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On Cycles Containing a Given Arc in Regular Multipartite Tournaments

Lin Qiang PAN
Department of Control Science and Engineering, Huazhong University of Science and Technology, Wuhan 430074*, P. R. China*
E-mail: lqpan@mail.hust.edu.cn

Ke Min ZHANG
Department of Mathematics, Nanjing University, Nanjing 210093*, P. R. China*
E-mail: zkmfl@hotmail.com

Abstract In this paper we prove that if T is a regular n-partite tournament with $n \geq 4$, then each arc of T lies on a cycle whose vertices are from exactly k partite sets for $k = 4, 5, \ldots, n$. Our result, in a sense, generalizes a theorem due to Alspach.

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1 Introduction

We use the terminology and notation of [1]. A *digraph* $D = (V(D), A(D))$ is determined by its set of vertices $V(D)$, and its set of arcs $A(D)$. If xy is an arc of a digraph D, then we say that x *dominates* y, denoted by $x \to y$. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \to B$. For a vertex $x \in V(D)$, the *outset* $N_D^+(x)$ (*inset* $N_D^-(x)$) is the set of vertices dominated by x (dominating x) in D. The numbers $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ are called *outdegree* and *indegree* of x, respectively. A *regular* digraph D is a digraph such that for each vertex v, $d^+(v) = d^-(v) = k$. By a *cycle* (*path,* resp.), we mean a directed cycle (directed path, resp.). A cycle of length k is called a *k-cycle.* A digraph D is *pancyclic* if it contains cycles of lengths $3, 4, \ldots, |V(D)|$. A digraph D is *vertex pancyclic* (*arc pancyclic*) if for all $v \in V(D)$ ($e \in A(D)$) it contains cycles of lengths $3, 4, \ldots, |V(D)|$, which all include the vertex v (arc e). The *converse* of D is a digraph D', where $V(D') = V(D)$, $xy \in A(D')$ if and only if $yx \in A(D)$.

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An *n-partite or multipartite tournament* is a digraph obtained from a complete n-partite graph by giving each edge an orientation. Let T be a multipartite tournament and $x \in V(T)$, we denote by $V^c(x)$ the partite set of T to which x belongs. A *k-outpath* of an arc xy in a multipartite tournament is a path with length k starting at xy such that x does not dominate the end vertex of the path.

The following two well-known theorems on tournaments are due to Moon and Alspach, respectively.

Theorem A (Moon [2]) *Every strong tournament is vertex pancyclic.*

Theorem B (Alspach [3]) *Every regular tournament is arc pancyclic.*

Many interesting results are obtained on the extension of Moon's theorem for multipartite tournaments (see [4–8]). For a survey on this topic, we refer the reader to [9–10].

The arc pancyclicity problem for multipartite tournaments seems to be difficult in general. There are relatively few papers on this topic.

Theorem C (Guo [6]) *Let T be a regular n-partite tournament with $n \geq 3$. Then every arc of T has a $(k-1)$-outpath for all $3 \leq k \leq n$.*

Theorem D (Zhou and Zhang [11]) *Let T be a regular n-partite tournament with $n \geq 6$. Then each arc of T lies on a k-cycle for $k = 4, 5, \ldots, n$.*

Theorem E (Guo and Kwak, Theorem 2.16 in [10]) *Let D be a regular n-partite tournament. If the cardinality of each partite set is odd, then every arc of D is in a cycle that contains vertices from exactly m partite sets for each $m \in \{3, 4, \ldots, n\}$.*

In this paper, no matter what is the parity of the cardinality of each partite set, we prove the following result, which, in a sense, is also a generalization of Alspach's theorem.

Main Theorem *Let T be a regular n-partite tournament with $n \geq 4$. Then every arc of T lies on a cycle which contains vertices from exactly k partite sets for $k = 4, 5, \ldots, n$.*

2 Proof of Theorem

Lemma 1 *Let T be a regular n-partite tournament with $n \geq 3$. Then every arc xy of T lies on one of the following three kinds of cycles*:

(1) *A 3-cycle*;

(2) *A 4-cycle $xyx'zx$, where x and x' are in the same partite set, y and z are in different partite sets* (*a 4-cycle of this kind is called a 4-cycle of type I*);

(3) *A 4-cycle $xyx'y'x$, where x and x'* (y *and* y') *are in the same partite set* (*a 4-cycle of this kind is called a 4-cycle of type II*).

Proof The partite sets of T are denoted by $V_1, V_2, \ldots, V_n$. By Lemma 2.10 in [10], $|V_1| = |V_2| = \cdots = |V_n|$. Assume $e = (x, y)$ to be any arc in T. Without loss of generality, we may assume that $x \in V_1$, $y \in V_2$. Denote $A = N^-(x) \cap (V(T) - V_1 - V_2)$, $B = N^+(y) \cap (V(T) - V_1 - V_2)$.

Suppose e is not on any 3-cycle. Then $A \cap B = \emptyset$, $A \subseteq N^-(y)$, $B \subseteq N^+(y)$. Moreover, it is easy to see that $A \neq \emptyset$, $B \neq \emptyset$. If $V_1 \to y$, then $V_1 \subseteq N^-(y)$, so $N^-(y) \supseteq A \cup V_1$ and $N^-(x) \subseteq A \cup (V_2 - y)$. Hence $|A| + |V_2| - 1 \geq |N^-(x)| = |N^-(y)| \geq |A| + |V_1|$, i.e., $|V_2| > |V_1|$, which is impossible. Therefore, $N^+(y) \cap V_1 \neq \emptyset$. Assume $x' \in N^+(y) \cap V_1$. Then $x \to y \to x'$, and $y \in N^+(x) \cap N^-(x')$. Denote $a = |N^+(x) \cap N^-(x')|$, $b = |N^+(x) \cap N^+(x')|$, $c = |N^-(x) \cap N^-(x')|$, $d = |N^-(x) \cap N^+(x')|$. Then $d^+(x) = a+b$, $d^-(x) = c+d$, $d^+(x') = b+d$, $d^-(x') = a + c$. Since T is regular, it follows that $a + b = c + d$, $b + d = a + c$. So $a = d$, $b = c$, and $a \geq 1$. Thus $N^-(x) \cap N^+(x') \neq \emptyset$. Assume that $y' \in N^-(x) \cap N^+(x')$. Then $x' \to y' \to x$. If $y' \in V_2$, then $xyx'y'x$ is a 4-cycle of type II. If $y' \notin V_2$, then $xyx'y'x$ is a 4-cycle of type I.

Here we remark that Zhou Guo-Fei has ever proved that if we let T be a regular n-partite tournament with $n \geq 3$, then every arc of T lies on a 3-cycle or a 4-cycle (private communication).

Lemma 2 *Let T be a regular n-partite tournament with $n \geq 4$. If the arc xy of T is not on any 3-cycle and 4-cycle of type I, then the arc xy is on a cycle whose vertices are from 3 or 4 partite sets.*

Proof The partite sets of T are denoted by $V_1, V_2, \ldots, V_n$. By Lemma 2.10 in [9], $|V_1| = |V_2| = \cdots = |V_n| = k$. Assume $e = (x, y)$ is an arc in T which is not on a 3-cycle and 4-cycle of type I. Without loss of generality, we may assume that $x \in V_1$ and $y \in V_2$. Denote $A = N^-(x) \cap (V(T) - V_1 - V_2)$, $B = N^+(x) \cap (V(T) - V_1 - V_2)$, $C = N^-(y) \cap V_1$, $D = N^+(y) \cap V_1$, $E = N^+(x) \cap V_2$, $F = N^-(x) \cap V_2$.

Since e is not on any 3-cycle and T is regular, it is easy to see that $A \neq \emptyset$, $B \neq \emptyset$, $D \neq \emptyset$, $F \neq \emptyset$, $A \cap B = \emptyset$, and $X \to b$.

If there is an arc $x' \to a$, where $x' \in D$, $a \in A$, then e is on a 4-cycle of type I $xyx'ax$, which is impossible. So $A \to D$. Similarly, we have $F \to B \cap N^+(y)$.

Let x' be an arbitrary vertex in D. Since $|N^+(x')| = |N^+(x)|$ and $x \to y \to x'$, there is a vertex v such that $x' \to v \to x$. By $A \to D$ and $F \to B \cap N^+(y)$, it is easy to see that $v \in F$. So $N^+(x') \cap F \neq \emptyset$ for every $x' \in D$. If there is an arc $u \to x'$, where $u \in B \cap N^+(y)$, $x' \in D$, then by $N^+(x') \cap F \neq \emptyset$ for every $x' \in D$, there is $y' \in N^+(x') \cap F$, hence $xyux'y'x$ is a cycle, whose vertices are from 3-partite sets. So we may assume $D \to B \cap N^+(y)$. Similarly, we can also assume that $A \to F$, $A \to C \to B \cap N^+(y)$, $A \to E \to B \cap N^+(y)$. Therefore, we have $A \to V_1 \cup V_2 \to B \cap B^+(y)$.

By Lemma 2.10 in [10], we have $|N^-(x)| = \frac{1}{2}(n-1)k$, so $\frac{1}{2}(n-1)k = |N^-(x)| = |A| + |F| \leq |A| + (k-1)$, $|A| \geq \frac{1}{2}(n-1)k - (k-1) \cdots (*)$. If $N(B \cap N^+(y)) \cap A = \emptyset$, then for any $u \in B \cap N^+(y)$, we have

$$\begin{aligned}|N^-(u)| &\geq |V_1 \cup V_2| + |N^-(u) \cap A| \\ &\geq 2k + |A - V^c(u)| = 2k + |A| - |V^c(u)| \\ &\geq 2k + \frac{1}{2}(n-1)k - (k-1) - k \quad (\text{by } *) \\ &= \frac{1}{2}(n-1)k + 1,\end{aligned}$$

which is impossible, by the regularity of T. Therefore $N^+(B \cap N^+(y)) \cap A \neq \emptyset$. Suppose uv is an arc, $u \in B \cap N^+(y)$, $v \in A$. It is easy to see that $xyuvx$ is a cycle, whose vertices are from 4-partite sets.

Proof of the Main Theorem Let $V_1, V_2, \ldots, V_n$ be the partite sets of T. By Lemmas 1 and 2, we may assume v_1v_2 is an arc of T and v_1v_2 lies on a cycle $C = v_1v_2 \cdots v_mv_1$ whose vertices are from k partite sets, where $3 \leq k \leq n-1$. It suffices to show that v_1v_2 is on a cycle whose vertices are from $k+1$ partite sets. Let

$$S = \{x|x \in V_i, V_i \cap V(C) = \emptyset\},$$
$$A = \{x|x \in S, x \to V(C)\},$$
$$B = \{x|x \in S, V(C) \to x\},$$
$$X = S - A - B.$$

We consider the following two cases:

Case 1 $A \neq \emptyset$ or $B \neq \emptyset$.

Without loss of generality, we may assume that $A \neq \emptyset$; for the case $B \neq \emptyset$; we need to consider only the converse of T.

Let m' be the subscript such that $v_1v_2 \cdots v_{m'}$ contains vertices from exactly $k-1$ partite sets. We denote the path $v_1v_2 \cdots v_{m'}$ by P. Let x be a vertex of A. By Lemma 1, $xv_{m'}$ lies on a 3-cycle $xv_{m'}yx$, a 4-cycle of type I $xv_{m'}x'zx$, or a 4-cycle of type II $xv_{m'}x'v'x$.

(1) $xv_{m'}$ lies on a 3-cycle $xv_{m'}yx$.

If $V^c(y) \cap V(C) \neq \emptyset$, then $v_1v_2 \cdots v_{m'}yxv_{m'+1} \cdots v_mv_1$ is a cycle whose vertices are from exactly $k+1$ partite sets.

If $V^c(y) \cap V(C) = \emptyset$, then $v_1v_2 \cdots v_{m'}yxv_1$ is a cycle whose vertices are from exactly $k+1$ partite sets.

(2) $xv_{m'}$ lies on a 4-cycle of type I $xv_{m'}x'zx$.

Clearly $x' \notin V(C)$. By the definition of A, $z \notin V(C)$.

If $V^c(z) \cap V(C) \neq \emptyset$, then $v_1v_2 \cdots v_{m'}x'zxv_{m'+1} \cdots v_mv_1$ is a cycle whose vertices are from exactly $k+1$ partite sets.

If $V^c(z) \cap V(C) = \emptyset$, then $v_1v_2 \cdots v_{m'}x'zxv_1$ is a cycle whose vertices are from exactly $k+1$ partite sets.

(3) $xv_{m'}$ lies on a 4-cycle of type II $xv_{m'}x'v'x$.

Clearly $x', v' \notin V(C)$. In this case the cycle $v_1v_2 \cdots v_{m'}x'v'xv_{m'+1} \cdots v_mv_1$ contains vertices from exactly $k+1$ partite sets.

Case 2 $A = \emptyset$ and $B = \emptyset$.

In this case we have $X \neq \emptyset$. Suppose there is a vertex $x \in X$ such that $v_2 \to x$. By the definition of X, there is a vertex $v_i \in V(C)$ such that $x \to v_i$. We can insert x into cycle C to obtain a cycle which contains vertices v_1, v_2, $\ldots$, v_m, x with the arc v_1v_2. It is easy to see that these vertices come from exactly $k+1$ partite sets. So we may assume $X \to v_2$. Similarly we assume $v_1 \to X$.

If there is a vertex $x \in X$ such that $x \to v_m$, then $x \to v_i$, $i \geq 2$, otherwise we can get a cycle containing vertices v_1, v_2, …, v_m, x with the arc v_1v_2. It is easy to see that these vertices come from exactly $k+1$ partite sets. Because of $X \to v_2$ and the regularity of T, xv_2 lies on a 3-cycle xv_2yx. We can check that $y \notin V(C) \cup X$, the cycle $v_1v_2yxv_3 \cdots v_mv_1$ contains vertices from exactly $k+1$ partite sets. So we may assume that $v_m \to X$.

By considering the converse of T, we may also assume that $X \to v_3$.

If there is an arc $yx \in A(T)$ such that $y \in N^+(v_2) - V(C) - V^c(v_1)$, $x \in X$, then clearly $y \notin X$, the cycle $v_1v_2yxv_3 \cdots v_mv_1$ contains vertices from exactly $k+1$ partite sets. So we may assume $X \to N^+(v_2) - V(C) - V^c(v_1)$. Similarly, $v_1 \to N^+(v_2) - V(C) - V^c(v_1)$, since otherwise there is a cycle $v_1v_2 \cdots v_mxyv_1$ which contains vertices from exactly $k+1$ partite sets.

If $v_2 \to v_i$, we may assume $v_{i-1} \not\to v_1$, since, otherwise, the cycle $v_1v_2v_i \cdots v_mx\, v_3 \cdots v_{i-1}v_1$ contains vertices from exactly $k+1$ partite sets, where $x \in X$. It follows that

$$|N^+(v_1) \cap V(C)| \geq |N^+(v_2) \cap V(C)| - |V^c(v_1) \cap V(C)| + 1.$$

Hence we have

$$\begin{aligned}
|N^+(v_1)| &\geq |X| + |N^+(v_2) - V(C) - V^c(v_1)| + |N^+(v_1) \cap V(C)| \\
&\geq |X| + |N^+(v_2) - V(C) - V^c(v_1)| + |N^+(v_2) \cap V(C)| - |V^c(v_1) \cap V(C)| + 1 \\
&\geq |X| + |N^+(v_2) - V(C)| - |N^+(v_2) \cap V^c(v_1) - V(C)| + |N^+(v_2) \cap V(C)| \\
&\quad - |V^c(v_1) \cap V(C)| + 1 \\
&\geq |X| + |N^+(v_2)| - |V^c(v_1)| + 1 \\
&\geq |N^+(v_2)| + 1.
\end{aligned}$$

This contradicts the regularity of T. This completes the proof of the main theorem.

From the main theorem, we can obtain the following corollary:

Corollary *Let T be a regular n-partite tournament with $n \geq 3$. Then every arc of T has an outpath whose vertices come from exactly k partite sets, for all $k = 3, 4, \ldots, n$.*

Lastly we give the following two problems:

Problem 1 *Which regular multipartite tournaments have the property that every arc lies on a 3-cycle?*

Problem 2 *Which regular multipartite tournaments have the property that every arc lies on a cycle whose vertices come from exactly 3 partite sets?*

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