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The Ramsey Numbers $R(T_n, W_6)$ for $\Delta(T_n) \geq n - 3$

YAOJUN CHEN*, YUNQING ZHANG AND KEMIN ZHANG
Department of Mathematics, Nanjing University
Nanjing 210093, P.R. China

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Abstract—Let T_n denote a tree of order n and W_m a wheel of order $m+1$. In a previous paper, we evaluated the Ramsey number $R(T_n, W_m)$ in the cases where T_n is the star of order n and $m = 6$ or m is odd and $n \geq m - 1 \geq 2$. In this paper, we determine $R(T_n, W_6)$ in the case where the maximum degree of T_n is at least $n - 3$. Our results show that a recent conjecture of Baskoro *et al.* is false.

Keywords—Ramsey number, Tree, Wheel.

1. INTRODUCTION

All graphs considered in this paper are finite simple graphs without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest positive integer n such that for any graph G of order n, either G contains G_1 or $\bar{G}$ contains G_2, where $\bar{G}$ is the complement of G. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G. The *minimum and maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a vertex $v \in V(G)$ and a subgraph H of G, $N_H(v)$ is the set of neighbors of v contained in H, i.e., $N_H(v) = N(v) \cap V(H)$. We let $d_H(v) = |N_H(v)|$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. Let U, V be two disjoint vertex sets. We use $E(U, V)$ to denote the set of edges between U and V. Let m be a positive integer. We use mG to denote m vertex disjoint copies of G. A path and a cycle of order n are denoted by P_n and C_n, respectively. A *star* S_n ($n \geq 3$) is a bipartite graph $K_{1,n-1}$. We use $K_{3,3,\ldots,3}$ to denote a balanced complete $n/3$-partite graph of order $n \equiv 0 \pmod 3$. A *wheel* $W_n = \{x\} + C_n$ is a graph of $n+1$ vertices, that is, a vertex x, called the *hub* of the wheel, adjacent to all vertices of C_n. $S_n(l, m)$ is a tree of order n obtained from $S_{n-l\times m}$ by subdividing each of l chosen edges m times. $S_n(l)$ is a tree of order n obtained from an S_l and an S_{n-l} by adding an edge joining the centers of them. A graph on n vertices is *pancyclic* if it contains cycles of every length l, $3 \leq l \leq n$.

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Many Ramsey numbers concerning wheel or star have been established, see for instance [1–4]. Recently, the following Ramsey numbers were obtained.

THEOREM A. *(See [5].)* $R(S_n, W_4) = 2n-1$ *for* $n \geq 3$ *and odd;* $R(S_n, W_4) = 2n+1$ *for* $n \geq 4$ *and even;* $R(S_n, W_5) = 3n-2$ *for* $n \geq 4$.

THEOREM B. *(See [6].)* $R(S_n, W_6) = 2n+1$ *for* $n \geq 3$.

THEOREM C. *(See [6].)* $R(S_n, W_m) = 3n-2$ *for* m *odd and* $n \geq m-1 \geq 2$.

THEOREM D. *(See [7].)* $R(P_n, W_m) = 3n-2$ *for* m *odd and* $n \geq m-1 \geq 2$; $R(P_n, W_m) = 2n-1$ *for* m *even and* $n \geq m-1 \geq 3$.

THEOREM E. *(See [8].) Let* T_n *be a tree of order* n *other than* S_n. *Then* $R(T_n, W_4) = 2n-1$ *for* $n \geq 3$; $R(T_n, W_5) = 3n-2$ *for* $n \geq 4$.

Motivated by Theorem E, Baskoro *et al.* furthermore posed in [8] the following.

CONJECTURE 1. *Let* T_n *be a tree other than* S_n *and* $n \geq m-1$. *Then* $R(T_n, W_m) = 2n-1$ *for* $m \geq 6$ *even;* $R(T_n, W_m) = 3n-2$ *for* $m \geq 7$ *and odd.*

In this paper, we consider the Ramsey numbers $R(T_n, W_6)$ for $\Delta(T_n) \geq n-3$. If $\Delta(T_n) \geq n-3$ and $T_n \neq S_n$, then T_n is isomorphic to one of the trees $S_n(1,1)$, $S_n(1,2)$, $S_n(2,1)$, or $S_n(3)$. The main results of this paper are the following.

THEOREM 1. $R(S_n(1,1), W_6) = 2n$ *for* $n \geq 4$.

THEOREM 2. $R(S_n(1,2), W_6) = 2n$ *for* $n \geq 6$ *and* $n \equiv 0 \pmod 3$.

THEOREM 3. $R(S_n(3), W_6) = R(S_n(2,1), W_6) = 2n-1$ *for* $n \geq 6$; $R(S_n(1,2), W_6) = 2n-1$ *for* $n \geq 6$ *and* $n \not\equiv 0 \pmod 3$.

REMARK. By Theorems 1 and 2, we can see that Conjecture 1 is not true when $m = 6$. In fact, if m is even, then $R(S_n(1,1), W_m)$ is a function related to both n and m as can be seen by the following examples.

Let $m \geq 6$ be an even integer, $n = km/2 + 3$, where $k \geq 2$ is an integer, and $G = H \cup K_{n-1}$, where $\bar{H} = (k+1)K_{m/2}$. Obviously, G is a graph of order $2n + m/2 - 4$. Since $\Delta(H) = km/2 = n-3$ and $\Delta(S_n(1,1)) = n-2$, we can see that G contains no $S_n(1,1)$. On the other hand, it is not difficult to check that $\bar{G}$ contains no W_m. Thus, we have $R(S_n(1,1), W_m) \geq 2n + m/2 - 3$ if $n = km/2 + 3$ for some integer $k \geq 2$.

In general, by modifying the examples above, we can show for even m, $R(T_n, W_m)$ is a function related to both n and m if $\Delta(T_n)$ is large enough. By Theorem D, we believe $R(T_n, W_m) = 2n-1$ for m even and $n \geq m-1$ if $\Delta(T_n)$ is small.

PROBLEM 1. *Characterize trees* T_n *with* $R(T_n, W_m) = 2n-1$ *for* m *even and* $n \geq m-1$.

2. SOME LEMMAS

In order to prove our results, we need the following lemmas.

LEMMA 1. *(See [9].) Let* G *be a graph of order* n. *If* $\delta(G) \geq n/2$, *then either* G *is pancyclic or* n *is even and* $G = K_{n/2,n/2}$.

LEMMA 2. *Let* G *be a graph of order* $2n-1 \geq 7$ *and* (U, V) *a partition of* $V(G)$ *with* $|U| \geq 3$ *and* $|V| \geq 4$. *Suppose* $u_i \in U$ *and* $N_V(u_i) = \emptyset$, *where* $1 \leq i \leq 3$. *If* $\bar{G}$ *contains no* W_6, *then* $\delta(G[V]) \geq |V| - 3$.

PROOF. If there is some vertex $v \in V$ such that $d_V(v) \leq |V| - 4$, say $v_1, v_2, v_3 \in V$ and $v_1, v_2, v_3 \notin N_V(v)$, then $\bar{G}[v, v_1, v_2, v_3, u_1, u_2, u_3]$ contains a W_6 with the hub v, a contradiction. ∎

LEMMA 3. *Let G be a graph of order $n \geq 6$ with $\delta(G) \geq n-3$. Then G contains $S_n(3)$ and $S_n(2,1)$. Furthermore, if $G \neq K_{3,3,\ldots,3}$, then G contains $S_n(1,2)$.*

PROOF. We first show G contains an $S_n(3)$. Since $\delta(G) \geq n-3$, G contains a tree $T = S_{n-2}$ with the center v_0. Set $V(G)-V(T) = U = \{u_1, u_2\}$. If $N(u_1) \cap N(u_2) \cap N_T(v_0) \neq \emptyset$, then G contains an $S_n(3)$. Hence, we may assume $N(u_1) \cap N(u_2) \cap N_T(v_0) = \emptyset$. Thus, if $u_1, u_2 \notin N(v_0)$ or $u_1u_2 \notin E(G)$, then since $\delta(G) \geq n-3$, we have $2(n-3) \leq d(u_1)+d(u_2) \leq n-3+2$ which implies $n \leq 5$, a contradiction. Hence, we may assume $v_0u_1, u_1u_2 \in E(G)$. Noting that $\delta(G) \geq n-3$ and $n \geq 6$, we have $d(u_1) \geq 3$ which implies $N(u_1) \cap N_T(v_0) \neq \emptyset$, and hence G contains an $S_n(3)$.

Next, we show G contains an $S_n(2,1)$. Let T be an $S_n(3)$ in G with $V(T) = \{v_0, v_1, \ldots, v_{n-3}, u_1, u_2\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq n-3\} \cup \{v_1u_1, v_1u_2\}$. If $N(u_i) \cap (N_T(v_0) - \{v_1\}) \neq \emptyset$ for some $i \in \{1,2\}$, then G contains an $S_n(2,1)$. Hence, we may assume $N(u_i) \cap (N_T(v_0) - \{v_1\}) = \emptyset$ for $i = 1,2$ which implies $u_2v_0 \in E(G)$ since $d(u_2) \geq 3$. Noting that $d(v_2) \geq 3$, there is some vertex $v \in \{v_3, v_4, \ldots, v_{n-3}\}$ such that $vv_2 \in E(G)$ which implies G contains an $S_n(2,1)$.

Finally, we show G contains an $S_n(1,2)$ if $G \neq K_{3,3,\ldots,3}$. If $\Delta(G) \geq n-2$, then G contains a star S_{n-1}. Let T be a star S_{n-1} with the center v and $V(G)-V(T) = \{u\}$. Since $d(u) \geq 3$, we have $|N(u) \cap N_T(v)| \geq 2$, and hence G contains an $S_n(1,2)$. Thus, we may assume $\Delta(G) = \delta(G) = n-3$. Let v be any vertex in $V(G)$ and $V(G)-N[v] = \{u_1, u_2\}$. If $u_1u_2 \in E(G)$, then since $d(u_1) \geq 3$, we have $N(u_1) \cap N(v) \neq \emptyset$, and hence, G contains an $S_n(1,2)$. Thus, $V(G)-N(v)$ is an independent set for any $v \in V(G)$ which implies $G = K_{3,3,\ldots,3}$. ∎

3. PROOFS OF THEOREMS

PROOF OF THEOREM 1. The example $G = \bar{H} \cup K_{n-1}$ shows that $R(S_n(1,1), W_6) \geq 2n$, where $H = C_n$ if $n \neq 6$ and $H = 2C_3$ if $n = 6$. We first consider the case $n \geq 5$. Assume that G is a graph of order $2n$ such that $S_n(1,1) \not\subset G$ and $W_6 \not\subset \bar{G}$. Let u be a vertex of degree $\Delta(G)$ and $N_G(u) = U$. Set $W = V(G) - (\{u\} \cup U)$. If $\Delta(G) = n-1$, then $S_n(1,1) \not\subset G$ implies U is an independent set and $E(U,W) = \emptyset$. But then we may take four vertices from U and three from W and find a W_6 in $\bar{G}$, a contradiction. By Theorem B, we are left to consider the case in which $\Delta(G) = n-2$. As in the preceding case, $E(U,W) = \emptyset$. Consider $\bar{G}[W]$. If some vertex therein has degree at least three, then this vertex as the hub together with three appropriate vertices from W and three arbitrary vertices from U give a W_6 in $\bar{G}$, a contradiction. Hence, $G[W]$ is regular of degree $n-2$. To see this is impossible, pick two nonadjacent vertices in $G[W]$, say v and w. Since $2(n-1) > n$, their neighborhoods have a nonempty intersection, thus producing $S_n(1,1) \subset G$, a contradiction. As for the case $n = 4$, we leave it to the reader. ∎

PROOF OF THEOREM 2. Let G be a graph of order $2n$. Suppose $\bar{G}$ contains no W_6. By Theorem 1, G contains an $S_n(1,1)$. Let T be an $S_n(1,1)$ with $V(T) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq n-2\} \cup \{v_1v_{n-1}\}$. Set $U = V(G) - V(T)$. Obviously, $|U| = n$. If G contains no $S_n(1,2)$, then $N(v_{n-1}) \subseteq \{v_0, v_1\}$. If $\Delta(G[U]) \leq 1$, then since $n \geq 6$, we have $\delta(\bar{G}[U]) \geq n-2 > n/2$ which implies $\bar{G}[U]$ contains a C_6 by Lemma 1, and hence, $\bar{G}$ contains a W_6 with the hub v_{n-1}, a contradiction. Thus, we have $\Delta(G[U]) \geq 2$, which implies $G[U]$ contains a P_3. Let $P = u_1u_2u_3$ be a P_3 in $G[U]$. Since G contains no $S_n(1,2)$, we have $u_iv_j \notin E(G)$ for $1 \leq i \leq 3$ and $2 \leq j \leq 4$. Thus, noting that $N(v_{n-1}) \subseteq \{v_0, v_1\}$, we can see $\bar{G}[v_{n-1}, u_1, u_2, u_3, v_2, v_3, v_4]$ contains a W_6 with hub v_{n-1}, a contradiction. Thus, we have $R(S_n(1,2), W_6) \leq 2n$. On the other hand, since $n \equiv 0 \pmod 3$, the graph $K_{n-1} \cup K_{3,3,\ldots,3}$ shows $R(S_n(1,2), W_6) \geq 2n$, and hence, we have $R(S_n(1,2), W_6) = 2n$. ∎

PROOF OF THEOREM 3. Obviously, $2K_{n-1}$ contains no tree of order n and its complement contains no wheels, and hence, we have $R(S_n(3), W_6) \geq 2n-1$, $R(S_n(2,1), W_6) \geq 2n-1$, and $R(S_n(1,2), W_6) \geq 2n-1$.

Let G be a graph of order $2n-1$. Suppose $\bar{G}$ contains no W_6. By Theorem B, G contains a star S_{n-1}. Let T be an S_{n-1} with the center v_0 and $N_T(v_0) = V = \{v_1, \ldots, v_{n-2}\}$. Set $U = V(G) - V(T)$. Obviously, $|U| = n$.

If $\delta(G[U]) \geq n-3$, then by Lemma 3, G contains $S_n(3)$, $S_n(2,1)$ and if $n \not\equiv 0 \pmod 3$, then $G \neq K_{3,3,\dots,3}$ which implies G contains $S_n(1,2)$. Thus, we may assume

$$\delta(G[U]) \leq n-4. \tag{1}$$

By Lemma 2 and (1), we have $E(U,V) \neq \emptyset$. Assume without loss of generality that $N_U(v_1) \neq \emptyset$, say $u_1 \in N_U(v_1)$.

We first show $R(S_n(3), W_6) \leq 2n-1$. If G contains no $S_n(3)$, then we have $N_V(v_1) = \emptyset$ and $d_U(v_i) \leq 1$ for any $v_i \in V$. Thus, since $n \geq 6$, there are three vertices $u_2, u_3, u_4 \in U - \{u_1\}$ such that $d_V(u_i) \leq 1$ for $2 \leq i \leq 4$ which implies $\bar{G}[v_1, v_2, v_3, v_4, u_2, u_3, u_4]$ contains a W_6 with the hub v_1, a contradiction. Thus, we have $R(S_n(3), W_6) \leq 2n-1$, and hence, $R(S_n(3), W_6) = 2n-1$.

Next, we show $R(S_n(2,1), W_6) \leq 2n-1$. Suppose to the contrary G contains no $S_n(2,1)$. If $d_U(v_1) \geq 2$, then $N_U(v_i) = \emptyset$ for $2 \leq i \leq n-2$ since otherwise G contains an $S_n(2,1)$. By Lemma 2, we have $\delta(G[U]) \geq n-3$ which contradicts (1), and hence, we have $N_U(v_1) = \{u_1\}$. In this case, we have $N_U(v_i) \subseteq \{u_1\}$ for $2 \leq i \leq n-2$, and hence, $\delta(G[V]) \geq (n-2)-3 \geq 1$ by Lemma 2. If $G[V - \{v_1\}]$ contains an edge, then G contains an $S_n(2,1)$, and hence, $V - \{v_1\}$ is an independent set. Thus, since $\delta(G[V]) \geq 1$, we have $v_1 v_i \in E(G)$ for $2 \leq i \leq n-2$ which implies $N_U(v_i) = \emptyset$ for $2 \leq i \leq n-2$ since otherwise G contains an $S_n(2,1)$. Noting that $n \geq 6$, we have $|V - \{v_1\}| \geq 3$. By Lemma 2, we have $\delta(G[U]) \geq n-3$ which contradicts (1). Thus, we have $R(S_n(2,1), W_6) \leq 2n-1$, and hence, $R(S_n(2,1), W_6) = 2n-1$.

Finally, we show $R(S_n(1,2), W_6) \leq 2n-1$ for $n \not\equiv 0 \pmod 3$. If G contains no $S_n(1,2)$, then we have

$$N_U(v_i) \cap N_U(v_j) = \emptyset, \qquad \text{for all } v_i, v_j \in V \quad \text{and} \quad i \neq j, \tag{2}$$

$$N_V(v_i) = \emptyset, \qquad \text{if } N_U(v_i) \neq \emptyset, \quad \text{for each } v_i \in V, \tag{3}$$

and

$$N_U(u) = \emptyset, \qquad \text{if } u \in N_U(v_i), \quad \text{for some } v_i \in V. \tag{4}$$

CLAIM 1. *$d_U(v) \leq 2$ for each $v \in V$.*

PROOF. Suppose that $d_U(v) \geq 3$ for some $v \in V$ and let $W = \{w_1, w_2, w_3\} \subseteq N_U(v)$. By (4), W is an independent set. Let $Z = \{z_1, z_2, z_3\} \subseteq V - \{v\}$. Then $E(Z, W) = \emptyset$ by (2). If $N_U(z_i) = \emptyset$ for $i = 1, 2, 3$, then by Lemma 2, we have $\delta(G[U]) \geq n-3$ which contradicts (1). Thus, without loss of generality, we may assume that $N_U(z_1) \neq \emptyset$. Thus, by (3) we must have $N_V(z_1) = N_V(v) \neq \emptyset$. We then see that $\bar{G}[z_1, z_2, z_3, v, w_1, w_2, w_3]$ contains a wheel W_6 with hub z_1, a contradiction. ∎

Since $n \geq 6$, by Claim 1, we can choose three vertices, say $u_2, u_3, u_4 \in U - \{u_1\}$ such that $|N(v_i) \cap \{u_2, u_3, u_4\}| \leq 1$ for $i = 2, 3, 4$. Thus, by (2) and (4), we can see $\bar{G}[v_2, v_3, v_4, u_1, u_2, u_3, u_4]$ contains a W_6 with the hub u_1, a contradiction. This implies $R(S_n(1,2), W_6) \leq 2n-1$, and hence, $R(S_n(1,2), W_6) = 2n-1$ for $n \not\equiv 0 \pmod 3$. ∎

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