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Note

# The Ramsey numbers of paths versus wheels ☆

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**Abstract**

For two given graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the smallest integer $n$ such that for any graph $G$ of order $n$, either $G$ contains $G_1$ or the complement of $G$ contains $G_2$. Let $P_n$ denote a path of order $n$ and $W_m$ a wheel of order $m+1$. In this paper, we show that $R(P_n, W_m)=2n-1$ for $m$ even and $n \geqslant m-1 \geqslant 3$ and $R(P_n, W_m)=3n-2$ for $m$ odd and $n \geqslant m-1 \geqslant 2$.


*Keywords:* Ramsey number; Path; Wheel

All graphs considered in this paper are finite simple graph without loops. For two given graphs $G_1$ and $G_2$, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer $n$ such that for any graph $G$ of order $n$, either $G$ contains $G_1$ or $\overline{G}$ contains $G_2$, where $\overline{G}$ is the complement of $G$. The *neighborhood* $N(v)$ of a vertex $v$ is the set of vertices adjacent to $v$ in $G$ and $N[v]=N(v)\cup\{v\}$. The *minimum degree* of $G$ is denoted by $\delta(G)$. $C_n$ and $P_n$ denote a cycle and a path of order $n$, respectively. A *Wheel* $W_n=\{x\}+C_n$ is a graph of $n+1$ vertices, namely, a vertex $x$, called the *hub* of the wheel, adjacent to all vertices of $C_n$. $mK_n$ denotes the union of $m$ vertex disjoint $K_n$. The lengths of the longest cycle and path of $G$ are denoted by $c(G)$ and $p(G)$, respectively.

Some results on Ramsey numbers concerning paths or wheels are obtained. See for instance [3,5]. For a survey, see [4]. In [2], Faudree et al. considered the Ramsey numbers for all path-cycle pairs and obtained the following.

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**Theorem 1** (*Faudree et al. [2]*). *(1) $R(P_n, C_m) = 2n - 1$ for $m$ odd and $n \geqslant m - 1 \geqslant 2$; (2) $R(P_n, C_m) = n + m/2 - 1$ for $m$ even and $n \geqslant m - 1 \geqslant 3$.*

In [6], Surahmat et al. obtained the Ramsey numbers of a path versus $W_4$ or $W_5$.

**Theorem 2** (*Surahmat et al. [6]*). *(1) $R(P_n, W_5) = 3n - 2$ for $n \geqslant 4$; (2) $R(P_n, W_4) = 2n - 1$ for $n \geqslant 3$.*

In this paper, we evaluate the Ramsey numbers of paths versus wheels in a more general situation. The main result of this paper is the following.

**Theorem 3.** *(1) $R(P_n, W_m) = 3n - 2$ for $m$ odd and $n \geqslant m - 1 \geqslant 2$; (2) $R(P_n, W_m) = 2n - 1$ for $m$ even and $n \geqslant m - 1 \geqslant 3$.*

For the case $n \leqslant m - 2$, Zhang et al. have determined all the values of $R(P_n, W_m)$ for $n \geqslant \lceil m/2 \rceil$ and established a best possible upper bound for $R(P_n, W_m)$ with $n \leqslant \lceil m/2 \rceil - 1$. The values of $R(P_n, W_m)$ for $n \leqslant \lceil m/2 \rceil - 1$ are still not known. It may be difficult to determine the values of $R(P_n, W_m)$ for $n \leqslant \lceil m/2 \rceil - 1$.

In order to prove Theorem 3, we need the following lemmas.

**Lemma 1** (*Dirac [1]*). *Let $G$ be a connected graph of order $n \geqslant 3$ with $\delta(G) = \delta$. Then $p(G) \geqslant \min\{2\delta, n - 1\}$.*

**Lemma 2.** *Let $G$ be a graph with $|G| \geqslant R(P_n, C_m) + 1$. If there is a vertex $v \in V(G)$ such that $|N[v]| \leqslant |G| - R(P_n, C_m)$ and $G$ contains no $P_n$, then $\overline{G}$ contains a $W_m$.*

**Proof.** Let $G' = G - N[v]$, then $|G'| \geqslant R(P_n, C_m)$. Since $G$ contains no $P_n$, we can see $G'$ contains no $P_n$ which implies $\overline{G'}$ contains a $C_m$ and hence $\overline{G}$ contains a $W_m$ with the hub $v$. □

**Proof of Theorem 3.** Let $G$ be a graph with $|G| \geqslant R(P_n, C_m) + 1$ and $H$ a maximum component of $G$. Suppose to the contrary neither $G$ contains a $P_n$ nor $\overline{G}$ contains a $W_m$. By Lemma 2, we may assume $|N[v]| \geqslant |G| - R(P_n, C_m) + 1$ for any vertex $v \in V(G)$ and hence

$$\delta(G) \geqslant |G| - R(P_n, C_m). \qquad (*)$$

(1) If $|G| = 3n - 2$, $m$ is odd and $n \geqslant m - 1 \geqslant 2$, then by Theorem 1(1) and $(*)$, we have $\delta(G) \geqslant n - 1$ which implies $|H| \geqslant n$. Thus, since $n \geqslant 3$, we have $p(G) \geqslant p(H) \geqslant \min\{2(n - 1), |H| - 1\} \geqslant n - 1$ by Lemma 1 and hence $G$ contains a $P_n$, a contradiction. Thus $R(P_n, W_m) \leqslant 3n - 2$. Noting that $G = 3K_{n-1}$ contains no $P_n$ and $\overline{G}$ contains no $W_m$ for $m$ is odd, we have $R(P_n, W_m) \geqslant 3n - 2$ and hence $R(P_n, W_m) = 3n - 2$.

(2) If $|G| = 2n - 1$, $m$ is even and $n \geqslant m - 1 \geqslant 3$, then by Theorem 1(2) and $(*)$, we have $\delta(G) \geqslant n - m/2$. If $|H| \geqslant n$, then since $n \geqslant m - 1$, we have $p(G) \geqslant p(H) \geqslant \min\{2(n - m/2), |H| - 1\} \geqslant n - 1$ by Lemma 1 and hence $G$ contains a $P_n$. Thus we may assume $|H| \leqslant n - 1$ which implies $G$ contains at least three components since $|G| = 2n - 1$. For any

component $H'$ of $G$, since $\delta(G) \geqslant n - m/2$ and $n \geqslant m-1$, we have $|H'| \geqslant m/2$. Thus we can see $\overline{G}$ contains a $W_m$ and hence $R(P_n, W_m) \leqslant 2n-1$. Since $G = 2K_{n-1}$ contains no $P_n$ and $\overline{G}$ contains no wheels, we have $R(P_n, W_m) \geqslant 2n-1$ and hence $R(P_n, W_m) = 2n-1$.

The proof of Theorem 3 is completed. □

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