leq \gamma_c$, $d_i^d \leq \gamma_d$, using the $t > B$ and (**), we have (7), (8).

Remark by [3] we know $R(3,4,4,) \leq 79$, $R(3,3,4) \leq 31$, now using (2) we have $R(4,4,4) \leq 2 + \min_{t>91}\frac{3(t-13)^2}{4(t-91)} = 2 + \frac{3(t-13)^2}{4(t-91)}\Big|_{t=169} = 236$.

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(Executive editor SHEN Mei-Fang)