The Ramsey Numbers $R(T_n, W_6)$ for Small n*

Yaojun CHEN†, Yunqing ZHANG and Kemin ZHANG
Department of Mathematics, Nanjing University,
Nanjing 210093, P.R. China

Abstract: Let T_n denote a tree of order n and W_m a wheel of order $m+1$. Baskoro et al. conjectured in [2] that if T_n is not a star, then $R(T_n, W_m) = 2n-1$ for $m \geq 6$ even and $n \geq m-1$. We disprove the Conjecture in [6]. In this paper, we determine $R(T_n, W_6)$ for $n \leq 8$ which is the first step for us to determine $R(T_n, W_6)$ for any tree T_n.

Key words: Ramsey number, Tree, Wheel

1. Introduction

All graphs considered in this paper are finite simple graph without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest positive integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. Let G be a graph. The *neighborhood* $N(v)$ of a

*This project was supported by NSFC under grant number 10201012
†This project was partially supported by Nanjing University Talent Development Foundation

vertex v is the set of vertices adjacent to v in G. The *minimum and maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a vertex $v \in V(G)$ and a subgraph H of G, $N_H(v)$ is the set of neighbors of v contained in H, i.e., $N_H(v) = N(v) \cap V(H)$. We let $d_H(v) = |N_H(v)|$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. Let U, V be two disjoint vertex set. We use $E(U, V)$ to denote the set of edges between U and V. Let m be a positive integer, we use mG to denote m vertex disjoint copies of G. A path and a cycle of order n are denoted by P_n and C_n respectively. A *Star* S_n $(n \geq 3)$ is a bipartite graph $K_{1,n-1}$. A *Wheel* $W_n = \{x\} + C_n$ is a graph of $n+1$ vertices, that is, a vertex x, called the *hub* of the wheel, adjacent to all vertices of C_n. $S_n(l, m)$ is a tree of order n obtained from $S_{n-l\times m}$ by subdividing each of its l edges m times. $S_n(l)$ is a tree of order n obtained from an S_l and an S_{n-l} by adding an edge joining the centers of them. $S_n[l]$ is a tree of order n obtained from an S_l and an S_{n-l} by adding an edge joining a vertex of degree one of S_l to the center of S_{n-l}. A graph on n vertices is *pancyclic* if it contains cycles of every length l, $3 \leq l \leq n$.

Many Ramsey numbers concerning wheel or star have been established, see for instance [3, 7, 8, 9]. Recently, the following Ramsey numbers were obtained.

Theorem A (Surahmat et al. [10]). $R(S_n, W_4) = 2n - 1$ for $n \geq 3$ and odd; $R(S_n, W_4) = 2n+1$ for $n \geq 4$ and even; $R(S_n, W_5) = 3n-2$ for $n \geq 4$.

Theorem B (Baskoro et al. [2]). Let T_n be a tree of order n other than S_n. Then $R(T_n, W_4) = 2n - 1$ for $n \geq 3$; $R(T_n, W_5) = 3n - 2$ for $n \geq 4$.

Motivated by Theorem B, Baskoro et al. [2] posed the following conjecture.

Conjecture 1. Let T_n be a tree other than S_n and $n \geq m-1$. Then

$R(T_n, W_m) = 2n - 1$ for $m \geq 6$ even; $R(T_n, W_m) = 3n - 2$ for $m \geq 7$ and odd.

In [6], we consider $R(T_n, W_6)$ for $T_n \neq S_n$ and $\Delta(T_n) \geq n - 3$ and obtain the following.

Theorem C (Chen et al. [6]). $R(S_n(1,1), W_6) = 2n$ for $n \geq 4$.

Theorem D (Chen et al. [6]). $R(S_n(1,2), W_6) = 2n$ for $n \geq 6$ and $n \equiv 0 \pmod 3$.

Theorem E (Chen et al. [6]). $R(S_n(3), W_6) = R(S_n(2,1), W_6) = 2n - 1$ for $n \geq 6$; $R(S_n(1,2), W_6) = 2n - 1$ for $n \geq 6$ and $n \not\equiv 0 \pmod 3$.

By Theorems C and D, we can see that Conjecture 1 is not true when m is even. However, we believe that for $n \geq 5$, if $T_n \neq S_n(1,1)$, $S_n(1,2)$, then $R(T_n, W_6) = 2n - 1$. In order to determine $R(T_n, W_6)$ for general tree T_n, this paper consider $R(T_n, W_6)$ for $n \leq 8$ as the first step. The main result is the following.

Theorem 1. Let T_n be a tree of order n other than S_n and $5 \leq n \leq 8$. If $T_n \neq S_n(1,1)$ and $T_n \neq S_n(1,2)$ for $n \equiv 0 \pmod 3$, then $R(T_n, W_6) = 2n - 1$.

2. Some Lemmas

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 (Bondy [1]). Let G be a graph of order n. If $\delta(G) \geq n/2$, then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.

Lemma 2 (Chen et al. [4]). $R(P_n, W_m) = 2n - 1$ for m even and $n \geq m - 1 \geq 3$.

Lemma 3 (Chen et al. [5]). $R(S_n, W_6) = 2n + 1$ for $n \geq 3$.

Lemma 4 (Chen et al. [6]). Let G be a graph of order $2n - 1 \geq 7$ and (U, W) a partition of $V(G)$ with $|U| \geq 3$ and $|W| \geq 4$. Suppose $u_i \in U$ and $N_W(u_i) = \emptyset$, where $1 \leq i \leq 3$. If $\overline{G}$ contains no W_6, then $\delta(G[W]) \geq |W| - 3$.

Lemma 5. Let G be a graph of order 7 and $\delta(G) \geq 4$. Then for any $v \in V(G)$, G contains a tree $T = S_7(3, 1)$ such that $d_T(v) = 3$.

Proof. Let $G' = G - v$. Then $\delta(G') \geq 3$ and hence G' contains a C_6 by Lemma 1. Since $d(v) \geq 4$, after an easy check, we can see G contains a tree $T = S_7(3, 1)$ such that $d_T(v) = 3$. ▮

Lemma 6. $R(S_n[4], W_6) = 2n - 1$ for $n \geq 8$.

Proof. Let G be a graph of order $2n - 1$. If $\overline{G}$ contains no W_6, then G contains an $S_n(3)$ by Theorem E. Let T be an $S_n(3)$ with $V(T) = V_0 \cup W$, where $V_0 = \{v_0, v_1, \ldots, v_{n-3}\}$, $W = \{w_1, w_2\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq n-3\} \cup \{v_1w_1, v_1w_2\}$. Set $U = V(G) - V(T)$. If G contains no $S_n[4]$, then we have

$$v_1v_i \notin E(G) \text{ for } 2 \leq i \leq n - 3, \tag{1}$$

and

$$\text{for } u \in U \text{ with } d_U(u) \geq 2,\ N(u) \cap (V_0 - \{v_0\}) = \emptyset. \tag{2}$$

Since $|U| = n - 1 \geq 7$ and $\overline{G}[U]$ contains no W_6, by Lemma 3, $G[U]$ contains an S_3 which implies

$$\Delta(G[U]) \geq 2. \tag{3}$$

Claim 1. $d_U(v_1) \leq n - 8$.

Proof. If $d_U(v_1) \geq n - 7$, then since G contains no $S_n[4]$, we have $\Delta(G[V_0 - \{v_0, v_1\}]) \leq 1$. If there are two vertices $u_1, u_2 \in U$ such that $d_U(u_i) \geq 2$ for $i = 1, 2$, then since $n \geq 8$ we can see $\overline{G}[\{v_1, v_2, v_3, v_4, v_5, u_1, u_2\}]$ contains a W_6 with the hub v_1 by (1) and (2), a contradiction. Thus, by (3), we conclude that there is only one vertex $u \in U$ such that $d_U(u) \geq 2$ which implies $N_U(u)$ is an independent set. If $d_U(u) \geq n-4$, then $N(v_i) \cap N_U(u) = \emptyset$ for $i = 1, 2, 3$ since otherwise G contains an $S_n[4]$. Thus, since $n \geq 8$, $\overline{G}[\{v_1, v_2, v_3, u_1, u_2, u_3, u_4\}]$ contains a W_6 with the hub u_1 for any four vertices $u_1, u_2, u_3, u_4 \in N_U(u)$, a contradiction. Hence we have $d_U(u) \leq n-5$. Let $U' = U - N(u)$, then $|U'| \geq 3$. If $G[U']$ contains an edge, say $u_1u_2 \in E(G[U'])$, then $|N(u_i) \cap (V_0 - \{v_0\})| \leq 1$ for $i = 1, 2$ since otherwise $G[V_0 \cup \{u_1, u_2\}]$ contains an $S_n[4]$. Thus, noting that $n \geq 8$ and $\Delta(G[V_0 - \{v_0, v_1\}]) \leq 1$, $\overline{G}[\{v_1, v_2, v_3, v_4, v_5, u_1, u\}]$ contains a W_6 with the hub u by (1) and (2). Hence we may assume U' is an independent set which implies $U - \{u\}$ is an independent set. If $n \geq 9$, then $\overline{G}[U - \{u\}] = K_{n-2}$ and hence $\overline{G}[U]$ contains a W_6, a contradiction. If $n = 8$, then $|U| = 7$ and $d_U(u) \leq 3$. It is easy to see $\overline{G}[U]$ contains a W_6 in this case, again a contradiction. ∎

Let $U' = U - N(v_1)$. By Claim 1, we have $|U'| \geq 7$. If $\Delta(G[U']) \leq 2$, then by Lemma 1, $\overline{G}[U']$ contains a C_6 and hence $\overline{G}$ contains a W_6 with the hub v_1, a contradiction. Thus there is some vertex $u \in U'$ such that $d_{U'}(u) \geq 3$. Assume $\{u_1, u_2, u_3\} \subseteq N_{U'}(u)$. If there is some u_i with $1 \leq i \leq 3$ such that $d_{V_0}(u_i) \geq 2$, then $G[V_0 \cup \{u, u_i\}]$ contains an $S_n[4]$. Hence we have $d_{V_0}(u_i) \leq 1$. If there is some vertex v_i with $2 \leq i \leq n-3$ such that $|N(v_i) \cap \{u_1, u_2, u_3\}| \geq 2$, then we have $|N(v_j) \cap \{u_1, u_2, u_3\}| \leq 1$ for any j with $2 \leq j \leq n-3$ and $j \neq i$. Thus, since $n - 3 \geq 6$ we can always choose three vertices, say v_2, v_3, v_4 such that $|N(v_i) \cap \{u_1, u_2, u_3\}| \leq 1$ for $2 \leq i \leq 4$. Noting that $d_{V_0}(u_i) \leq 1$ for $1 \leq i \leq 3$, we can see that $\overline{G}[\{u_1, u_2, u_3, v_2, v_3, v_4\}]$ contains a C_6 and hence $\overline{G}$ contains a W_6 with the hub v_1 by (1), a contradiction. Thus we have $R(S_n[4], W_6) \leq 2n-1$. On the other hand, the graph $G = 2K_{n-1}$ shows $R(S_n[4], W_6) \geq 2n-1$ and hence we have $R(S_n[4], W_6) = 2n-1$. ∎

Lemma 7. $R(S_n(1,3), W_6) = 2n-1$ for $n \geq 8$.

Proof. Let G be a graph of order $2n-1$. If $\overline{G}$ contains no W_6, then G contains an $S_n[4]$ by Lemma 6. Let T be an $S_n[4]$ with $V(T) = V_0 \cup W$, where $V_0 = \{v_0, v_1, \ldots, v_{n-4}\}$, $W = \{w_1, w_2, w_3\}$ and $E(T) = \{v_0 v_i \mid 1 \leq i \leq n-4\} \cup \{v_1 w_1, w_1 w_2, w_1 w_3\}$. Let $U = V(G) - V(T)$. If G contains no $S_n(1,3)$, then we have

$$N(w_i) \cap (V_0 \cup U - \{v_0\}) = \emptyset \text{ for } i = 2, 3, \tag{4}$$

and if $u \in U$ and $d_U(u) \geq 2$, then

$$N(u_i) \cap (V_0 - \{v_0\}) = \emptyset \text{ for any } u_i \in N_U(u). \tag{5}$$

If $\Delta(G[U]) \leq 2$, then since $|U| = n-1 \geq 7$, $\overline{G}[U]$ contains a C_6 by Lemma 1 and hence $\overline{G}$ contains a W_6 with the hub w_2 by (4), a contradiction. Thus there is some vertex $u \in U$ such that $d_U(u) \geq 3$. Assume $\{u_1, u_2, u_3\} \subseteq N_U(u)$. By (4) and (5), we can see $\overline{G}[\{w_2, v_1, v_2, v_3, u_1, u_2, u_3\}]$ contains a W_6 with the hub w_2, again a contradiction. Thus we have $R(S_n(1,3), W_6) \leq 2n-1$. On the other hand, the graph $G = 2K_{n-1}$ shows $R(S_n(1,3), W_6) \geq 2n-1$ and hence we have $R(S_n(1,3), W_6) = 2n-1$. ∎

Lemma 8. $R(S_n(3,1), W_6) = 2n-1$ for $n \geq 8$.

Proof. Let G be a graph of order $2n-1$. If $\overline{G}$ contains no W_6, then by Theorem E, G contains an $S_n(2,1)$. Let $T = S_n(2,1)$ with $V(T) = V_0 = \{v_0, \ldots, v_{n-3}, w_1, w_2\}$ and $E(T) = \{v_0 v_i \mid 1 \leq i \leq n-3\} \cup \{v_1 w_1, v_2 w_2\}$. Set $U = V(G) - V_0$. Obviously, $|U| = n-1 \geq 7$.

If G contains no $S_n(3,1)$, then $N_U(v_i) = \emptyset$ for $3 \leq i \leq n-3$ and $\{v_3, \ldots, v_{n-3}\}$ is an independent set. If $n \geq 9$, then $\overline{G}[\{v_3, v_4, v_5, v_6, u_1, u_2, u_3\}]$ contains a W_6 with the hub v_3 for any three vertices $u_1, u_2, u_3 \in U$, a contradiction. Hence we have $n = 8$. By Lemma 4, we have $\delta(G[U]) \geq 4$. By Lemma 5, we have $N_{V_0}(u) = \emptyset$ for any $u \in U$ which implies $\delta(G[V_0]) \geq 5$ by Lemma 4. Noting that $\{v_3, v_4, v_5\}$ is an independent set, we have $\{v_1, w_1\} \subseteq N(v_3) \cap N(v_4)$ which implies G contains an $S_n(3,1)$, a contradiction. Thus we have $R(S_n(3,1), W_6) \leq 2n-1$. On the other hand, the graph $G = 2K_{n-1}$

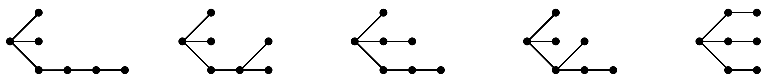
shows $R(S_n(3,1), W_6) \geq 2n-1$ and hence we have $R(S_n(3,1), W_6) = 2n-1$. ▮

3. The Ramsey Numbers $R(T_n, W_6)$ for $\Delta(T_n) = 3$ and $n = 7$

In this section, we determine $R(T_n, W_6)$ for $n = 7$ and $\Delta(T_n) = 3$.

Theorem 2. $R(T_n, W_6) = 13$ for $n = 7$ and $\Delta(T_n) = 3$.

Proof. Let T be tree with order 7 and $\Delta(T) = 3$, then it is not difficult to see T must be isomorphic to one of the five trees of order 7 below.



T_{7a} T_{7b} T_{7c} T_{7d} T_{7e}

Thus we need only to show $R(T, W_6) = 13$ for $T = T_{7a}, T_{7b}, T_{7c}, T_{7d}$ and T_{7e}.

Let G be a graph of order 13. Suppose $\overline{G}$ contains no W_6. Since $2K_6$ contains no trees of order 7 and its complement contains no W_6, we have $R(T, W_6) \geq 13$ for each tree T with $|T| = 7$. In the following proof, we need only to prove $R(T, W_6) \leq 13$ for each $T \in \{T_{7a}, T_{7b}, T_{7c}, T_{7d}, T_{7e}\}$.

We first show $R(T_{7a}, W_6) = R(T_{7c}, W_6) = 13$. By Theorem E, G contains an $S_7(1,2)$. Let T be an $S_7(1,2)$ in G with $V(T) = V = \{v_0, \ldots, v_4, w_1, w_2\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, w_1w_2\}$. Set $U = V(G) - V$. Obviously, $|U| = 6$.

If G contains no T_{7a}, then $N(w_2) \cap (U \cup \{v_2, v_3, v_4\}) = \emptyset$. If $d_U(u) \leq 2$ for each $u \in U$, then $\overline{G}[U]$ contains a C_6 by Lemma 1 and hence $\overline{G}$ contains a W_6 with the hub w_2, a contradiction.

Thus there is some vertex $u \in U$ such that $d_U(u) \geq 3$. Assume $u_1, u_2, u_3 \in N_U(u)$. Since G contains no T_{7a}, we have $v_i u_j \notin E(G)$ for $2 \leq i \leq 4$ and $1 \leq j \leq 3$. Thus $\overline{G}[\{w_2, u_1, u_2, u_3, v_2, v_3, v_4\}]$ contains a W_6 with the hub w_2, a contradiction. Thus we have $R(T_{7a}, W_6) \leq 13$.

If G contains no T_{7c}, then $N_U(v_i) = \emptyset$ for $2 \leq i \leq 4$ and $\{v_2, v_3, v_4\}$ is an independent set. Thus by Lemma 4, we have $\delta(G[U]) \geq 3$ which implies $G[U]$ contains a C_6. In this case, we have $N_V(u) = \emptyset$ for any $u \in U$ since otherwise G contains a T_{7c}. By Lemma 4, we have $\delta(G[V]) \geq 4$ which implies $N_V(v_i) = \{v_0, v_1, w_1, w_2\}$ for $i = 2, 3, 4$ and hence $G[V]$ contains a T_{7c}, a contradiction. Thus we have $R(T_{7c}, W_6) \leq 13$.

Next, we show $R(T_{7b}, W_6) = R(T_{7d}, W_6) = 13$. By Theorem E, G contains an $S_7(3)$. Let $T = S_7(3)$, $V(T) = \{v_0, \ldots, v_4, w_1, w_2\}$, $E(T) = \{v_0 v_i \mid 1 \leq i \leq 4\} \cup \{v_1 w_1, v_1 w_2\}$. Set $U = V(G) - V(T)$. Obviously, $|U| = 6$.

If G contains no T_{7b}, we have $v_1 v_i \notin E(G)$ for $2 \leq i \leq 4$. For any $u \in U$, if $u \in N(v_i)$ for some i with $1 \leq i \leq 4$, then $d_U(u) \leq 1$. Thus, if there are three vertices $u_1, u_2, u_3 \in U$ such that $d_U(u_i) \geq 2$ for $1 \leq i \leq 3$, then $\overline{G}[\{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a W_6 with the hub v_1, a contradiction. If there is some v_i with $2 \leq i \leq 4$ such that $d_U(v_i) \geq 2$, then G contains a T_{7b} and hence we have $d_U(v_i) \leq 1$ for $2 \leq i \leq 4$. If $G[\{v_2, v_3, v_4\}]$ contains two edges, then G contains a T_{7b} and hence we may assume $v_2 v_3, v_2 v_4 \notin E(G)$. Let $U = \{u_i \mid 1 \leq i \leq 6\}$. Assume $N_U(v_2) \subseteq \{u_1\}$ and $N_U(v_3) \cup N_U(v_4) \subseteq \{u_1, u_2, u_3\}$. If $u_2 \notin N_U(v_3) \cup N_U(v_4)$, then since U contains at most two vertices with degree not less than 2, we can see $\overline{G}[\{u_2, u_4, u_5, u_6\}]$ contains a P_3, say $u_2 u_4 u_5$ is a P_3. Then $v_3 u_2 u_4 u_5 v_4 u_6 v_3$ is a C_6 in $\overline{G}$ and hence $\overline{G}$ contains a W_6 with the hub v_2. If $u_2 \in N_U(v_3) \cup N_U(v_4)$, then since $d_U(u_2) \leq 1$, we may assume $u_2 u_4, u_2 u_5 \notin E(G)$ which implies $v_3 u_4 u_2 u_5 v_4 u_6 v_3$ is a C_6 in $\overline{G}$ and hence $\overline{G}$ contains a W_6 with the hub v_2, again a contradiction. Thus we have $R(T_{7b}, W_6) \leq 13$.

If G contains no T_{7d}, then $N_U(w_1) = N_U(v_i) = \emptyset$ for $2 \le i \le 4$ and $N(w_1) \cap \{v_2, v_3, v_4\} = \emptyset$. Thus $\overline{G}[\{w_1, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a W_6 with the hub w_1 for any three vertices $u_1, u_2, u_3 \in U$, a contradiction. Thus we have $R(T_{7d}, W_6) \le 13$.

Finally, we show $R(T_{7e}, W_6) = 13$. By Theorem E, G contains an $S_7(2,1)$. Let $T = S_7(2,1)$, $V(T) = V = \{v_0, \dots, v_4, w_1, w_2\}$ and $E(T) = \{v_0v_i \mid 1 \le i \le 4\} \cup \{v_1w_1, v_2w_2\}$. Set $U = V(G) - V$. Obviously, $|U| = 6$.

If G contains no T_{7e}, then $N_U(v_i) = \emptyset$ for $i = 3, 4$ and $v_3v_4 \notin E(G)$. If $d_U(v_0) = 0$, then by Lemma 4, we have $\delta(G[U]) \ge 3$ which implies $G[U]$ contains a C_6 by Lemma 1 and hence $N_V(u) = \emptyset$ for any $u \in U$ since G contains no T_{7e}. Thus by Lemma 4, we have $\delta(G[V]) \ge 4$. Noting that $v_3v_4 \notin E(G)$, after an easy check, we can see $G[V]$ contains a T_{7e} and hence we may assume $d_U(v_0) \ge 1$.

For any $u \in U$, if $uv_0 \in E(G)$, then $d_U(u) = 0$ for otherwise G contains a T_{7e}. If $d_U(v_0) \ge 2$, say $u_1, u_2 \in N_U(v_0)$, then $\{v_3, v_4, u_1, u_2\}$ is an independent set and for any $u \in U - \{u_1, u_2\}$, $N(u) \cap \{v_3, v_4, u_1, u_2\} = \emptyset$. In this case, we can see $\overline{G}[\{v_3, v_4, u_1, u_2, u_3, u_4, u_5\}]$ contains a W_6 with the hub v_3 for any three vertices $u_3, u_4, u_5 \in U - \{u_1, u_2\}$, a contradiction. Hence we may assume $d_U(v_0) = 1$.

Let $N_U(v_0) = \{u_1\}$ and $U' = U - \{u_1\} = \{u_2, u_3, u_4, u_5, u_6\}$. If $G[U']$ contains a C_5, then for any $u \in U'$, $N_V(u) = \emptyset$ for otherwise G contains a T_{7e}. By Lemma 4, we have $\delta(G[V]) \ge 4$. Noting that $v_3v_4 \notin E(G)$, after an easy check, we can see $G[V]$ contains a T_{7e}. Hence we may assume $G[U']$ contains no C_5. By Lemma 1, there is some vertex $u \in U'$ such that $d_{U'}(u) \le 2$ which implies $\overline{G}[U']$ contains a P_3, say $u_2u_3u_4$ is a P_3 in $\overline{G}$. Then $v_4u_2u_3u_4u_1u_5v_4$ is a C_6 in $\overline{G}$ and hence $\overline{G}$ contains a W_6 with the hub v_3, also a contradiction. Thus we have $R(T_{7e}, W_6) \le 13$.

The proof of Theorem 2 is completed. ∎

4. The Ramsey Numbers $R(T_n, W_6)$ for $3 \leq \Delta(T_n) \leq 4$ and $n = 8$

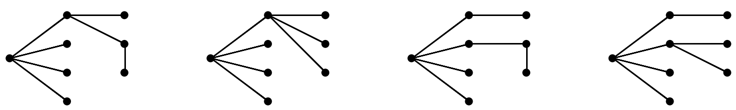
In this section, we determine $R(T_n, W_6)$ for $n = 8$ and $3 \leq \Delta(T_n) \leq 4$.

Theorem 3. $R(T_n, W_6) = 15$ for $n = 8$ and $3 \leq \Delta(T_n) \leq 4$.

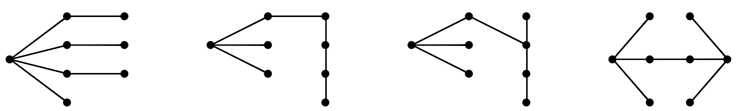
Proof. Let T be a tree with order 8 and $3 \leq \Delta(T) \leq 4$. Since $2K_7$ contains no trees of order 8 and its complement contains no W_6, we have $R(T, W_6) \geq 15$. Thus, in order to prove $R(T, W_6) = 15$, we need only to show $R(T, W_6) \leq 15$.

If $T = S_8[4]$, then Theorem 3 holds by Lemma 6. If $T = S_8(1,3)$, then Theorem 3 holds by Lemma 7. If $T \neq S_8[4], S_8(1,3)$, then we have the following.

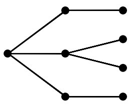
Proposition 1. Let T be a tree of order 8 with $3 \leq \Delta(T) \leq 4$. If $T \neq S_8[4], S_8(1,3)$, then T must be isomorphic to one of the fifteen trees of order 8 below.



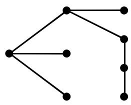
T_{8a} T_{8b} T_{8c} T_{8d}



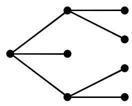
T_{8e} T_{8f} T_{8g} T_{8h}



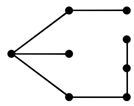
T_{8i}



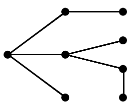
T_{8j}



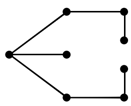
T_{8k}



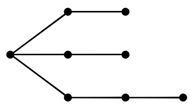
T_{8l}



T_{8m}



T_{8n}



T_{8o}

Let G be a graph of order 15. Suppose $\overline{G}$ contains no W_6. By Proposition 1, we will complete the proof by showing the following theorems.

Theorem 4. $R(T, W_6) = 15$ for $T = T_{8a}, T_{8b}, T_{8d}$ or T_{8k}.

Proof. By Theorem E, G contains an $S_8(3)$. Let T be an $S_8(3)$ with $V(T) = V = \{v_0, \ldots, v_5, w_1, w_2\}$ and $E(T) = \{v_0v_i \mid 1 \le i \le 5\} \cup \{v_1w_1, v_1w_2\}$. Set $U = V(G) - V$. Obviously, $|U| = 7$.

If G contains no T_{8a}, then $w_1v_i \notin E(G)$ for $2 \le i \le 5$ and $N_U(w_1) = \emptyset$. If there is some vertex v_i with $2 \le i \le 5$ such that $d_U(v_i) \ge 2$, say $u_1, u_2 \in N_U(v_2)$, then $v_2v_i \notin E(G)$ for $3 \le i \le 5$ and $v_iu_j \notin E(G)$ for $3 \le i \le 5$ and $j = 1, 2$ since otherwise G contains a T_{8a}. Thus, $\overline{G}[\{w_1, v_2, u_1, u_2, v_3, v_4, v_5\}]$ contains a W_6 with the hub w_1, a contradiction. Hence we may assume $d_U(v_i) \le 1$ for $2 \le i \le 5$ which implies there are three vertices $u_1, u_2, u_3 \in U$ such that $v_iu_j \notin E(G)$ for $2 \le i \le 5$ and $1 \le j \le 3$. Thus $\overline{G}[\{w_1, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a W_6 with the hub w_1, again a contradiction. Hence we have $R(T_{8a}, W_6) \le 15$.

If G contains no T_{8b}, then we have $N_U(v_1) = \emptyset$, $v_1v_i \notin E(G)$ for $2 \le i \le 5$ and $d_U(v_i) \le 2$ for $2 \le i \le 5$. Thus, since $|U| = 7$, we can choose three vertices $u_1, u_2, u_3 \in U$ such that $d_V(u_i) \le 1$ for $1 \le i \le 3$ and $N_V(u_i) \cap N_V(u_j) = \emptyset$ for $i, j \in \{1, 2, 3\}$ and $i \ne j$, and hence $\overline{G}[\{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a W_6 with the hub v_1, a contradiction. Thus we have $R(T_{8b}, W_6) \leq 15$.

If G contains no T_{8d}, then $v_2 v_i \notin E(G)$ for $3 \leq i \leq 5$ and $N_U(v_i) = \emptyset$ for $2 \leq i \leq 5$. Thus for any three vertices $u_1, u_2, u_3 \in U$, $\overline{G}[\{v_2, v_3, v_4, v_5, u_1, u_2, u_3\}]$ contains a W_6 with the hub v_2, a contradiction. Hence we have $R(T_{8d}, W_6) \leq 15$.

If G contains no T_{8k}, then $d_U(v_i) \leq 1$ for $2 \leq i \leq 5$. Let $V' = \{v_2, v_3, v_4, v_5\}$. Since $|U| = 7$, there are three vertices $u_1, u_2, u_3 \in U$ such that $N_{V'}(u_i) = \emptyset$ for $1 \leq i \leq 3$. If $\delta(G[V']) = 0$, say $d_{V'}(v_2) = 0$, then $\overline{G}[\{v_2, v_3, v_4, v_5, u_1, u_2, u_3\}]$ contains a W_6 with the hub v_2, a contradiction. Hence we have $\delta(G[V']) \geq 1$. If $\Delta(G[V']) \geq 2$, then G contains a T_{8k}. Thus we may assume $E(G[V']) = \{v_2v_3, v_4v_5\}$. In this case, we have $N_U(v_i) = \emptyset$ for $2 \leq i \leq 5$. By Lemmas 1 and 4, $G[U]$ contains a C_7. If $N_U(w_1) \neq \emptyset$ or $\{v_2, v_3\} \subseteq N(w_1)$, then G contains a T_{8k}. Hence we may assume $N_U(w_1) = \emptyset$ and $v_2 \notin N(w_1)$. Thus, $\overline{G}[\{v_2, w_1, v_4, v_5, u_1, u_2, u_3\}]$ contains a W_6 with the hub v_2 for any three vertices $u_1, u_2, u_3 \in U$, a contradiction. Hence we have $R(T_{8k}, W_6) \leq 15$. ∎

Theorem 5. $R(T, W_6) = 15$ for $T = T_{8c}, T_{8n}$.

Proof. By Theorem E, G contains an $S_8(1,2)$. Let $T = S_8(1,2)$ with $V(T) = V = \{v_0, \ldots, v_5, w_1, w_2\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 5\} \cup \{v_1w_1, w_1w_2\}$. Set $V' = \{v_2, v_3, v_4, v_5\}$ and $U = V(G) - V$. Obviously, $|U| = 7$.

If G contains no T_{8c}, then $v_2 v_i \notin E(G)$ for $3 \leq i \leq 5$ and $N_U(v_i) = \emptyset$ for $2 \leq i \leq 5$. Thus for any three vertices $u_1, u_2, u_3 \in U$, $\overline{G}[\{v_2, v_3, v_4, v_5, u_1, u_2, u_3\}]$ contains a W_6 with the hub v_2, a contradiction. Hence we have $R(T_{8c}, W_6) \leq 15$.

Since $|U| = 7$ and $\overline{G}[U]$ contains no W_6, By Lemma 3, $G[U]$ contains an S_3. Assume $u_1, u_2, u_3 \in U$ and $u_1u_2, u_2u_3 \in E(G)$. If G contains no T_{8n}, then $N_{V'}(u_i) = \emptyset$ for $1 \leq i \leq 3$. If $\delta(G[V']) = 0$, say $d_{V'}(v_2) = 0$, then $\overline{G}[\{v_2, v_3, v_4, v_5, u_1, u_2, u_3\}]$ contains a W_6

with the hub v_2, a contradiction. Hence we have $\delta(G[V']) \geq 1$. If $\Delta(G[V']) \geq 2$, then G contains a T_{8n}. Thus we may assume $E(G[V']) = \{v_2v_3, v_4v_5\}$. In this case, we have $N_U(v_i) = \emptyset$ for $2 \leq i \leq 5$. By Lemmas 1 and 4, $G[U]$ contains a C_7 and hence $N_V(u) = \emptyset$ for any $u \in U$ since otherwise G contains a T_{8n}. By Lemma 4, we have $\delta(G[V]) \geq 5$ which implies $v_3w_1, v_5w_2 \in E(G)$ and hence G contains a T_{8n}. Thus we have $R(T_{8n}, W_6) \leq 15$. ▮

Theorem 6. $R(T, W_6) = 15$ for $T = T_{8f}, T_{8h}, T_{8j}, T_{8l}$.

Proof. By Lemma 7, G contains an $S_8(1,3)$. Let T be an $S_8(1,3)$ in G with $V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, w_1w_2, w_2w_3\}$. Set $U = V(G) - V$. Obviously, $|U| = 7$.

If G contains no T_{8f}, then $w_3v_i \notin E(G)$ for $2 \leq i \leq 4$, $N_U(w_3) = \emptyset$ and $d_U(v_i) \leq 1$ for $2 \leq i \leq 4$. Thus, since $|U| = 7$, we can choose three vertices $u_1, u_2, u_3 \in U$ such that $v_iu_j \notin E(G)$ for $2 \leq i \leq 4$ and $1 \leq j \leq 3$ and hence $\overline{G}[\{w_3, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a W_6 with the hub w_2, a contradiction. Thus we have $R(T_{8f}, W_6) \leq 15$.

If G contains no T_{8h}, then $w_2v_i \notin E(G)$ for $2 \leq i \leq 4$ and $N_U(w_2) = \emptyset$. If $\Delta(G[U]) \leq 2$, then since $|U| = 7$, $\overline{G}[U]$ contains a C_6 by Lemma 1 and hence $\overline{G}$ contains a W_6 with the hub w_2, a contradiction. Hence we may assume $u \in U$ and $u_1, u_2, u_3 \in N_U(u)$. In this case, we have $v_iu_j \notin E(G)$ for $2 \leq i \leq 4$ and $1 \leq j \leq 3$ for otherwise G contains a T_{8h} and hence $\overline{G}[\{w_2, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a W_6 with the hub w_2, again a contradiction. Thus we have $R(T_{8h}, W_6) \leq 15$.

If G contains no T_{8j}, then $v_1v_i \notin E(G)$ for $2 \leq i \leq 4$, $N_U(v_1) = \emptyset$ and $d_U(v_i) \leq 1$ for $2 \leq i \leq 4$. Thus, since $|U| = 7$, we can choose three vertices $u_1, u_2, u_3 \in U$ such that $v_iu_j \notin E(G)$ for $2 \leq i \leq 4$ and $1 \leq j \leq 3$ and hence $\overline{G}[\{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a W_6 with the hub w_2, a contradiction. Thus we have $R(T_{8j}, W_6) \leq 15$.

If G contains no T_{8l}, then $N_U(v_i) = \emptyset$ for $2 \leq i \leq 4$ and $\{v_2, v_3, v_4\}$ is an independent set. By Lemma 4, we have $\delta(G[U]) \geq 4$

and hence $G[U]$ contains a C_7 by Lemma 1. In this case, we have $d_U(v) = 0$ for each $v \in V(T)$ which implies $\delta(G[V]) \geq 5$ by Lemma 4. Noting that $\{v_2, v_3, v_4\}$ is an independent set, we have $v_2v_1, v_3w_3 \in E(G)$ and hence G contains a T_{8l}. Thus we have $R(T_{8l}, W_6) \leq 15$. ∎

Theorem 7. $R(T_{8g}, W_6) = 15$.

Proof. By Lemma 6, G contains an $S_8[4]$. Let T be an $S_8[4]$ with $V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, w_1w_2, w_1w_3\}$. Set $U = V(G) - V(T)$. If G contains no T_{8g}, then $w_2v_i \notin E(G)$ for $2 \leq i \leq 4$, $N_U(w_2) = \emptyset$ and $d_U(v_i) = 0$ for $2 \leq i \leq 4$. Thus, for any three vertices $u_1, u_2, u_3 \in U$, $\overline{G}[\{w_2, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a W_6 with the hub w_2, a contradiction. Hence we have $R(T_{8g}, W_6) \leq 15$. ∎

Theorem 8. $R(T_{8i}, W_6) = 15$.

Proof. By Lemma 8, G contains an $S_8(3,1)$. Let T be an $S_8(3,1)$ in G with $V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, v_2w_2, v_3w_3\}$. Set $U = V(G) - V(T)$. Obviously, $|U| = 7$. If G contains no T_{8i}, then we have $v_4v_i \notin E(G)$ for $1 \leq i \leq 3$, $N_U(v_i) = \emptyset$ for $1 \leq i \leq 3$ and $d_U(v_4) \leq 1$. Thus, since $|U| = 7$, there is three vertices $u_1, u_2, u_3 \in U$ such that $u_1, u_2, u_3 \notin N_U(v_4)$ and hence $\overline{G}[\{v_4, v_1, v_2, v_3, u_1, u_2, u_3\}]$ contains a W_6 with the hub v_4, a contradiction. Thus we have $R(T_{8i}, W_6) \leq 15$. ∎

Theorem 9. $R(T_{8o}, W_6) = 15$.

Proof. By Theorem 8, G contains a T_{8i}. Let T be a T_{8i} with $V(T) = \{v_0, \ldots, v_3, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 3\} \cup \{v_1w_1, v_2w_2, v_3w_3, v_3w_4\}$. Set $U = V(G) - V(T)$. If G contains no T_{8o}, then we have $w_4w_i \notin E(G)$ for $1 \leq i \leq 3$ and $N_U(w_i) = \emptyset$ for $1 \leq i \leq 4$. Thus, for any three vertices $u_1, u_2, u_3 \in U$, $\overline{G}[\{w_4, w_1, w_2, w_3, u_1, u_2, u_3\}]$ contains a W_6 with the hub w_4, a contradiction. Hence we have $R(T_{8o}, W_6) \leq 15$. ∎

Theorem 10. $R(T_{8m}, W_6) = 15$.

Proof. By Theorem 4, G contains a T_{8a}. Let T be a T_{8a} with $V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, v_1w_2, w_2w_3\}$. Set $U = V(G) - V(T)$. Obviously, $|U| = 7$. If G contains no T_{8m}, then $\{v_2, v_3, v_4\}$ is an independent set and $N_U(v_i) = \emptyset$ for $2 \leq i \leq 4$. By Lemmas 1 and 4, $G[U]$ contains a C_7. This implies $N_U(w_2) = \emptyset$ for otherwise G contains a T_{8m}. If $\{v_2, v_3, v_4\} \subseteq N(w_2)$, then G contains a T_{8m}. Hence we may assume $v_2w_2 \notin E(G)$. Thus, for any three vertices $u_1, u_2, u_3 \in U$, $\overline{G}[\{v_2, v_3, v_4, w_2, u_1, u_2, u_3\}]$ contains a W_6 with the hub v_2, a contradiction. Hence we have $R(T_{8o}, W_6) \leq 15$. ∎

The proof of Theorem 3 is completed. ∎

5. Proof of Theorem 1

Proof of Theorem 1. If $\Delta(T_n) = 2$, then Theorem 1 holds by Lemma 2. Hence we may assume $\Delta(T_n) \geq 3$. If $n = 5$, then $T_5 = S_5(1, 1)$ and hence Theorem 1 holds. If $n \geq 6$ and $\Delta(T_n) \geq n - 3$, then Theorem 1 holds by Theorems D and E. Thus we may assume $3 \leq \Delta(T_n) \leq n - 4$. In this case, we have $n \geq 7$. If $n = 7$, then Theorem 1 holds by Theorem 2. If $n = 8$, then Theorem 1 holds by Theorem 3. ∎

References

[1] J.A. Bondy, Pancyclic graphs, Journal of Combinatorial Theory, Series B, 11(1971), 80-84.

[2] E.T. Baskoro, Surahmat, S.M. Nababan and M. Miller, On Ramsey numbers for trees versus wheels of five or six vertices, Graphs and Combinatorics, 18(2002), 717-721.

[3] G.T. Chen, A result on C_4-star Ramsey number, Discrete Mathematics, 163(1997), 243-246.

[4] Y.J. Chen, Y.Q. Zhang and K.M. Zhang, The Ramsey numbers of paths versus wheels, Discrete Mathematics, 290(2005), 85-87.

[5] Y.J. Chen, Y.Q. Zhang and K.M. Zhang, The Ramsey numbers of stars versus wheels, European Journal of Combinatorics, 25(2004), 1067-1075.

[6] Y.J. Chen, Y.Q. Zhang and K.M. Zhang, The Ramsey numbers $R(T_n, W_6)$ for $\Delta(T_n) \geq n-3$, Applied Mathematics Letters, 17(2004), 281-285.

[7] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs, III. Small off diagonal numbers, Pacific Journal of Mathematics, 41(1972), 335-345.

[8] G.R.T. Henry, The Ramsey numbers $R(K_2 + \overline{K_3}, K_4)$ and $R(K_1 + C_4, K_4)$, Utilitas Mathematica, 41(1992), 181-203.

[9] S.P. Radziszowski and J. Xia, Paths, cycles and wheels without antitriangles, Australasian Journal of Combinatorics, 9(1994), 221-232.

[10] Surahmat and E.T. Baskoro, On the Ramsey number of path or star versus W_4 or W_5, Proceedings of the 12th Australasian Workshop on Combinatorial Algorithms, Bandung, Indonesia, July 14-17(2001), 174-179.