The Ramsey numbers of stars versus wheels

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Abstract

For two given graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the smallest positive integer $n$ such that for any graph $G$ of order $n$, either $G$ contains $G_1$ or the complement of $G$ contains $G_2$. Let $S_n$ denote a star of order $n$ and $W_m$ a wheel of order $m+1$. This paper shows that $R(S_n, W_6) = 2n + 1$ for $n \geq 3$ and $R(S_n, W_m) = 3n - 2$ for $m$ odd and $n \geq m - 1 \geq 2$.

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1. Introduction

All graphs considered in this paper are finite simple graph without loops. For two given graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the smallest positive integer $n$ such that for any graph $G$ of order $n$, either $G$ contains $G_1$ or the complement of $G$ contains $G_2$, where $\overline{G}$ is the complement of $G$. Let $G = (V(G), E(G))$ be a graph. The neighborhood of vertex $v$ is denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$. For a vertex $v \in V(G)$ and a subgraph $H$ of $G$, $N_H(v) = N(v) \cap V(H)$. Let $d_H(v) = |N_H(v)|$. For two vertex disjoint sets $S$ and $T$, we define $d_T(S) = \sum_{s \in S} d_T(s)$. The connectivity, independence number, maximum degree and minimum degree of $G$ are denoted by $\kappa(G)$, $\alpha(G)$, $\Delta(G)$ and $\delta(G)$, respectively. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by $S$ in $G$. A complete graph of order $n$ is denoted by $K_n$. A complete bipartite graph of order $m + n$ is denoted by $K_{m,n}$ and a Star $S_n$ is $K_{1,n-1}$. A path and a cycle of order $n$ are denoted by $P_n$ and $C_n$, respectively. Let $m$ be a positive integer and $G$ a graph, we use $mG$ to denote $m$ vertex disjoint copies of $G$. A Wheel $W_n = \{x\} + C_n$ is a graph of $n + 1$ vertices, $x$ called the hub of the wheel. The length of a shortest and longest cycle of $G$ are denoted by $g(G)$ and $c(G)$, respectively.
A graph on \( n \) vertices is \textit{pancyclic} if it contains cycles of every length \( l, 3 \leq l \leq n \) and \textit{weakly pancyclic} if it contains cycles of every length \( l, g(G) \leq l \leq c(G) \).

Ramsey theory studies conditions when a combinatorial object contains necessarily some smaller given objects. The role of Ramsey number is to quantify some of the general existential theorems in Ramsey theory. The classical Ramsey number is \( R(k, l) \) for complete graphs. Since it is very difficult to determine \( R(k, l) \), people turn to consider Ramsey numbers concerning general graph results, such as Ramsey numbers of path versus cycle, cycle versus star, tree versus wheel and so on, see for instance [1, 4–6, 8]. Recently, the following results are obtained.

**Theorem A** (Surahmat and Baskoro [9]). \( R(S_n, W_4) = 2n - 1 \) for \( n \geq 3 \) and \( n \equiv 1 \) (mod 2) and \( R(S_n, W_4) = 2n + 1 \) for \( n \geq 4 \) and \( n \equiv 0 \) (mod 2).

**Theorem B** (Surahmat and Baskoro [9]). \( R(S_n, W_5) = 3n - 2 \) for \( n \geq 4 \).

**Theorem C** (Baskoro et al. [1]). Let \( T_n \) be a tree other than \( S_n \), then \( R(T_n, W_4) = 2n - 1 \) for \( n \geq 3 \) and \( R(T_n, W_5) = 3n - 2 \) for \( n \geq 4 \).

Furthermore, motivated by **Theorem C**, Baskoro et al. [1] posed the following.

**Conjecture 1.** Let \( T_n \) be a tree other than \( S_n \) and \( n \geq m - 1 \). Then \( R(T_n, W_m) = 2n - 1 \) for \( m \geq 6 \) and even, and \( R(T_n, W_m) = 3n - 2 \) for \( m \geq 7 \) and odd.

In this paper, we consider the Ramsey numbers of star versus wheel in a more general situation. The main results of this paper are the following.

**Theorem 1.** \( R(S_n, W_6) = 2n + 1 \) for \( n \geq 3 \).

**Theorem 2.** \( R(S_n, W_m) = 3n - 2 \) for \( m \) odd and \( n \geq m - 1 \geq 2 \).

**Remark.** By **Theorem 2**, we can see that \( R(S_n, W_m) \) is a function of \( n \) if \( m \) is odd. However, it is not the case when \( m \) is even. In fact, if \( m \) is even, then \( R(S_n, W_m) \) is a function related to both \( n \) and \( m \) as can be seen by the following examples.

Let \( m \geq 6 \) be an even integer, \( n = km/2 + 2 \), where \( k \geq 2 \) is an integer, and \( G = H \cup K_{n-1} \), where \( H = (k + 1)K_{m/2} \). Obviously, \( G \) is a graph of order \( 2n + m/2 - 3 \) and \( \Delta(G) = n - 2 \) and hence \( G \) contains no \( S_n \). It is not difficult to see \( G \) contains no \( W_m \). Thus we have \( R(S_n, W_m) \geq 2n + m/2 - 2 \) if \( n = km/2 + 2 \) for some integer \( k \geq 2 \).

**Problem 1.** Determine \( R(S_n, W_m) \) for \( m \) even and \( n \geq m - 1 \geq 7 \).

## 2. Some lemmas

In order to prove our results, we need the following lemmas.

**Lemma 1** (Bondy [2]). Let \( G \) be a graph of order \( n \). If \( \delta(G) \geq n/2 \), then either \( G \) is pancyclic or \( n \) is even and \( G = K_{n/2,n/2} \).

**Lemma 2** (Dirac [7]). Let \( G \) be a 2-connected graph of order \( n \geq 3 \) with \( \delta(G) = \delta \). Then \( c(G) \geq \min(2\delta, n) \).
Lemma 3 (Brandt [3]). Every non-bipartite graph $G$ of order $n$ with $\delta(G) \geq (n + 2)/3$ is weakly pancyclic.

Lemma 4. Let $G$ be a 2-connected graph of order 8 with $\delta(G) = 3$. Then $G$ contains a $C_6$.

**Proof.** Let $V(G) = \{v_i \mid 1 \leq i \leq 8\}$ and $C = v_1v_2\cdots v_k$ a longest cycle of $G$. By Lemma 2, $k \geq 6$. If $k = 6$, we are done. If $k = 7$, then by the maximality of $C$, $v_9$ has no two consecutive neighbors on $C$. Since $\delta(G) = 3$, we may assume $N(v_9) = \{v_1, v_3, v_5\}$. Thus $v_1v_2v_3v_4v_5v_6v_1$ is a $C_6$. If $k = 8$, we assume $d(v_1) = 3$ and hence $C$ has a chord $v_1v_7$. If $i \in \{4, 6\}$, then $G$ contains a $C_6$. Hence $i \in \{3, 5, 7\}$. By symmetry, we may assume $i \in \{3, 5\}$. Since $\delta(G) = 3$, $C$ has a chord $v_5v_7$. By an analogous argument as above, we have $j \in \{1, 3, 7\}$. If $i = 3$, then $j \neq 1$ and hence $j \in \{3, 7\}$ which implies $G$ contains a $C_6$. Hence we have $i = 5$. In this case, $v_5v_7 \notin E(G)$ for otherwise $v_1v_2v_3v_7v_8v_1$ is a $C_6$. If $\{v_2, v_4\} \cap N(v_7) \neq \emptyset$, then $G$ contains a $C_6$. Hence we may assume $v_2v_7, v_4v_7 \notin E(G)$. Thus noting that $d(v_1) = 3$ and $\delta(G) = 3$, we have $v_7v_5 \in E(G)$. By symmetry, we have $v_3v_5 \in E(G)$ which implies $v_1v_2v_3v_5v_7v_8v_1$ is a $C_6$. \[\Box\]

Lemma 5. Let $G$ be a 2-connected graph of order 9 with $\delta(G) = 4$. Then $G$ contains a $C_6$.

**Proof.** Let $V(G) = \{v_i \mid 1 \leq i \leq 9\}$ and $C = v_1v_2\cdots v_k$ a longest cycle of $G$. By Lemma 2, $k \geq 8$. If $k = 8$, then by the maximality of $C$, $v_9$ has no two consecutive neighbors on $C$. Since $\delta(G) = 4$, we may assume $N(v_9) = \{v_1, v_3, v_5, v_7\}$. Thus $g(G) \leq 4$. If $G$ is non-bipartite, then $G$ contains a $C_6$ by Lemma 3. If $G$ is bipartite, then since $\delta(G) = 4$, it is not difficult to see that $G = K_{4,5}$ and hence $G$ contains a $C_6$. If $k = 9$, then $G$ is non-bipartite. Since $\delta(G) = 4$, $C$ has a chord which implies $g(G) \leq 5$. Thus $G$ contains a $C_6$ by Lemma 3. \[\Box\]

3. **Proof of Theorem 1**

**Proof of Theorem 1.** Let $n \geq 3$ be an integer and $G = H \cup K_{n-1}$, where $\overline{H} = C_{n+1}$ if $n \neq 5$ and $\overline{H} = 2C_3$ if $n = 5$. Obviously, $|G| = 2n$. It is not difficult to see neither $G$ contains a star $S_n$ nor $\overline{G}$ contains a $W_6$ and hence $R(S_n, W_6) \geq 2n + 1$.

In order to show $R(S_n, W_6) \geq 2n + 1$, we use induction on $n$. Let $G$ be a graph of order $2n + 1$. As the basis of induction, we first show $R(S_n, W_6) = 2n + 1$ for $3 \leq n \leq 6$.

Suppose $G$ contains no $S_n$. Then $\Delta(G) \leq n - 2$ which implies $\delta(\overline{G}) \geq n + 2$. Let $v \in V(G)$ be a vertex such that $d_{\overline{G}}(v) = d = \Delta(\overline{G}) = n + 2 + k$, where $k \geq 0$. Set $N_{\overline{G}}(v) = V_0$, $U = V(G) - V_0 \cup \{v\}$ and $F = \overline{G}[V_0]$. It is not difficult to see that $\delta(F) \geq 3 + k$. If $v_1 \in V_0$ and $d_F(v_1) = 3 + k$, then we must have

$$U \subseteq N_{\overline{G}}(v_1). \hspace{1cm} (1)$$

If $n = 3$, then we can see $\overline{G}$ contains a $W_6$. If $n = 4$, then $\delta(F) \geq 3 + k \geq (6 + k)/2 = |V_0|/2$ which implies $F$ contains a $C_6$ by Lemma 1. If $n = 5$, then $\delta(\overline{G}) \geq 7$. We have $d \geq 8$ since the number of vertices of odd degree is even, which implies $k \geq 1$ and hence
we have $\delta(F) \geq 3 + k \geq (7 + k)/2 = |V_0|/2$ which implies $F$ contains a $C_6$ by Lemma 1. Thus $G$ contains a $W_6$ with the hub $v$ when $n = 4, 5$ and hence we may assume $n = 6$.

If $k \geq 2$, then $\delta(F) \geq 3 + k \geq (8 + k)/2 = |V_0|/2$ which implies $F$ contains a $C_6$ by Lemma 1 and hence $G$ contains a $W_6$. Thus we may assume $k \leq 1$.

If $k = 0$, then $d = 8$. If $\delta(F) \geq 4$, then by Lemma 1, $F$ contains a $C_6$ and hence $G$ contains a $W_6$. Thus we have $\delta(F) = 3$. If $F$ is not connected, then since $\delta(F) = 3$, we can see $F = 2K_4$. By (1), $U \subseteq \mathcal{N}_G(v_i)$ for any $v_i \in V_0$ and hence $G$ contains a $W_6$ with the hub $v_i$ for any $v_i \in V_0$. If $\kappa(F) = 1$, we let $w$ be a cut-vertex and $H_1$ a component of $F - w$ such that $|H_1|$ is as small as possible. Then $|H_1| = 3$, $V(H_1) \cup \{w\}$ is a 4-clique and $d_F(h) = 3$ for any $h \in V(H_1)$. Let $V(H_1) = \{h_1, h_2, h_3\}$. If $|\mathcal{N}_G(w) \cap U| \leq 1$, then $\mathcal{G}[U]$ contains at least two edges since $\delta(G) \geq 8$. Let $U = \{u_1, u_2, u_3, u_4\}$. If $\mathcal{G}[U]$ contains a $P_3$, say $F = u_1u_2u_3$, then by (1), $h_2u_1h_3u_2\bar{h}h_2$ is a $C_6$ and hence $G$ contains a $W_6$ with the hub $h_1$. If $\mathcal{G}[U]$ contains no $P_3$, then $\mathcal{G}[U] = 2K_2$. Assume $E(\mathcal{G}[U]) = \{u_1u_2, u_3u_4\}$. Then by (1), $h_2u_1h_3u_2h_1h_2$ is a $C_6$ and hence $G$ contains a $W_6$ with the hub $h_1$. Thus we may assume $|N_G(w) \cap U| \geq 2$. Let $u_1, u_2 \in N_G(w)$, then $h_2u_1u_2h_3u_2h_2$ is a $C_6$ and hence $G$ contains a $W_6$ with the hub $v$. Thus, we have $R(S_n, W_6) = 2n + 1$ for $3 \leq n \leq 6$.

Now, assume $n \geq 7$ and Theorem 1 holds for smaller values of $n$.

If $\mathcal{G}$ contains no $W_6$, then we have $\alpha(G) \leq 6$. If $\alpha(G) \leq 2$, then $\Delta(G) \geq n$ which implies $G$ contains a star $S_n$. Hence we may assume $3 \leq \alpha(G) \leq 6$ and consider the following three cases separately.

**Case 1.** $\alpha(G) = 3$.

We consider the following two subcases separately.

**Subcase 1.1.** $G$ contains an induced subgraph $G_0 = 3K_2$.

Let $V(G_0) = V_0 = \{a_i | 1 \leq i \leq 6\}$ and $E(G_0) = \{a_1a_2, a_3a_4, a_5a_6\}$. Since $n \geq 7$, we have $n - 3 \geq 4$. By induction hypothesis, $G - V_0$ contains a star $S_{n-3}$ with center $v_1$. Since $\alpha(G) = 3$ and both $\{a_1, a_3, a_5\}$ and $\{a_2, a_4, a_6\}$ are independent sets, we have $|N_{V_0}(v_1)| \geq 2$. If $d_{V_1}(v_1) \geq 3$, then $G$ contains a star $S_n$ with center $v_1$. Hence we may assume $|N_{V_0}(v_1)| = 2$. Assume without loss of generality that $a_1 \in N(v_1)$. Then $a_2 \in N(v_1)$ for otherwise we can obtain an independent set of order 4. Thus we have $N_{V_0}(v_1) = \{a_1, a_2\}$.

Let $V_1 = V_0 \cup \{v_1\}$. Obviously, $G[V_1] = 2K_2 \cup K_3$. Since $n \geq 7$, we have $n - 4 \geq 3$. By induction hypothesis, $G - V_1$ contains a star $S_{n-4}$ with center $v_2$. For the same reason as above, we have $d_{V_1}(v_2) = 2$ or 3 and if $d_{V_1}(v_2) = 2$, then $N_{V_1}(v_2) = \{a_3, a_4\}$ or $\{a_5, a_6\}$. Assume $N_{V_1}(v_2) = \{a_3, a_4\}$, then it is no difficult to see that $\mathcal{G}[V_1 - \{a_6\} \cup \{v_2\}]$ contains
a $W_6$ with the hub $a_5$, a contradiction. Hence we have $d_{V_1}(v_2) = 3$. Let $U = \{a_1, a_2, v_1\}$. If $d_{U}(v_2) = 0$, we may assume that $N_{V_1}(v_2) = \{a_3, a_4, a_5\}$. If $d_{U}(v_2) = 1$, then since $\alpha(G) = 3$, we may assume $a_3, a_4 \in N_{V_1}(v_2)$. Thus we can see $\overrightarrow{G}[V_1 \cup \{v_2\} - \{a_5\}]$ contains a $W_6$ with the hub $a_6$ if $d_{U}(v_2) \leq 1$, a contradiction. If $d_{U}(v_2) = 2$, we may assume $N_{V_1}(v_2) = \{a_1, a_2, a_3\}$. Thus, $\{v_1, v_2, a_3, a_5\}$ is an independent set which contradicts $\alpha(G) = 3$. Hence we have $d_{U}(v_2) = 3$.

Let $V_2 = V_1 \cup \{v_2\}$. Clearly, $G[V_2] = 2K_2 \cup K_4$. Since $n \geq 7$, we have $n - 4 \geq 3$. By induction hypothesis, $G - V_2$ contains a star $S_{n-4}$ with center $v$. For the same reason as above, we have $d_{V_2}(v) = 3$. Let $U_1 = U \cup \{v_2\}$. If $d_{U_1}(v) = 0$, then we may assume $N_{V_2}(v) = \{a_3, a_4, a_5\}$. If $d_{U_1}(v) = 1$, say $a_1 \in N(v)$, then since $\alpha(G) = 3$, we may assume $a_3, a_4 \in N_{V_2}(v)$. Thus we see $\overrightarrow{G}[V_2 \cup \{v\} - \{a_1, a_3\}]$ contains a $W_6$ with the hub $a_6$ if $d_{U_1}(v) \leq 1$, a contradiction. If $d_{U_1}(v) = 2$, we may assume that $N_{V_2}(v) = \{a_1, a_2, a_3\}$. Thus, $\{v_1, a_4, a_5, v\}$ is an independent set which contradicts $\alpha(G) = 3$. If $d_{U_1}(v) = 3$, say $N_{V_2}(v) = \{v_1, v_2, a_1\}$, then $\{a_2, a_3, a_5, v\}$ is an independent set which contradicts $\alpha(G) = 3$.

### Subcase 1.2.

$G$ does not contain an induced subgraph $3K_2$.

Let $A = \{a_1, a_2, a_3\}$ be a maximum independent set of $G$. Since $n \geq 7$, we have $n - 2 \geq 5$. By induction hypothesis, $G - A$ contains a star $S_{n-2}$ with center $v_1$. If $d_A(v_1) \geq 2$, then $G$ contains a star $S_n$. Hence $d_A(v_1) \leq 1$. Assume $N_A(v_1) = \{a_1\}$ and $A_1 = A \cup \{v_1\}$. Since $n \geq 7$, we have $n - 2 \geq 5$. By induction hypothesis, $G - A_1$ contains a star $S_{n-2}$ with center $v_2$. For the same reason as above, we have $d_{A_1}(v_2) = 1$. If $N_{A_1}(v_2) \cap \{a_2, a_3\} = \emptyset$, then $A \cup \{v_2\}$ or $\{a_2, a_3, v_1, v_2\}$ is an independent set which contradicts $\alpha(G) = 3$. Thus we may assume $N_{A_1}(v_2) = \{a_2\}$.

Let $X = \{a_1, a_2, v_1, v_2\}$ and $Y = V(G) - N[a_3] \cup X$. Since $\overrightarrow{G}$ contains no $W_6$, we have the following claims.

**Claim 1.** For any vertex $y \in Y$, $d_{X}(y) \geq 2$ and if $d_{X}(y) = 2$, then $N_X(y) = \{a_1, v_1\}$ or $\{a_2, v_2\}$.

**Proof.** If $d_{X}(y) \leq 1$, say $N_X(y) \cap (X - \{a_1\}) = \emptyset$, then $\{v_1, v_2, a_3, y\}$ is an independent set which contradicts $\alpha(G) = 3$. As for the latter part, the proof is similar. □

**Claim 2.** For any vertex $y \in Y$, there is some vertex $y' \in Y$ such that $yy' \notin E(G)$.

**Proof.** Since $G$ contains no $S_n$, we have $|N[a_3]| \leq n - 1$. Noting that $|X| = 4$, we have $|Y| \geq n - 2$. If there is some vertex $y \in Y$ such that $Y - \{y\} \subseteq N(y)$, then by Claim 1, we have $d(y) \geq n - 1$ which implies $G$ contains a star $S_n$, a contradiction. □

**Claim 3.** For any two vertices $y_1, y_2 \in Y$ with $y_1 y_2 \notin E(G)$, $d_{X}(y_1) + d_{X}(y_2) \geq 6$.

**Proof.** Assume $d_{X}(y_1) \leq d_{X}(y_2)$. If $d_{X}(y_1) + d_{X}(y_2) \leq 5$, then $d_{X}(y_1) \leq 2$. Thus by Claim 1 we may assume $N_X(y_1) = \{a_1, v_1\}$. Since $\alpha(G) = 3$ and $y_1 y_2 \notin E(G)$, $\{a_3, y_1, y_2\}$ is a maximum independent set of $G$ which implies $\{a_2, v_2\} \subseteq N_X(y_2)$. Since $d_{X}(y_1) + d_{X}(y_2) \leq 5$, we have $\{a_1, v_1\} \not\subseteq N_X(y_2)$. Assume $a_1 \notin N_X(y_2)$, then
\[y_1 y_2 a_1 a_2 v_1 v_2 y_3\] is a \(C_6\) in \(G\). Noting that \(X \cup \{y_1, y_2\} \subseteq V(G) - N[a_3],\) \(\overline{G}\) contains a \(W_6\) with the hub \(a_3\), a contradiction. \(\square\)

Let \(Y_0 = \{y \mid y \in Y \text{ and } d_X(y) = 2\}\).

Claim 4. For any two vertices \(y_1, y_2 \in Y_0\), \(N_X(y_1) = N_X(y_2)\).

Proof. Otherwise we may assume \(N_X(y_i) = \{a_i, v_i\}\) by Claim 1, where \(i = 1, 2\). In this case, it is not difficult to see that \(\overline{G}\) contains a \(W_6\) with \(V(W_6) = X \cup \{a_3, y_1, y_2\}\) and the hub \(a_3\), a contradiction. \(\square\)

Claim 5. \(d_Y(X) \leq 3|Y| - 3\).

Proof. Let \(N(a_3) = B\). Since \(G\) does not contain an induced subgraph \(3K_2\), we have \(d_X(b) \geq 1\) for any \(b \in B\). Thus we have \(d_B(X) \geq |B|\).

If \(d_Y(X) \geq 3|Y| - 2\), then since \(d_B(X) \geq |B|\), we have \(d_Y(X) + d_B(X) \geq 3|Y| + |B| - 2\). Noting that \(|X| = 4\), we have \(d_Y(X) + d_B(X) \geq 3|Y| + |B| - 2 = 3(2n - 4 - |B|) + |B| - 2 = 6n - 14 - 2|B|\). Since \(G\) contains no star \(S_n\), we have \(|B| \leq n - 2\). Thus we have \(d_Y(X) + d_B(X) \geq 6n - 14 - 2(n - 2) = 4n - 10\) which implies there is some vertex \(x \in X\) such that \(d_Y(x) + d_B(x) \geq n - 2\). Since \(d_X(x) = 1\), we have \(d(x) \geq n - 1\) which implies \(G\) contains a star \(S_n\), a contradiction. \(\square\)

If \(|Y_0| \leq 2\), then by Claim 1, we have \(d_Y(X) \geq 3|Y| - 2\) which contradicts Claim 5. Hence \(|Y_0| \geq 3\). If \(\overline{G}[Y]\) contains a matching \(M\) which satursates \(Y_0\), then by Claim 3, we have \(d_Y(X) \geq \sum_{y \in V(M)} d_X(y) + \sum_{y \in Y - V(M)} d_X(y) \geq 3|Y|\) which contradicts Claim 5.

Hence \(\overline{G}[Y]\) contains no matching \(M\) which satursates \(Y_0\). Thus by Claim 2, there are two vertices \(y_1, y_2 \in Y_0\) and a vertex \(y_0 \in Y\) such that \(y_0 y_1, y_0 y_2 \notin E(G)\). By Claim 4, we may assume \(N_X(y) = \{a_1, v_1\}\) for any vertex \(y \in Y_0\). Since \(|Y_0| \geq 3\), we can choose a vertex \(y_3 \in Y_0 - \{y_1, y_2\}\). It is not difficult to see that \(y_0 y_1 a_2 v_2 y_2 y_3\) is a \(C_6\) in \(\overline{G}[X \cup Y]\). Thus, noting that \(X \cup Y = V(G) - N[a_3]\), we can see that \(G\) contains a \(W_6\) with the hub \(a_3\), a contradiction.

Case 2. \(\alpha(G) = 4\).

In this case, we first show the following claim.

Claim 6. \(G\) has at least one of the following graphs as an induced subgraph: \(3K_1 \cup K_3, 2K_1 \cup P_4\) and \(2K_1 \cup 2K_2\).

Proof. Let \(A = \{a_1, a_2, a_3, a_4\}\) be a maximum independent set of \(G\). Then \(d_A(v) \geq 1\) for any vertex \(v \in V(G) - A\). If there is at most one vertex, say \(v \in V(G) - A\) such that \(d_A(v) = 1\), then \(d(A) \geq 2(2n - 3) - 1 = 4n - 7\) which implies there is at least one vertex \(a \in A\) such that \(d(a) \geq n - 1\) and hence \(G\) contains a star \(S_n\), a contradiction. Thus there are at least two vertices in \(V(G) - A\), say \(v_1, v_2\), such that \(d_A(v_1) = d_A(v_2) = 1\). If \(N_A(v_1) = N_A(v_2)\), then \(G\) contains \(3K_1 \cup K_3\) as an induced subgraph. If \(N_A(v_1) \neq N_A(v_2)\) and \(v_1 v_2 \in E(G)\), then \(G\) contains \(2K_1 \cup P_4\) as an induced subgraph. If \(N_A(v_1) \neq N_A(v_2)\) and \(v_1 v_2 \notin E(G)\), then \(G\) contains \(2K_1 \cup 2K_2\) as an induced subgraph. \(\square\)

By Claim 6, we need to consider the following three cases separately.
Subcase 2.1. $G$ contains an induced subgraph $G_0 = 3K_1 \cup K_3$.

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Assume that $V(G_0) = V_0 = A \cup B$ and $E(G_0) = \{a_1a_2, a_2a_3, a_3a_1\}$. Since $n \geq 7$, we have $n - 3 \geq 4$. By induction hypothesis, $G - V_0$ contains a star $S_{n-3}$ with center $v$. It is easy to see that $1 \leq d_{V_0}(v) \leq 2$.

If $d_{V_0}(v) = 1$, then since $\alpha(G) = 4$, we have $N_{V_0}(v) \cap A = \emptyset$ and hence we may assume $N_{V_0}(v) = \{b_1\}$. If $d_{V_0}(v) = 2$, then since $\alpha(G) = 4$, we have $d_A(v) \leq 1$. If $d_A(v) = 1$, we may assume $N_{V_0}(v) = \{a_1, b_1\}$. If $d_A(v) = 0$, we may assume $N_{V_0}(v) = \{b_1, b_2\}$. Thus $\overline{G}[V_0 \cup \{v\}]$ contains a $W_6$ with the hub $b_2$ in any case, a contradiction.

Subcase 2.2. $G$ contains an induced subgraph $G_0 = 2K_1 \cup P_4$.

Let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2\}$. Assume $V(G_0) = V_0 = A \cup B$ and $E(G_0) = \{a_1a_2, a_2a_3, a_3a_4\}$. Since $n \geq 7$, we have $n - 3 \geq 4$. By induction hypothesis, $G - V_0$ contains a star $S_{n-3}$ with center $v_1$. It is easy to see that $1 \leq d_{V_0}(v_1) \leq 2$.

If $d_{V_0}(v_1) = 1$, then since $\alpha(G) = 4$, we have $N_{V_0}(v_1) \cap A = \emptyset$ and hence we may assume $N_{V_0}(v_1) = \{b_1\}$. Thus $\overline{G}[V_0 \cup \{v_1\}]$ contains a $W_6$ with the hub $b_2$, a contradiction. Hence we may assume $d_{V_0}(v_1) = 2$. If $d_B(v_1) = 0$, then since $\alpha(G) = 4$, $A - N_{V_0}(v_1)$ must be a clique. Thus by symmetry we may assume $N_{V_0}(v_1) = \{a_1, a_2\}$ or $\{a_1, a_4\}$. If $d_B(v_1) = 1$, then by symmetry we may assume $N_{V_0}(v_1) = \{a_1, b_1\}$ or $\{a_2, b_1\}$. Thus, it is not difficult to see $\overline{G}[V_0 \cup \{v_1\}]$ contains a $W_6$ with the hub $b_2$ if $d_B(v_1) \leq 1$, a contradiction. Hence we may assume $d_B(v_1) = 2$.

Let $V_1 = V_0 \cup \{v_1\}$, then $G[V_1] = P_3 \cup P_4$. Since $n \geq 7$, we have $n - 4 \geq 3$. By induction hypothesis, $G - V_1$ contains a star $S_{n-4}$ with center $v_2$. Obviously, $1 \leq d_{V_1}(v_2) \leq 3$. If $d_{V_1}(v_2) = 1$, then since $\alpha(G) = 4$, we have $N_{V_1}(v_2) \subseteq B$. Assume $N_{V_1}(v_2) = \{b_1\}$, then $\overline{G}[V_0 \cup \{v_2\}]$ contains a $W_6$ with the hub $b_2$, a contradiction. Hence we may assume $d_{V_1}(v_2) \geq 2$. Now, let $d_{V_1}(v_2) = 2$. If $d_A(v_2) = 2$, then since $\alpha(G) = 4$, we may assume $N_{V_1}(v_2) = \{a_1, a_2\}$ or $\{a_1, a_4\}$. If $d_A(v_2) = 1$, then since $\alpha(G) = 4$, we have $N_{V_1}(v_2) \cap B \neq \emptyset$. By symmetry we may assume $N_{V_1}(v_2) = \{a_1, b_1\}$ or $\{a_2, b_1\}$. Thus, it is not difficult to see $\overline{G}[V_0 \cup \{v_2\}]$ contains a $W_6$ with the hub $b_2$ if $d_A(v_2) \geq 1$, a contradiction. If $d_A(v_2) = 0$, then by symmetry we may assume $N_{V_1}(v_2) = \{b_1, v_1\}$ or $\{b_1, b_2\}$. Thus, $\overline{G}[V_0 \cup \{v_2\}]$ contains a $W_6$ with the hub $b_2$ in the former case and $\overline{G}[V_1 \cup \{v_2\} - \{a_2\}]$ contains a $W_6$ with the hub $a_1$ in the latter case, a contradiction. Therefore we have $d_{V_1}(v_2) = 3$.

If $d_A(v_2) = 3$, then we may assume $N_{V_1}(v_2) = \{a_1, a_3, a_4\}$ or $\{a_1, a_2, a_4\}$. Thus, $\overline{G}[V_1 \cup \{v_2\} - \{a_2\}]$ contains a $W_6$ with the hub $a_4$ in the former case and $\overline{G}[V_0 \cup \{v_2\}]$ contains a $W_6$ with the hub $b_2$ in the latter case, a contradiction.

If $d_A(v_2) = 2$ and $v_1 \in N_{V_1}(v_2)$, then since $\alpha(G) = 4$, $A - N(v_2)$ must be a clique. Thus, we may assume by symmetry that $N_{V_1}(v_2) = \{a_1, a_2, v_1\}$ or $\{a_1, a_4, v_1\}$. If $d_A(v_2) = 2$ and $v_1 \notin N_{V_1}(v_2)$, then by symmetry we may assume $N_{V_1}(v_2) = \{a_1, a_2, b_1\}$ or $\{a_1, a_3, b_1\}$ or $\{a_1, a_4, b_1\}$ or $\{a_2, a_3, b_1\}$. Thus, $\overline{G}[V_0 \cup \{v_2\}]$ contains a $W_6$ with the hub $b_2$ in all the cases above, a contradiction.

If $d_A(v_2) = 1$, then by symmetry we may assume $N_{V_1}(v_2) = \{a_1, b_1, v_1\}$ or $\{a_2, b_1, v_1\}$ or $\{a_1, b_1, b_2\}$ or $\{a_2, b_1, b_2\}$. It is not difficult to check that $\overline{G}[V_1 \cup \{v_2\} - \{a_3\}]$ contains a $W_6$ with the hub $a_4$ in all the cases above, a contradiction.
4. Proof of Theorem 2

Proof of Theorem 2. Let $G$ be a graph of order $3n - 2$. If $G$ contains no $S_n$, then $\Delta(G) \leq n - 2$ which implies $\delta(\overline{G}) \geq (3n - 3) - (n - 2) = 2n - 1$. Let $v$ be any vertex of $V(G)$ and $d_{\overline{G}}(v) = (2n - 1) + k$, where $k \geq 0$. Assume $F = \overline{G}[N_{\overline{G}}(v)]$. We now show $F$ is pancyclic. Since $|F| = (2n - 1) + k$ and $\delta(\overline{G}) \geq 2n - 1$, we have $\delta(F) \geq 2n - 1 - [(3n - 2) - (2n - 1 + k)] = n + k$. Noting that $k \geq 0$, we have $\delta(F) \geq n + k > (2n - 1 + k)/2 = |F|/2$ which implies $F$ is pancyclic by Lemma 1, that is, $F$ contains $C_i$ for $3 \leq i \leq 2n - 1$. Since $m \leq n + 1$, we can see $\overline{G}$ contains a $W_m$ with the hub $v$ and hence $R(S_n, W_m) \leq 3n - 2$. On the other hand, it is not difficult to see neither $3K_{n-1}$ contains $S_n$ nor its complement contains $W_m$ for odd $m$. Thus we have $R(S_n, W_m) \geq 3n - 2$ and hence $R(S_n, W_m) = 3n - 2$. □

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