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The Ramsey numbers of stars versus wheels

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Abstract

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest positive integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let S_n denote a star of order n and W_m a wheel of order $m+1$. This paper shows that $R(S_n, W_6) = 2n+1$ for $n \geq 3$ and $R(S_n, W_m) = 3n-2$ for m odd and $n \geq m-1 \geq 2$.

Keywords: Ramsey number; Star; Wheel

1. Introduction

All graphs considered in this paper are finite simple graph without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest positive integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. Let $G = (V(G), E(G))$ be a graph. The *neighborhood* of vertex v is denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$. For a vertex $v \in V(G)$ and a subgraph H of G, $N_H(v) = N(v) \cap V(H)$. Let $d_H(v) = |N_H(v)|$. For two vertex disjoint sets S and T, we define $d_T(S) = \sum_{s \in S} d_T(s)$. The *connectivity, independence number, maximum degree* and *minimum degree* of G are denoted by $\kappa(G)$, $\alpha(G)$, $\Delta(G)$ and $\delta(G)$, respectively. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. A *complete graph* of order n is denoted by K_n. A *complete bipartite graph* of order $m+n$ is denoted by $K_{m,n}$ and a *Star* S_n is $K_{1,n-1}$. A *path* and a *cycle* of order n are denoted by P_n and C_n, respectively. Let m be a positive integer and G a graph, we use mG to denote m vertex disjoint copies of G. A *Wheel* $W_n = \{x\} + C_n$ is a graph of $n+1$ vertices, x called the *hub* of the wheel. The length of a shortest and longest cycle of G are denoted by $g(G)$ and $c(G)$, respectively.

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A graph on n vertices is *pancyclic* if it contains cycles of every length l, $3 \le l \le n$ and *weakly pancyclic* if it contains cycles of every length l, $g(G) \le l \le c(G)$.

Ramsey theory studies conditions when a combinatorial object contains necessarily some smaller given objects. The role of Ramsey number is to quantify some of the general existential theorems in Ramsey theory. The classical Ramsey number is $R(k, l)$ for complete graphs. Since it is very difficult to determine $R(k, l)$, people turn to consider Ramsey numbers concerning general graph results, such as Ramsey numbers of path versus cycle, cycle versus star, tree versus wheel and so on, see for instance [1, 4–6, 8]. Recently, the following results are obtained.

Theorem A (Surahmat and Baskoro [9]). *$R(S_n, W_4) = 2n - 1$ for $n \ge 3$ and $n \equiv 1 \pmod 2$ and $R(S_n, W_4) = 2n + 1$ for $n \ge 4$ and $n \equiv 0 \pmod 2$.*

Theorem B (Surahmat and Baskoro [9]). *$R(S_n, W_5) = 3n - 2$ for $n \ge 4$.*

Theorem C (Baskoro et al. [1]). *Let T_n be a tree other than S_n, then $R(T_n, W_4) = 2n-1$ for $n \ge 3$ and $R(T_n, W_5) = 3n - 2$ for $n \ge 4$.*

Furthermore, motivated by Theorem C, Baskoro et al. [1] posed the following.

Conjecture 1. *Let T_n be a tree other than S_n and $n \ge m - 1$. Then $R(T_n, W_m) = 2n - 1$ for $m \ge 6$ and even, and $R(T_n, W_m) = 3n - 2$ for $m \ge 7$ and odd.*

In this paper, we consider the Ramsey numbers of star versus wheel in a more general situation. The main results of this paper are the following.

Theorem 1. *$R(S_n, W_6) = 2n + 1$ for $n \ge 3$.*

Theorem 2. *$R(S_n, W_m) = 3n - 2$ for m odd and $n \ge m - 1 \ge 2$.*

Remark. By Theorem 2, we can see that $R(S_n, W_m)$ is a function of n if m is odd. However, it is not the case when m is even. In fact, if m is even, then $R(S_n, W_m)$ is a function related to both n and m as can be seen by the following examples.

Let $m \ge 6$ be an even integer, $n = km/2 + 2$, where $k \ge 2$ is an integer, and $G = H \cup K_{n-1}$, where $\overline{H} = (k+1)K_{m/2}$. Obviously, G is a graph of order $2n + m/2 - 3$ and $\Delta(G) = n - 2$ and hence G contains no S_n. It is not difficult to see $\overline{G}$ contains no W_m. Thus we have $R(S_n, W_m) \ge 2n + m/2 - 2$ if $n = km/2 + 2$ for some integer $k \ge 2$.

Problem 1. Determine $R(S_n, W_m)$ for m even and $n \ge m - 1 \ge 7$.

2. Some lemmas

In order to prove our results, we need the following lemmas.

Lemma 1 (Bondy [2]). *Let G be a graph of order n. If $\delta(G) \ge n/2$, then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.*

Lemma 2 (Dirac [7]). *Let G be a 2-connected graph of order $n \ge 3$ with $\delta(G) = \delta$. Then $c(G) \ge \min\{2\delta, n\}$.*

Lemma 3 (Brandt [3]). *Every non-bipartite graph G of order n with $\delta(G) \geq (n+2)/3$ is weakly pancyclic.*

Lemma 4. *Let G be a 2-connected graph of order 8 with $\delta(G) = 3$. Then G contains a C_6.*

Proof. Let $V(G) = \{v_i \mid 1 \leq i \leq 8\}$ and $C = v_1 v_2 \cdots v_k$ a longest cycle of G. By Lemma 2, $k \geq 6$. If $k = 6$, we are done. If $k = 7$, then by the maximality of C, v_8 has no two consecutive neighbors on C. Since $\delta(G) = 3$, we may assume $N(v_8) = \{v_1, v_3, v_5\}$. Thus $v_1 v_2 v_3 v_4 v_5 v_8 v_1$ is a C_6. If $k = 8$, we assume $d(v_1) = 3$ and hence C has a chord $v_1 v_i$. If $i \in \{4, 6\}$, then G contains a C_6. Hence $i \in \{3, 5, 7\}$. By symmetry, we may assume $i \in \{3, 5\}$. Since $\delta(G) = 3$, C has a chord $v_5 v_j$. By an analogous argument as above, we have $j \in \{1, 3, 7\}$. If $i = 3$, then $j \neq 1$ and hence $j \in \{3, 7\}$ which implies G contains a C_6. Hence we have $i = 5$. In this case, $v_3 v_7 \notin E(G)$ for otherwise $v_1 v_2 v_3 v_7 v_6 v_5 v_1$ is a C_6. If $\{v_2, v_4\} \cap N(v_7) \neq \emptyset$, then G contains a C_6. Hence we may assume $v_2 v_7, v_4 v_7 \notin E(G)$. Thus noting that $d(v_1) = 3$ and $\delta(G) = 3$, we have $v_7 v_5 \in E(G)$. By symmetry, we have $v_3 v_5 \in E(G)$ which implies $v_1 v_2 v_3 v_5 v_7 v_8 v_1$ is a C_6. □

Lemma 5. *Let G be a 2-connected graph of order 9 with $\delta(G) = 4$. Then G contains a C_6.*

Proof. Let $V(G) = \{v_i \mid 1 \leq i \leq 9\}$ and $C = v_1 v_2 \cdots v_k$ a longest cycle of G. By Lemma 2, $k \geq 8$. If $k = 8$, then by the maximality of C, v_9 has no two consecutive neighbors in C. Since $\delta(G) = 4$, we may assume $N(v_9) = \{v_1, v_3, v_5, v_7\}$. Thus $g(G) \leq 4$. If G is non-bipartite, then G contains a C_6 by Lemma 3. If G is bipartite, then since $\delta(G) = 4$, it is not difficult to see that $G = K_{4,5}$ and hence G contains a C_6. If $k = 9$, then G is non-bipartite. Since $\delta(G) = 4$, C has a chord which implies $g(G) \leq 5$. Thus G contains a C_6 by Lemma 3. □

3. Proof of Theorem 1

Proof of Theorem 1. Let $n \geq 3$ be an integer and $G = H \cup K_{n-1}$, where $\overline{H} = C_{n+1}$ if $n \neq 5$ and $\overline{H} = 2C_3$ if $n = 5$. Obviously, $|G| = 2n$. It is not difficult to see neither G contains a star S_n nor $\overline{G}$ contains a W_6 and hence $R(S_n, W_6) \geq 2n + 1$.

In order to show $R(S_n, W_6) \leq 2n + 1$, we use induction on n. Let G be a graph of order $2n + 1$. As the basis of induction, we first show $R(S_n, W_6) = 2n + 1$ for $3 \leq n \leq 6$.

Suppose G contains no S_n. Then $\Delta(G) \leq n - 2$ which implies $\delta(\overline{G}) \geq n + 2$. Let $v \in V(G)$ be a vertex such that $d_{\overline{G}}(v) = d = \Delta(\overline{G}) = n + 2 + k$, where $k \geq 0$. Set $N_{\overline{G}}(v) = V_0$, $U = V(G) - V_0 \cup \{v\}$ and $F = \overline{G}[V_0]$. It is not difficult to see that $\delta(F) \geq 3 + k$. If $v_i \in V_0$ and $d_F(v_i) = 3 + k$, then we must have

$$U \subseteq N_{\overline{G}}(v_i). \tag{1}$$

If $n = 3$, then we can see $\overline{G}$ contains a W_6. If $n = 4$, then $\delta(F) \geq 3 + k \geq (6 + k)/2 = |V_0|/2$ which implies F contains a C_6 by Lemma 1. If $n = 5$, then $\delta(\overline{G}) \geq 7$. We have $d \geq 8$ since the number of vertices of odd degree is even, which implies $k \geq 1$ and hence we have $\delta(F) \geq 3+k \geq (7+k)/2 = |V_0|/2$ which implies F contains a C_6 by Lemma 1. Thus $\overline{G}$ contains a W_6 with the hub v when $n = 4, 5$ and hence we may assume $n = 6$.

If $k \geq 2$, then $\delta(F) \geq 3+k \geq (8+k)/2 = |V_0|/2$ which implies F contains a C_6 by Lemma 1 and hence $\overline{G}$ contains a W_6. Thus we may assume $k \leq 1$.

If $k = 0$, then $d = 8$. If $\delta(F) \geq 4$, then by Lemma 1, F contains a C_6 and hence $\overline{G}$ contains a W_6. Thus we have $\delta(F) = 3$. If F is not connected, then since $\delta(F) = 3$, we can see $F = 2K_4$. By (1), $U \subseteq N_{\overline{G}}(v_i)$ for any $v_i \in V_0$ and hence $\overline{G}$ contains a W_6 with the hub v_i for any $v_i \in V_0$. If $\kappa(F) = 1$, we let w be a cut-vertex and H_1 a component of $F - w$ such that $|H_1|$ is as small as possible. Then $|H_1| = 3$, $V(H_1) \cup \{w\}$ is a 4-clique and $d_F(h) = 3$ for any $h \in V(H_1)$. Let $V(H_1) = \{h_1, h_2, h_3\}$. If $|N_{\overline{G}}(w) \cap U| \leq 1$, then $\overline{G}[U]$ contains at least two edges since $\delta(\overline{G}) \geq 8$. Let $U = \{u_1, u_2, u_3, u_4\}$. If $\overline{G}[U]$ contains a P_3, say $P = u_1u_2u_3$, then by (1), $h_2u_1u_2u_3h_3u_4h_2$ is a C_6 and hence $\overline{G}$ contains a W_6 with the hub h_1. If $\overline{G}[U]$ contains no P_3, then $\overline{G}[U] = 2K_2$. Assume $E(\overline{G}[U]) = \{u_1u_2, u_3u_4\}$, then by (1), $h_2u_1u_2h_3u_3u_4h_2$ is a C_6 and hence $\overline{G}$ contains a W_6 with the hub h_1. Thus we may assume $|N_{\overline{G}}(w) \cap U| \geq 2$. Let $u_1, u_2 \in N_{\overline{G}}(w)$, then $h_2u_1wu_2h_3u_3h_2$ is a C_6 and hence $\overline{G}$ contains a W_6 with the hub h_1. If $\kappa(F) \geq 2$, then by Lemma 4, F contains a C_6 and hence $\overline{G}$ contains a W_6 with the hub v.

If $k = 1$, then $d = 9$. By Lemma 1, we may assume $\delta(F) = 4$. Since $\delta(F) = 4$ and $d = 9$, we have $\kappa(F) \geq 1$. If $\kappa(F) = 1$, then it is not difficult to see that F is two K_5's with one vertex, say w, in common. Obviously $F - w = 2K_4$. Take a K_4 in $F - w$ and let $V_1 = V(K_4) = \{v_1, v_2, v_3, v_4\}$. It is not difficult to see $d_F(v_i) = 4$ for any $v_i \in V_1$. Thus by (1), we can see $\overline{G}[U \cup V_1]$ contains a W_6 with the hub v_1. If $\kappa(F) \geq 2$, then by Lemma 5, F contains a C_6 and hence $\overline{G}$ contains a W_6 with the hub v. Thus, we have $R(S_n, W_6) = 2n+1$ for $3 \leq n \leq 6$.

Now, assume $n \geq 7$ and Theorem 1 holds for smaller values of n.

If $\overline{G}$ contains no W_6, then we have $\alpha(G) \leq 6$. If $\alpha(G) \leq 2$, then $\Delta(G) \geq n$ which implies G contains a star S_n. Hence we may assume $3 \leq \alpha(G) \leq 6$ and consider the following three cases separately.

Case 1. $\alpha(G) = 3$.

We consider the following two subcases separately.

Subcase 1.1. G contains an induced subgraph $G_0 = 3K_2$.

Let $V(G_0) = V_0 = \{a_i \mid 1 \leq i \leq 6\}$ and $E(G_0) = \{a_1a_2, a_3a_4, a_5a_6\}$. Since $n \geq 7$, we have $n - 3 \geq 4$. By induction hypothesis, $G - V_0$ contains a star S_{n-3} with center v_1. Since $\alpha(G) = 3$ and both $\{a_1, a_3, a_5\}$ and $\{a_2, a_4, a_6\}$ are independent sets, we have $|N_{V_0}(v_1)| \geq 2$. If $d_{V_0}(v_1) \geq 3$, then G contains a star S_n with center v_1. Hence we may assume $|N_{V_0}(v_1)| = 2$. Assume without loss of generality that $a_1 \in N(v_1)$. Then $a_2 \in N(v_1)$ for otherwise we can obtain an independent set of order 4. Thus we have $N_{V_0}(v_1) = \{a_1, a_2\}$.

Let $V_1 = V_0 \cup \{v_1\}$. Obviously, $G[V_1] = 2K_2 \cup K_3$. Since $n \geq 7$, we have $n-4 \geq 3$. By induction hypothesis, $G - V_1$ contains a star S_{n-4} with center v_2. For the same reason as above, we have $d_{V_1}(v_2) = 2$ or 3 and if $d_{V_1}(v_2) = 2$, then $N_{V_1}(v_2) = \{a_3, a_4\}$ or $\{a_5, a_6\}$. Assume $N_{V_1}(v_2) = \{a_3, a_4\}$, then it is no difficult to see that $\overline{G}[(V_1 - \{a_6\}) \cup \{v_2\}]$ contains a W_6 with the hub a_5, a contradiction. Hence we have $d_{V_1}(v_2) = 3$. Let $U = \{a_1, a_2, v_1\}$. If $d_U(v_2) = 0$, we may assume that $N_{V_1}(v_2) = \{a_3, a_4, a_5\}$. If $d_U(v_2) = 1$, then since $\alpha(G) = 3$, we may assume $a_3, a_4 \in N_{V_1}(v_2)$. Thus we can see $\overline{G}[V_1 \cup \{v_2\} - \{a_5\}]$ contains a W_6 with the hub a_6 if $d_U(v_2) \leq 1$, a contradiction. If $d_U(v_2) = 2$, we may assume $N_{V_1}(v_2) = \{a_1, a_2, a_3\}$. Thus, $\{v_1, v_2, a_4, a_5\}$ is an independent set which contradicts $\alpha(G) = 3$. Hence we have $d_U(v_2) = 3$.

Let $V_2 = V_1 \cup \{v_2\}$. Clearly, $G[V_2] = 2K_2 \cup K_4$. Since $n \geq 7$, we have $n - 4 \geq 3$. By induction hypothesis, $G - V_2$ contains a star S_{n-4} with center v. For the same reason as above, we have $d_{V_2}(v) = 3$. Let $U_1 = U \cup \{v_2\}$. If $d_{U_1}(v) = 0$, then we may assume $N_{V_2}(v) = \{a_3, a_4, a_5\}$. If $d_{U_1}(v) = 1$, say $a_1 \in N(v)$, then since $\alpha(G) = 3$, we may assume $a_3, a_4 \in N_{V_1}(v)$. Thus we see $\overline{G}[V_2 \cup \{v\} - \{a_1, a_5\}]$ contains a W_6 with the hub a_6 if $d_{U_1}(v) \leq 1$, a contradiction. If $d_{U_1}(v) = 2$, we may assume that $N_{V_2}(v) = \{a_1, a_2, a_3\}$. Thus, $\{v_1, a_4, a_5, v\}$ is an independent set which contradicts $\alpha(G) = 3$. If $d_{U_1}(v) = 3$, say $N_{V_2}(v) = \{v_1, v_2, a_1\}$, then $\{a_2, a_3, a_5, v\}$ is an independent set which contradicts $\alpha(G) = 3$.

Subcase 1.2. G does not contain an induced subgraph $3K_2$.

Let $A = \{a_1, a_2, a_3\}$ be a maximum independent set of G. Since $n \geq 7$, we have $n - 2 \geq 5$. By induction hypothesis, $G - A$ contains a star S_{n-2} with center v_1. If $d_A(v_1) \geq 2$, then G contains a star S_n. Hence $d_A(v_1) \leq 1$. Since A is a maximum independent set of G, we have $d_A(v_1) = 1$. Assume $N_A(v_1) = \{a_1\}$ and $A_1 = A \cup \{v_1\}$. Since $n \geq 7$, we have $n - 2 \geq 5$. By induction hypothesis, $G - A_1$ contains a star S_{n-2} with center v_2. For the same reason as above, we have $d_{A_1}(v_2) = 1$. If $N_{A_1}(v_2) \cap \{a_2, a_3\} = \emptyset$, then $A \cup \{v_2\}$ or $\{a_2, a_3, v_1, v_2\}$ is an independent set which contradicts $\alpha(G) = 3$. Thus we may assume $N_{A_1}(v_2) = \{a_2\}$.

Let $X = \{a_1, a_2, v_1, v_2\}$ and $Y = V(G) - N[a_3] \cup X$. Since $\overline{G}$ contains no W_6, we have the following claims.

Claim 1. *For any vertex $y \in Y$, $d_X(y) \geq 2$ and if $d_X(y) = 2$, then $N_X(y) = \{a_1, v_1\}$ or $\{a_2, v_2\}$.*

Proof. If $d_X(y) \leq 1$, say $N_X(y) \cap (X - \{a_1\}) = \emptyset$, then $\{v_1, v_2, a_3, y\}$ is an independent set which contradicts $\alpha(G) = 3$. As for the latter part, the proof is similar. □

Claim 2. *For any vertex $y \in Y$, there is some vertex $y' \in Y$ such that $yy' \notin E(G)$.*

Proof. Since G contains no S_n, we have $|N[a_3]| \leq n - 1$. Noting that $|X| = 4$, we have $|Y| \geq n - 2$. If there is some vertex $y \in Y$ such that $Y - \{y\} \subseteq N(y)$, then by Claim 1, we have $d(y) \geq n - 1$ which implies G contains a star S_n, a contradiction. □

Claim 3. *For any two vertices $y_1, y_2 \in Y$ with $y_1y_2 \notin E(G)$, $d_X(y_1) + d_X(y_2) \geq 6$.*

Proof. Assume $d_X(y_1) \leq d_X(y_2)$. If $d_X(y_1) + d_X(y_2) \leq 5$, then $d_X(y_1) \leq 2$. Thus by Claim 1 we may assume $N_X(y_1) = \{a_1, v_1\}$. Since $\alpha(G) = 3$ and $y_1y_2 \notin E(G)$, $\{a_3, y_1, y_2\}$ is a maximum independent set of G which implies $\{a_2, v_2\} \subseteq N_X(y_2)$. Since $d_X(y_1) + d_X(y_2) \leq 5$, we have $\{a_1, v_1\} \nsubseteq N_X(y_2)$. Assume $a_1 \notin N_X(y_2)$, then

$y_1y_2a_1a_2v_1v_2y_1$ is a C_6 in $\overline{G}$. Noting that $X \cup \{y_1, y_2\} \subseteq V(G) - N[a_3]$, $\overline{G}$ contains a W_6 with the hub a_3, a contradiction. □

Let $Y_0 = \{y \mid y \in Y \text{ and } d_X(y) = 2\}$.

Claim 4. *For any two vertices $y_1, y_2 \in Y_0$, $N_X(y_1) = N_X(y_2)$.*

Proof. Otherwise we may assume $N_X(y_i) = \{a_i, v_i\}$ by Claim 1, where $i = 1, 2$. In this case, it is not difficult to see that $\overline{G}$ contains a W_6 with $V(W_6) = X \cup \{a_3, y_1, y_2\}$ and the hub a_3, a contradiction. □

Claim 5. $d_Y(X) \le 3|Y| - 3$.

Proof. Let $N(a_3) = B$. Since G does not contain an induced subgraph $3K_2$, we have $d_X(b) \ge 1$ for any $b \in B$. Thus we have $d_B(X) \ge |B|$.

If $d_Y(X) \ge 3|Y| - 2$, then since $d_B(X) \ge |B|$, we have $d_Y(X) + d_B(X) \ge 3|Y| + |B| - 2$. Noting that $|X| = 4$, we have $d_Y(X) + d_B(X) \ge 3|Y| + |B| - 2 = 3(2n - 4 - |B|) + |B| - 2 = 6n - 14 - 2|B|$. Since G contains no star S_n, we have $|B| \le n - 2$. Thus we have $d_Y(X) + d_B(X) \ge 6n - 14 - 2(n - 2) = 4n - 10$ which implies there is some vertex $x \in X$ such that $d_Y(x) + d_B(x) \ge n - 2$. Since $d_X(x) = 1$, we have $d(x) \ge n - 1$ which implies G contains a star S_n, a contradiction. □

If $|Y_0| \le 2$, then by Claim 1, we have $d_Y(X) \ge 3|Y| - 2$ which contradicts Claim 5. Hence $|Y_0| \ge 3$. If $\overline{G}[Y]$ contains a matching M which saturates Y_0, then by Claim 3, we have $d_Y(X) \ge \sum_{y \in V(M)} d_X(y) + \sum_{y \in Y - V(M)} d_X(y) \ge 3|Y|$ which contradicts Claim 5. Hence $\overline{G}[Y]$ contains no matching M which saturates Y_0. Thus by Claim 2, there are two vertices $y_1, y_2 \in Y_0$ and a vertex $y_0 \in Y$ such that $y_0y_1, y_0y_2 \notin E(G)$. By Claim 4, we may assume $N_X(y) = \{a_1, v_1\}$ for any vertex $y \in Y_0$. Since $|Y_0| \ge 3$, we can choose a vertex $y_3 \in Y_0 - \{y_1, y_2\}$. It is not difficult to see that $y_0y_1a_2y_3v_2y_2y_0$ is a C_6 in $\overline{G}[X \cup Y]$. Thus, noting that $X \cup Y = V(G) - N[a_3]$, we can see that $\overline{G}$ contains a W_6 with the hub a_3, a contradiction.

Case 2. $\alpha(G) = 4$.

In this case, we first show the following claim.

Claim 6. *G has at least one of the following graphs as an induced subgraph: $3K_1 \cup K_3$, $2K_1 \cup P_4$ and $2K_1 \cup 2K_2$.*

Proof. Let $A = \{a_1, a_2, a_3, a_4\}$ be a maximum independent set of G. Then $d_A(v) \ge 1$ for any vertex $v \in V(G) - A$. If there is at most one vertex, say v in $V(G) - A$ such that $d_A(v) = 1$, then $d(A) \ge 2(2n - 3) - 1 = 4n - 7$ which implies there is at least one vertex $a \in A$ such that $d(a) \ge n - 1$ and hence G contains a star S_n, a contradiction. Thus there are at least two vertices in $V(G) - A$, say v_1, v_2, such that $d_A(v_1) = d_A(v_2) = 1$. If $N_A(v_1) = N_A(v_2)$, then G contains $3K_1 \cup K_3$ as an induced subgraph. If $N_A(v_1) \ne N_A(v_2)$ and $v_1v_2 \in E(G)$, then G contains $2K_1 \cup P_4$ as an induced subgraph. If $N_A(v_1) \ne N_A(v_2)$ and $v_1v_2 \notin E(G)$, then G contains $2K_1 \cup 2K_2$ as an induced subgraph. □

By Claim 6, we need to consider the following three cases separately.

Subcase 2.1. G contains an induced subgraph $G_0 = 3K_1 \cup K_3$.

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Assume that $V(G_0) = V_0 = A \cup B$ and $E(G_0) = \{a_1a_2, a_2a_3, a_1a_3\}$. Since $n \geq 7$, we have $n - 3 \geq 4$. By induction hypothesis, $G - V_0$ contains a star S_{n-3} with center v. It is easy to see that $1 \leq d_{V_0}(v) \leq 2$. If $d_{V_0}(v) = 1$, then since $\alpha(G) = 4$, we have $N_{V_0}(v) \cap A = \emptyset$ and hence we may assume $N_{V_0}(v) = \{b_1\}$. If $d_{V_0}(v) = 2$, then since $\alpha(G) = 4$, we have $d_A(v) \leq 1$. If $d_A(v) = 1$, we may assume $N_{V_0}(v) = \{a_1, b_1\}$. If $d_A(v) = 0$, we may assume $N_{V_0}(v) = \{b_1, b_2\}$. Thus $\overline{G}[V_0 \cup \{v\}]$ contains a W_6 with the hub b_3 in any case, a contradiction.

Subcase 2.2. G contains an induced subgraph $G_0 = 2K_1 \cup P_4$.

Let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2\}$. Assume $V(G_0) = V_0 = A \cup B$ and $E(G_0) = \{a_1a_2, a_2a_3, a_3a_4\}$. Since $n \geq 7$, we have $n - 3 \geq 4$. By induction hypothesis, $G - V_0$ contains a star S_{n-3} with center v_1. It is easy to see that $1 \leq d_{V_0}(v_1) \leq 2$. If $d_{V_0}(v_1) = 1$, then since $\alpha(G) = 4$, we have $N_{V_0}(v_1) \cap A = \emptyset$ and hence we may assume $N_{V_0}(v_1) = \{b_1\}$. Thus $\overline{G}[V_0 \cup \{v_1\}]$ contains a W_6 with the hub b_2, a contradiction. Hence we may assume $d_{V_0}(v_1) = 2$. If $d_B(v_1) = 0$, then since $\alpha(G) = 4$, $A - N_{V_0}(v_1)$ must be a clique. Thus by symmetry we may assume $N_{V_0}(v_1) = \{a_1, a_2\}$ or $\{a_1, a_4\}$. If $d_B(v_1) = 1$, then by symmetry we may assume $N_{V_0}(v_1) = \{a_1, b_1\}$ or $\{a_2, b_1\}$. Thus, it is not difficult to see $\overline{G}[V_0 \cup \{v_1\}]$ contains a W_6 with the hub b_2 if $d_B(v_1) \leq 1$, a contradiction. Hence we may assume $d_B(v_1) = 2$.

Let $V_1 = V_0 \cup \{v_1\}$, then $G[V_1] = P_3 \cup P_4$. Since $n \geq 7$, we have $n - 4 \geq 3$. By induction hypothesis, $G - V_1$ contains a star S_{n-4} with center v_2. Obviously, $1 \leq d_{V_1}(v_2) \leq 3$. If $d_{V_1}(v_2) = 1$, then since $\alpha(G) = 4$, we have $N_{V_1}(v_2) \subseteq B$. Assume $N_{V_1}(v_2) = \{b_1\}$, then $\overline{G}[V_0 \cup \{v_2\}]$ contains a W_6 with the hub b_2, a contradiction. Hence we have $d_{V_1}(v_2) \geq 2$. Now, let $d_{V_1}(v_2) = 2$. If $d_A(v_2) = 2$, then since $\alpha(G) = 4$, we may assume $N_{V_1}(v_2) = \{a_1, a_2\}$ or $\{a_1, a_4\}$. If $d_A(v_2) = 1$, then since $\alpha(G) = 4$, we have $N_{V_1}(v_2) \cap B \neq \emptyset$. By symmetry we may assume $N_{V_1}(v_2) = \{a_1, b_1\}$ or $\{a_2, b_1\}$. Thus, it is not difficult to see $\overline{G}[V_0 \cup \{v_2\}]$ contains a W_6 with the hub b_2 if $d_A(v_2) \geq 1$, a contradiction. If $d_A(v_2) = 0$, then by symmetry we may assume $N_{V_1}(v_2) = \{b_1, v_1\}$ or $\{b_1, b_2\}$. Thus, $\overline{G}[V_0 \cup \{v_2\}]$ contains a W_6 with the hub b_2 in the former case and $\overline{G}[V_1 \cup \{v_2\} - \{a_2\}]$ contains a W_6 with the hub a_1 in the latter case, a contradiction. Therefore we have $d_{V_1}(v_2) = 3$.

If $d_A(v_2) = 3$, then we may assume $N_{V_1}(v_2) = \{a_1, a_2, a_3\}$ or $\{a_1, a_2, a_4\}$. Thus, $\overline{G}[V_1 \cup \{v_2\} - \{a_3\}]$ contains a W_6 with the hub a_4 in the former case and $\overline{G}[V_0 \cup \{v_2\}]$ contains a W_6 with the hub b_2 in the latter case, a contradiction.

If $d_A(v_2) = 2$ and $v_1 \in N_{V_1}(v_2)$, then since $\alpha(G) = 4$, $A - N(v_2)$ must be a clique. Thus, we may assume by symmetry that $N_{V_1}(v_2) = \{a_1, a_2, v_1\}$ or $\{a_1, a_4, v_1\}$. If $d_A(v_2) = 2$ and $v_1 \notin N_{V_1}(v_2)$, then by symmetry we may assume $N_{V_1}(v_2) = \{a_1, a_2, b_1\}$ or $\{a_1, a_3, b_1\}$ or $\{a_1, a_4, b_1\}$ or $\{a_2, a_3, b_1\}$. Thus, $\overline{G}[V_0 \cup \{v_2\}]$ contains a W_6 with the hub b_2 in all the cases above, a contradiction.

If $d_A(v_2) = 1$, then by symmetry we may assume $N_{V_1}(v_2) = \{a_1, b_1, v_1\}$ or $\{a_2, b_1, v_1\}$ or $\{a_1, b_1, b_2\}$ or $\{a_2, b_1, b_2\}$. It is not difficult to check that $\overline{G}[V_1 \cup \{v_2\} - \{a_3\}]$ contains a W_6 with the hub a_4 in all the cases above, a contradiction.

If $d_A(v_2) = 0$, then $N_{V_1}(v_2) = \{b_1, v_1, b_2\}$. In this case, we let $V_2 = V_1 \cup \{v_2\}$. Since $n \geq 7$, we have $n - 4 \geq 3$. By induction hypothesis, $G - V_2$ contains a star S_{n-4} with center v. Obviously, $1 \leq d_{V_1}(v) \leq 3$. By the analogous argument as before, we can obtain $N_{V_2}(v) = \{b_1, v_1, b_2\}$ which implies $v_2 v \notin E(G)$, otherwise G contains a star S_n. Thus, $\overline{G}[V_2 \cup \{v\} - \{a_2, v_1\}]$ contains a W_6 with the hub a_1, a contradiction.

Subcase 2.3. G contains an induced subgraph $G_0 = 2K_1 \cup 2K_2$.

Using an analogous argument as Subcase 2.2, we can see $\overline{G}$ contains a W_6, a contradiction.

Case 3. $\alpha(G) = 5$ or 6.

Let $A = \{a_i \mid 1 \leq i \leq k\}$ be a maximum independent set of G. Since $n \geq 7$, we have $n - 3 \geq 4$. By induction hypothesis, $G - A$ contains a star S_{n-3} with the center u. Obviously $d_A(u) = 1$ or 2.

If $k = 5$, we let $A_1 = A \cup \{u\}$. If $d_A(u) = 1$, we assume $a_1 u \in E(G)$. If $d_A(u) = 2$, we assume $a_1 u, a_2 u \in E(G)$. By induction hypothesis, $G - A_1$ contains a star S_{n-3} with the center v. Since $d_{A_1}(v) = 1$ or 2, it is not difficult to check that $\overline{G}[A_1 \cup \{v\}]$ contains a W_6 in any case, a contradiction.

If $k = 6$, then since $d_A(u) = 1$ or 2, we can see $\overline{G}[A \cup \{u\}]$ contains a W_6, again a contradiction.

Up to now, we have $R(S_n, W_6) \leq 2n + 1$ and hence $R(S_n, W_6) = 2n + 1$.

The proof of Theorem 1 is completed. □

4. Proof of Theorem 2

Proof of Theorem 2. Let G be a graph of order $3n - 2$. If G contains no S_n, then $\Delta(G) \leq n - 2$ which implies $\delta(\overline{G}) \geq (3n - 3) - (n - 2) = 2n - 1$. Let v be any vertex of $V(G)$ and $d_{\overline{G}}(v) = (2n - 1) + k$, where $k \geq 0$. Assume $F = \overline{G}[N_{\overline{G}}(v)]$. We now show F is pancyclic. Since $|F| = (2n - 1) + k$ and $\delta(\overline{G}) \geq 2n - 1$, we have $\delta(F) \geq 2n - 1 - [(3n - 2) - (2n - 1 + k)] = n + k$. Noting that $k \geq 0$, we have $\delta(F) \geq n + k > (2n - 1 + k)/2 = |F|/2$ which implies F is pancyclic by Lemma 1, that is, F contains C_i for $3 \leq i \leq 2n - 1$. Since $m \leq n + 1$, we can see $\overline{G}$ contains a W_m with the hub v and hence $R(S_n, W_m) \leq 3n - 2$. On the other hand, it is not difficult to see neither $3K_{n-1}$ contains S_n nor its complement contains W_m for odd m. Thus we have $R(S_n, W_m) \geq 3n - 2$ and hence $R(S_n, W_m) = 3n - 2$. □

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