The Ramsey Numbers $R(T_n, W_6)$ for Small $n$*

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Abstract: Let $T_n$ denote a tree of order $n$ and $W_m$ a wheel of order $m + 1$. Baskoro et al. conjectured in [2] that if $T_n$ is not a star, then $R(T_n, W_m) = 2n - 1$ for $m \geq 6$ even and $n \geq m - 1$. We disprove the Conjecture in [6]. In this paper, we determine $R(T_n, W_6)$ for $n \leq 8$ which is the first step for us to determine $R(T_n, W_6)$ for any tree $T_n$.

Key words: Ramsey number, Tree, Wheel

1. Introduction

All graphs considered in this paper are finite simple graph without loops. For two given graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the smallest positive integer $n$ such that for any graph $G$ of order $n$, either $G$ contains $G_1$ or $\overline{G}$ contains $G_2$, where $\overline{G}$ is the complement of $G$. Let $G$ be a graph. The neighborhood $N(v)$ of a

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vertex \( v \) is the set of vertices adjacent to \( v \) in \( G \). The minimum and maximum degree of \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \), respectively.

For a vertex \( v \in V(G) \) and a subgraph \( H \) of \( G \), \( N_H(v) \) is the set of neighbors of \( v \) contained in \( H \), i.e., \( N_H(v) = N(v) \cap V(H) \). We let \( d_H(v) = |N_H(v)| \). For \( S \subseteq V(G) \), \( G[S] \) denotes the subgraph induced by \( S \) in \( G \). Let \( U, V \) be two disjoint vertex set. We use \( E(U, V) \) to denote the set of edges between \( U \) and \( V \). Let \( m \) be a positive integer, we use \( mG \) to denote \( m \) vertex disjoint copies of \( G \). A path and a cycle of order \( n \) are denoted by \( P_n \) and \( C_n \) respectively. A Star \( S_n \) \((n \geq 3)\) is a bipartite graph \( K_{1,n-1} \). A Wheel \( W_n = \{x\} + C_n \) is a graph of \( n + 1 \) vertices, that is, a vertex \( x \), called the hub of the wheel, adjacent to all vertices of \( C_n \). \( S_n(l, m) \) is a tree of order \( n \) obtained from \( S_{n-l} \times m \) by subdividing each of its \( l \) edges \( m \) times. \( S_n(l) \) is a tree of order \( n \) obtained from an \( S_l \) and an \( S_{n-l} \) by adding an edge joining the centers of them. \( S_n[l] \) is a tree of order \( n \) obtained from an \( S_l \) and an \( S_{n-l} \) by adding an edge joining a vertex of degree one of \( S_l \) to the center of \( S_{n-l} \). A graph on \( n \) vertices is pancyclic if it contains cycles of every length \( l \), \( 3 \leq l \leq n \).

Many Ramsey numbers concerning wheel or star have been established, see for instance [3, 7, 8, 9]. Recently, the following Ramsey numbers were obtained.

**Theorem A** (Surahmat et al. [10]). \( R(S_n, W_4) = 2n - 1 \) for \( n \geq 3 \) and odd; \( R(S_n, W_4) = 2n + 1 \) for \( n \geq 4 \) and even; \( R(S_n, W_5) = 3n - 2 \) for \( n \geq 4 \).

**Theorem B** (Baskoro et al. [2]). Let \( T_n \) be a tree of order \( n \) other than \( S_n \). Then \( R(T_n, W_4) = 2n - 1 \) for \( n \geq 3 \); \( R(T_n, W_5) = 3n - 2 \) for \( n \geq 4 \).

Motivated by Theorem B, Baskoro et al. [2] posed the following conjecture.

**Conjecture 1.** Let \( T_n \) be a tree other than \( S_n \) and \( n \geq m - 1 \). Then
\[ R(T_n, W_m) = 2n - 1 \text{ for } m \geq 6 \text{ even}; \ R(T_n, W_m) = 3n - 2 \text{ for } m \geq 7 \text{ and odd}. \]

In [6], we consider \( R(T_n, W_6) \) for \( T_n \neq S_n \) and \( \Delta(T_n) \geq n - 3 \) and obtain the following.

**Theorem C** (Chen et al. [6]). \( R(S_n(1, 1), W_6) = 2n \) for \( n \geq 4 \).

**Theorem D** (Chen et al. [6]). \( R(S_n(1, 2), W_6) = 2n \) for \( n \geq 6 \) and \( n \equiv 0 \pmod{3} \).

**Theorem E** (Chen et al. [6]). \( R(S_n(3), W_6) = R(S_n(2, 1), W_6) = 2n - 1 \text{ for } n \geq 6; \ R(S_n(1, 2), W_6) = 2n - 1 \text{ for } n \geq 6 \text{ and } n \not\equiv 0 \pmod{3}. \)

By Theorems C and D, we can see that Conjecture 1 is not true when \( m \) is even. However, we believe that for \( n \geq 5 \), if \( T_n \neq S_n(1, 1), S_n(1, 2), \) then \( R(T_n, W_6) = 2n - 1. \) In order to determine \( R(T_n, W_6) \) for general tree \( T_n \), this paper consider \( R(T_n, W_6) \) for \( n \leq 8 \) as the first step. The main result is the following.

**Theorem 1.** Let \( T_n \) be a tree of order \( n \) other than \( S_n \) and \( 5 \leq n \leq 8. \) If \( T_n \neq S_n(1, 1) \) and \( T_n \neq S_n(1, 2) \) for \( n \equiv 0 \pmod{3}, \) then \( R(T_n, W_6) = 2n - 1. \)

2. Some Lemmas

In order to prove Theorem 1, we need the following lemmas.

**Lemma 1** (Bondy [1]). Let \( G \) be a graph of order \( n. \) If \( \delta(G) \geq n/2, \) then either \( G \) is pancyclic or \( n \) is even and \( G = K_{n/2,n/2}. \)

**Lemma 2** (Chen et al. [4]). \( R(P_n, W_m) = 2n - 1 \) for \( m \) even and \( n \geq m - 1 \geq 3. \)
Lemma 3 (Chen et al. [5]). \( R(S_n, W_6) = 2n + 1 \) for \( n \geq 3 \).

Lemma 4 (Chen et al. [6]). Let \( G \) be a graph of order \( 2n - 1 \geq 7 \) and \( (U, W) \) a partition of \( V(G) \) with \( |U| \geq 3 \) and \( |W| \geq 4 \). Suppose \( u_i \in U \) and \( N_W(u_i) = \emptyset \), where \( 1 \leq i \leq 3 \). If \( G \) contains no \( W_6 \), then \( \delta(G[W]) \geq |W| - 3 \).

Lemma 5. Let \( G \) be a graph of order 7 and \( \delta(G) \geq 4 \). Then for any \( v \in V(G) \), \( G \) contains a tree \( T = S_7(3, 1) \) such that \( d_T(v) = 3 \).

Proof. Let \( G' = G - v \). Then \( \delta(G') \geq 3 \) and hence \( G' \) contains a \( C_6 \) by Lemma 1. Since \( d(v) \geq 4 \), after an easy check, we can see \( G \) contains a tree \( T = S_7(3, 1) \) such that \( d_T(v) = 3 \).

Lemma 6. \( R(S_n[4], W_6) = 2n - 1 \) for \( n \geq 8 \).

Proof. Let \( G \) be a graph of order \( 2n - 1 \). If \( \overline{G} \) contains no \( W_6 \), then \( G \) contains an \( S_n(3) \) by Theorem E. Let \( T \) be an \( S_n(3) \) with \( V(T) = V_0 \cup W \), where \( V_0 = \{v_0, v_1, \ldots, v_{n-3}\} \), \( W = \{w_1, w_2\} \) and \( E(T) = \{v_0v_i \mid 1 \leq i \leq n-3\} \cup \{v_1w_1, v_1w_2\} \). Set \( U = V(G) - V(T) \). If \( G \) contains no \( S_n[4] \), then we have

\[
v_1v_i \notin E(G) \text{ for } 2 \leq i \leq n-3,
\]

and

\[
\text{for } u \in U \text{ with } d_U(u) \geq 2, N(u) \cap (V_0 - \{v_0\}) = \emptyset.
\]

Since \( |U| = n - 1 \geq 7 \) and \( \overline{G[U]} \) contains no \( W_6 \), by Lemma 3, \( G[U] \) contains an \( S_3 \) which implies

\[
\Delta(G[U]) \geq 2.
\]

Claim 1. \( d_U(v_1) \leq n - 8 \).

Proof. If \( d_U(v_1) \geq n - 7 \), then since \( G \) contains no \( S_n[4] \), we have \( \Delta(G[V_0 - \{v_0, v_1\}]) \leq 1 \). If there are two vertices \( u_1, u_2 \in U \) such that \( d_U(u_i) \geq 2 \) for \( i = 1, 2 \), then since \( n \geq 8 \) we can see \( \overline{G}[\{v_1, v_2, v_3, v_4, v_5, u_1, u_2\}] \) contains a \( W_6 \) with the hub \( v_1 \) by (1).
and (2), a contradiction. Thus, by (3), we conclude that there is only one vertex \( u \in U \) such that \( d_{U'}(u) \geq 2 \) which implies \( N_U(u) \) is an independent set. If \( d_{U'}(u) \geq n - 4 \), then \( N(v_i) \cap N_U(u) = \emptyset \) for \( i = 1, 2, 3 \) since otherwise \( G \) contains an \( S_n[4] \). Thus, since \( n \geq 8 \),
\[
\overline{G} \left[ \{v_1, v_2, v_3, u_1, u_2, u_3, u_4\} \right]
\]contains a \( W_6 \) with the hub \( u_1 \) for any four vertices \( u_1, u_2, u_3, u_4 \in N_U(u) \), a contradiction. Hence we have \( d_{U'}(u) \leq n - 5 \).

Let \( U' = U - N(u) \), then \( |U'| \geq 3 \). If \( G[U'] \) contains an edge, say \( u_1u_2 \in E(G[U']) \), then \( |N(v_i) \cap (V_0 \setminus \{v_0\})| \leq 1 \) for \( i = 1, 2 \) since otherwise \( G[V_0 \cup \{u_1, u_2\}] \) contains an \( S_n[4] \). Thus, noting that \( n \geq 8 \) and \( \Delta(G[V_0 \setminus \{v_0, v_1\}]) \leq 1 \),
\[
\overline{G} \left[ \{v_1, v_2, v_3, v_4, v_5, u_1, u_2\} \right]
\]contains a \( W_6 \) with the hub \( u \) by (1) and (2). Hence we may assume \( U' \) is an independent set which implies \( U - \{u\} \) is an independent set. If \( n \geq 9 \), then \( \overline{G}[U - \{u\}] = K_{n-2} \) and hence \( \overline{G}[U] \) contains a \( W_6 \), a contradiction. If \( n = 8 \), then \( |U| = 7 \) and \( d_{U'}(u) \leq 3 \). It is easy to see \( \overline{G}[U] \) contains a \( W_6 \) in this case, again a contradiction.

Let \( U' = U - N(v_1) \). By Claim 1, we have \( |U'| \geq 7 \). If \( \Delta(G[U']) \leq 2 \), then by Lemma 1, \( \overline{G}[U'] \) contains a \( C_6 \) and hence \( \overline{G} \) contains a \( W_6 \) with the hub \( v_1 \), a contradiction. Thus there is some vertex \( u \in U' \) such that \( d_{U'}(u) \geq 3 \). Assume \( \{u_1, u_2, u_3\} \subseteq N_U(u) \). If there is some \( u_i \) with \( 1 \leq i \leq 3 \) such that \( d_{V_0}(u_i) \geq 2 \), then \( G[V_0 \cup \{u, u_i\}] \) contains an \( S_n[4] \). Hence we have \( d_{V_0}(u_i) \leq 1 \). If there is some vertex \( v_i \) with \( 2 \leq i \leq n - 3 \) such that \( |N(v_i) \cap \{u_1, u_2, u_3\}| \geq 2 \), then we have \( |N(v_j) \cap \{u_1, u_2, u_3\}| \leq 1 \) for any \( j \) with \( 2 \leq j \leq n - 3 \) and \( j \neq i \). Thus, since \( n - 3 \geq 6 \) we can always choose three vertices, say \( v_2, v_3, v_4 \) such that \( |N(v_i) \cap \{u_1, u_2, u_3\}| \leq 1 \) for \( 2 \leq i \leq 4 \). Noting that \( d_{V_0}(u_i) \leq 1 \) for \( 1 \leq i \leq 3 \), we can see that \( \overline{G}[\{u_1, u_2, u_3, v_2, v_3, v_4\}] \) contains a \( C_6 \) and hence \( \overline{G} \) contains a \( W_6 \) with the hub \( v_1 \) by (1), a contradiction. Thus we have \( R(S_n[4], W_6) \leq 2n - 1 \). On the other hand, the graph \( G = 2K_{n-1} \) shows \( R(S_n[4], W_6) \geq 2n - 1 \) and hence we have \( R(S_n[4], W_6) = 2n - 1 \).

**Lemma 7.** \( R(S_n(1, 3), W_6) = 2n - 1 \) for \( n \geq 8 \).
**Proof.** Let \( G \) be a graph of order \( 2n - 1 \). If \( \overline{G} \) contains no \( W_6 \), then \( G \) contains an \( S_n[4] \) by Lemma 6. Let \( T \) be an \( S_n[4] \) with \( V(T) = V_0 \cup W \), where \( V_0 = \{v_0, v_1, \ldots, v_{n-4}\} \), \( W = \{w_1, w_2, w_3\} \) and \( E(T) = \{v_0v_i \mid 1 \leq i \leq n-4\} \cup \{v_1w_1, w_1w_2, w_1w_3\} \). Let \( U = V(G) - V(T) \). If \( G \) contains no \( S_n(1, 3) \), then we have

\[
N(w_i) \cap (V_0 \cup U - \{v_0\}) = \emptyset \quad \text{for} \quad i = 2, 3,
\]

and if \( u \in U \) and \( d_U(u) \geq 2 \), then

\[
N(u_i) \cap (V_0 - \{v_0\}) = \emptyset \quad \text{for any} \quad u_i \in N_U(u).
\]

If \( \Delta(G[U]) \leq 2 \), then since \( |U| = n - 1 \geq 7 \), \( \overline{G}[U] \) contains a \( C_6 \) by Lemma 1 and hence \( G \) contains a \( W_6 \) with the hub \( u_2 \) by (4), a contradiction. Thus there is some vertex \( u \in U \) such that \( d_U(u) \geq 3 \). Assume \( \{u_1, u_2, u_3\} \subseteq N_U(u) \). By (4) and (5), we can see \( \overline{G}[[w_2, v_1, v_2, v_3, u_1, u_2, u_3]] \) contains a \( W_6 \) with the hub \( u_2 \), again a contradiction. Thus we have \( R(S_n(1, 3), W_6) \leq 2n - 1 \). On the other hand, the graph \( G = 2K_{n-1} \) shows \( R(S_n(1, 3), W_6) \geq 2n - 1 \) and hence we have \( R(S_n(1, 3), W_6) = 2n - 1 \).

**Lemma 8.** \( R(S_n(3, 1), W_6) = 2n - 1 \) for \( n \geq 8 \).

**Proof.** Let \( G \) be a graph of order \( 2n - 1 \). If \( \overline{G} \) contains no \( W_6 \), then by Theorem E, \( G \) contains an \( S_n(2, 1) \). Let \( T = S_n(2, 1) \) with \( V(T) = V_0 = \{v_0, \ldots, v_{n-3}, w_1, w_2\} \) and \( E(T) = \{v_0v_i \mid 1 \leq i \leq n-3\} \cup \{v_1w_1, v_2w_2\} \). Set \( U = V(G) - V_0 \). Obviously, \( |U| = n - 1 \geq 7 \).

If \( G \) contains no \( S_n(3, 1) \), then \( N_U(v_i) = \emptyset \) for \( 3 \leq i \leq n - 3 \) and \( \{v_3, \ldots, v_{n-3}\} \) is an independent set. If \( n \geq 9 \), then \( \overline{G}[[v_3, v_4, v_5, v_6, u_1, u_2, u_3]] \) contains a \( W_6 \) with the hub \( v_3 \) for any three vertices \( u_1, u_2, u_3 \in U \), a contradiction. Hence we have \( n = 8 \). By Lemma 4, we have \( \delta(G[U]) \geq 4 \). By Lemma 5, we have \( N_U(u) = \emptyset \) for any \( u \in U \) which implies \( \delta(G[V_0]) \geq 5 \) by Lemma 4. Noting that \( \{v_3, v_4, v_5\} \) is an independent set, we have \( \{v_1, v_2\} \subseteq N(v_3) \cap N(v_4) \) which implies \( G \) contains an \( S_n(3, 1) \), a contradiction. Thus we have \( R(S_n(3, 1), W_6) \leq 2n - 1 \). On the other hand, the graph \( G = 2K_{n-1} \)
shows $R(S_n(3, 1), W_6) \geq 2n - 1$ and hence we have $R(S_n(3, 1), W_6) = 2n - 1$.

3. The Ramsey Numbers $R(T_n, W_6)$ for $\Delta(T_n) = 3$ and $n = 7$

In this section, we determine $R(T_n, W_6)$ for $n = 7$ and $\Delta(T_n) = 3$.

**Theorem 2.** $R(T_n, W_6) = 13$ for $n = 7$ and $\Delta(T_n) = 3$.

**Proof.** Let $T$ be a tree with order 7 and $\Delta(T) = 3$, then it is not difficult to see $T$ must be isomorphic to one of the five trees of order 7 below.

Thus we need only to show $R(T, W_6) = 13$ for $T = T_{7a}, T_{7b}, T_{7c}, T_{7d}$ and $T_{7e}$.

Let $G$ be a graph of order 13. Suppose $\overline{G}$ contains no $W_6$. Since $2K_6$ contains no trees of order 7 and its complement contains no $W_6$, we have $R(T, W_6) \geq 13$ for each tree $T$ with $|T| = 7$. In the following proof, we need only to prove $R(T, W_6) \leq 13$ for each $T \in \{T_{7a}, T_{7b}, T_{7c}, T_{7d}, T_{7e}\}$.

We first show $R(T_{7a}, W_6) = R(T_{7c}, W_6) = 13$. By Theorem E, $G$ contains an $S_7(1, 2)$. Let $T$ be an $S_7(1, 2)$ in $G$ with $V(T) = V = \{v_0, \ldots, v_4, w_1, w_2\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, w_1w_2\}$. Set $U = V(G) - V$. Obviously, $|U| = 6$.

If $G$ contains no $T_{7a}$, then $N(w_2) \cap (U \cup \{v_2, v_3, v_4\}) = \emptyset$. If $d_{T_U}(u) \leq 2$ for each $u \in U$, then $\overline{G}[U]$ contains a $C_6$ by Lemma 1 and hence $\overline{G}$ contains a $W_6$ with the hub $w_2$, a contradiction.
Thus there is some vertex \( u \in U \) such that \( d_U(u) \geq 3 \). Assume \( u_1, u_2, u_3 \in N_U(u) \). Since \( G \) contains no \( T_{7a} \), we have \( v_i u_j \notin E(G) \) for \( 2 \leq i \leq 4 \) and \( 1 \leq j \leq 3 \). Thus \( G[\{w_2, u_1, u_2, u_3, v_2, v_3, v_4\}] \) contains a \( W_6 \) with the hub \( w_2 \), a contradiction. Thus we have \( R(T_{7a}, W_6) \leq 13 \).

If \( G \) contains no \( T_{7c} \), then \( N_U(v_i) = \emptyset \) for \( 2 \leq i \leq 4 \) and \( \{v_2, v_3, v_4\} \) is an independent set. Thus by Lemma 4, we have \( \delta(G[U]) \geq 3 \) which implies \( G[U] \) contains a \( C_6 \). In this case, we have \( N_V(u) = \emptyset \) for any \( u \in U \) since otherwise \( G \) contains a \( T_{7c} \). By Lemma 4, we have \( \delta(G[V]) \geq 4 \) which implies \( N_V(v_i) = \{v_0, v_1, w_1, w_2\} \) for \( i = 2, 3, 4 \) and hence \( G[V] \) contains a \( T_{7c} \), a contradiction. Thus we have \( R(T_{7c}, W_6) \leq 13 \).

Next, we show \( R(T_{7b}, W_6) = R(T_{7d}, W_6) = 13 \). By Theorem E, \( G \) contains an \( S_7(3) \). Let \( T = S_7(3) \), \( V(T) = \{v_0, \ldots, v_4, w_1, w_2\} \), \( E(T) = \{v_0 v_i \mid 1 \leq i \leq 4\} \cup \{v_1 w_1, v_1 w_2\} \). Set \( U = V(G) - V(T) \). Obviously, \( |U| = 6 \).

If \( G \) contains no \( T_{7b} \), we have \( v_1 v_i \notin E(G) \) for \( 2 \leq i \leq 4 \). For any \( u \in U \), if \( u \in N(v_i) \) for some \( i \) with \( 1 \leq i \leq 4 \), then \( d_U(u) \leq 1 \). Thus, if there are three vertices \( u_1, u_2, u_3 \in U \) such that \( d_U(u_i) \geq 2 \) for \( 1 \leq i \leq 3 \), then \( \overline{G}[\{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}] \) contains a \( W_6 \) with the hub \( v_1 \), a contradiction. If there is some \( v_i \) with \( 2 \leq i \leq 4 \) such that \( d_U(v_i) \geq 2 \), then \( G \) contains a \( T_{7b} \) and hence we have \( d_U(v_i) \leq 1 \) for \( 2 \leq i \leq 4 \). If \( G[\{v_2, v_3, v_4\}] \) contains two edges, then \( G \) contains a \( T_{7b} \) and hence we may assume \( v_2 v_3, v_2 v_4 \notin E(G) \). Let \( U = \{u_i \mid 1 \leq i \leq 6\} \). Assume \( N_U(v_2) \subseteq \{u_1\} \) and \( N_U(v_2) \cup N_U(v_4) \subseteq \{u_1, u_2, u_3\} \). If \( u_2 \notin N_U(v_3) \cup N_U(v_4) \), then since \( U \) contains at most two vertices with degree not less than 2, we can see \( \overline{G}[\{u_2, u_4, u_5, u_6\}] \) contains a \( P_3 \), say \( u_2 u_4 u_5 \) is a \( P_3 \). Then \( v_3 u_2 u_4 u_5 v_4 u_6 v_3 \) is a \( C_6 \) in \( \overline{G} \) and hence \( \overline{G} \) contains a \( W_6 \) with the hub \( v_2 \). If \( u_2 \in N_U(v_3) \cup N_U(v_4) \), then since \( d_U(u_2) \leq 1 \), we may assume \( u_2 u_4, u_2 u_5 \notin E(G) \) which implies \( v_3 u_4 u_2 u_5 v_4 u_6 v_3 \) is a \( C_6 \) in \( \overline{G} \) and hence \( \overline{G} \) contains a \( W_6 \) with the hub \( v_2 \), again a contradiction. Thus we have \( R(T_{7b}, W_6) \leq 13 \).
If $G$ contains no $T_{7d}$, then $N_U(w_1) = N_U(v_i) = \emptyset$ for $2 \leq i \leq 4$ and $N(w_1) \cap \{v_2, v_3, v_4\} = \emptyset$. Thus $\overline{G}[\{w_1, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a $W_6$ with the hub $w_1$ for any three vertices $u_1, u_2, u_3 \in U$, a contradiction. Thus we have $R(T_{7d}, W_6) \leq 13$.

Finally, we show $R(T_{7e}, W_6) = 13$. By Theorem E, $G$ contains an $S_7(2,1)$. Let $T = S_7(2,1)$, $V(T) = V = \{v_0, \ldots, v_4, w_1, w_2\}$ and $E(T) = \{v_iv_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, v_2w_2\}$. Set $U = V(G) - V$. Obviously, $|U| = 6$.

If $G$ contains no $T_{7e}$, then $N_U(v_i) = \emptyset$ for $i = 3, 4$ and $v_3v_4 \notin E(G)$. If $d_U(v_0) = 0$, then by Lemma 4, we have $\delta(G[U]) \geq 3$ which implies $G[U]$ contains a $C_6$ by Lemma 1 and hence $N_U(v) = \emptyset$ for any $u \in U$ since $G$ contains no $T_{7e}$. Thus by Lemma 4, we have $\delta(G[V]) \geq 4$. Noting that $v_3v_4 \notin E(G)$, after an easy check, we can see $G[V]$ contains a $T_{7e}$ and hence we may assume $d_U(v_0) \geq 1$.

For any $u \in U$, if $uv_0 \in E(G)$, then $d_U(u) = 0$ for otherwise $G$ contains a $T_{7e}$. If $d_U(v_0) \geq 2$, say $u_1, u_2 \in N_U(v_0)$, then $\{v_3, v_4, u_1, u_2\}$ is an independent set and for any $u \in U - \{u_1, u_2\}$, $N(u) \cap \{v_3, v_4, u_1, u_2\} = \emptyset$. In this case, we can see $\overline{G}[\{v_3, v_4, u_1, u_2, u_3, u_4, u_5\}]$ contains a $W_6$ with the hub $v_3$ for any three vertices $u_3, u_4, u_5 \in U - \{u_1, u_2\}$, a contradiction. Hence we may assume $d_U(v_0) = 1$.

Let $N_U(v_0) = \{u_1\}$ and $U' = U - \{u_1\} = \{u_2, u_3, u_4, u_5, u_6\}$. If $G[U']$ contains a $C_5$, then for any $u \in U'$, $N_U(u) = \emptyset$ for otherwise $G$ contains a $T_{7e}$. By Lemma 4, we have $\delta(G[V]) \geq 4$. Noting that $v_3v_4 \notin E(G)$, after an easy check, we can see $G[V]$ contains a $T_{7e}$. Hence we may assume $G[U']$ contains no $C_5$. By Lemma 1, there is some vertex $u \in U'$ such that $d_U(u) \leq 2$ which implies $\overline{G}[U']$ contains a $P_3$, say $u_2u_3u_4$ is a $P_3$ in $\overline{G}$. Then $v_4u_2u_3u_4u_1u_3v_3$ is a $C_6$ in $\overline{G}$ and hence $\overline{G}$ contains a $W_6$ with the hub $v_3$, also a contradiction. Thus we have $R(T_{7e}, W_6) \leq 13$.

The proof of Theorem 2 is completed. □
4. The Ramsey Numbers \( R(T_n, W_6) \) for \( 3 \leq \Delta(T_n) \leq 4 \) and \( n = 8 \)

In this section, we determine \( R(T_n, W_6) \) for \( n = 8 \) and \( 3 \leq \Delta(T_n) \leq 4 \).

**Theorem 3.** \( R(T_n, W_6) = 15 \) for \( n = 8 \) and \( 3 \leq \Delta(T_n) \leq 4 \).

**Proof.** Let \( T \) be a tree with order 8 and \( 3 \leq \Delta(T) \leq 4 \). Since \( 2K_7 \) contains no trees of order 8 and its complement contains no \( W_6 \), we have \( R(T, W_6) \geq 15 \). Thus, in order to prove \( R(T, W_6) = 15 \), we need only to show \( R(T, W_6) \leq 15 \).

If \( T = S_8[4] \), then Theorem 3 holds by Lemma 6. If \( T = S_8(1, 3) \), then Theorem 3 holds by Lemma 7. If \( T \neq S_8[4], S_8(1, 3) \), then we have the following.

**Proposition 1.** Let \( T \) be a tree of order 8 with \( 3 \leq \Delta(T) \leq 4 \). If \( T \neq S_8[4], S_8(1, 3) \), then \( T \) must be isomorphic to one of the fifteen trees of order 8 below.
Let $G$ be a graph of order 15. Suppose $\overline{G}$ contains no $W_6$. By Proposition 1, we will complete the proof by showing the following theorems.

**Theorem 4.** $R(T, W_6) = 15$ for $T = T_{8a}, T_{8b}, T_{8d}$ or $T_{8k}$.

**Proof.** By Theorem E, $G$ contains an $S_8(3)$. Let $T$ be an $S_8(3)$ with $V(T) = V = \{v_0, \ldots, v_5, w_1, w_2\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 5\} \cup \{v_1w_1, v_1w_2\}$. Set $U = V(G) - V$. Obviously, $|U| = 7$.

If $G$ contains no $T_{8a}$, then $w_1v_i \notin E(G)$ for $2 \leq i \leq 5$ and $N_U(v_1) = \emptyset$. If there is some vertex $v_i$ with $2 \leq i \leq 5$ such that $d_U(v_i) \geq 2$, say $u_1, u_2 \in N_U(v_2)$, then $v_1v_i \notin E(G)$ for $3 \leq i \leq 5$ and $v_1u_j \notin E(G)$ for $3 \leq i \leq 5$ and $j = 1, 2$ since otherwise $G$ contains a $T_{8a}$. Thus, $\overline{G}[\{w_1, v_2, u_1, u_2, v_3, v_4, v_5\}]$ contains a $W_6$ with the hub $w_1$, a contradiction. Hence we may assume $d_U(v_i) \leq 1$ for $2 \leq i \leq 5$ which implies there are three vertices $u_1, u_2, u_3 \in U$ such that $v_1u_j \notin E(G)$ for $2 \leq i \leq 5$ and $1 \leq j \leq 3$. Thus $\overline{G}[\{w_1, v_2, v_3, v_4, u_1, u_2, u_3\}]$ contains a $W_6$ with the hub $w_1$, again a contradiction. Hence we have $R(T_{8a}, W_6) \leq 15$.

If $G$ contains no $T_{8b}$, then we have $N_U(v_1) = \emptyset$, $v_1v_i \notin E(G)$ for $2 \leq i \leq 5$ and $d_U(v_i) \leq 2$ for $2 \leq i \leq 5$. Thus, since $|U| = 7$, we can choose three vertices $u_1, u_2, u_3 \in U$ such that $d_U(u_i) \leq 1$ for $1 \leq i \leq 3$ and $N_U(u_i) \cap N_U(u_j) = \emptyset$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$, and
By Theorem E, \(d_5\) contains no \(G\) with the hub \(v_1\), a contradiction. Thus we have \(R(T_{8d}, W_6) \leq 15\).

If \(G\) contains no \(T_{8d}\), then \(v_2v_i \notin E(G)\) for \(3 \leq i \leq 5\) and \(N_U(v_i) = \emptyset\) for \(2 \leq i \leq 5\). Thus for any three vertices \(u_1, u_2, u_3 \in U\), \(\overline{G}[[v_2, v_3, v_4, v_5, u_1, u_2, u_3]]\) contains a \(W_6\) with the hub \(v_2\), a contradiction. Hence we have \(R(T_{8d}, W_6) \leq 15\).

If \(G\) contains no \(T_{8k}\), then \(d_U(v_i) \leq 1\) for \(2 \leq i \leq 5\). Let \(V' = \{v_2, v_3, v_4, v_5\}\). Since \(|U| = 7\), there are three vertices \(u_1, u_2, u_3 \in U\) such that \(N_U(v_i) = \emptyset\) for \(1 \leq i \leq 3\). If \(\delta(G[V']) = 0\), say \(d_U(v_2) = 0\), then \(\overline{G}[[v_2, v_3, v_4, v_5, u_1, u_2, u_3]]\) contains a \(W_6\) with the hub \(v_2\), a contradiction. Hence we have \(\delta(G[V']) \geq 1\). If \(\Delta(G[V']) \geq 2\), \(G\) contains a \(T_{8k}\). Thus we may assume \(E(G[V']) = \{v_2v_3, v_4v_5\}\). In this case, we have \(N_U(v_i) = \emptyset\) for \(2 \leq i \leq 5\). By Lemmas 1 and 4, \(G[U]\) contains a \(C_7\). If \(N_U(w_1) \neq \emptyset\) or \(\{v_2, v_3\} \subseteq N(w_1)\), then \(G\) contains a \(T_{8k}\). Hence we may assume \(N_U(v_1) = \emptyset\) and \(v_2 \notin N(w_1)\).

Thus, \(\overline{G}[[v_2, w_1, v_4, v_5, u_1, u_2, u_3]]\) contains a \(W_6\) with the hub \(v_2\) for any three vertices \(u_1, u_2, u_3 \in U\), a contradiction. Hence we have \(R(T_{8k}, W_6) \leq 15\).

**Theorem 5.** \(R(T, W_6) = 15\) for \(T = T_{8c}, T_{8n}\).

**Proof.** By Theorem E, \(G\) contains an \(S_8(1, 2)\). Let \(T = S_8(1, 2)\) with \(V(T) = V = \{v_0, \ldots, v_5, w_1, w_2\}\) and \(E(T) = \{v_0v_i \mid 1 \leq i \leq 5\} \cup \{v_1w_1, w_1w_2\}\). Set \(V' = \{v_2, v_3, v_4, v_5\}\) and \(U = V(G) - V\). Obviously, \(|U| = 7\).

If \(G\) contains no \(T_{8c}\), then \(v_2v_i \notin E(G)\) for \(3 \leq i \leq 5\) and \(N_U(v_i) = \emptyset\) for \(2 \leq i \leq 5\). Thus for any three vertices \(u_1, u_2, u_3 \in U\), \(\overline{G}[[v_2, v_3, v_4, v_5, u_1, u_2, u_3]]\) contains a \(W_6\) with the hub \(v_2\), a contradiction. Hence we have \(R(T_{8c}, W_6) \leq 15\).

Since \(|U| = 7\) and \(\overline{G[U]}\) contains no \(W_6\), By Lemma 3, \(G[U]\) contains an \(S_3\). Assume \(u_1, u_2, u_3 \in U\) and \(u_1u_2, u_2u_3 \in E(G)\). If \(G\) contains no \(T_{8n}\), then \(N_U(v_i) = \emptyset\) for \(1 \leq i \leq 3\). If \(\delta(G[V']) = 0\), say \(d_U(v_2) = 0\), then \(\overline{G}[[v_2, v_3, v_4, v_5, u_1, u_2, u_3]]\) contains a \(W_6\).
with the hub \( v_2 \), a contradiction. Hence we have \( \delta(G[V']) \geq 1 \).

If \( \Delta(G[V']) \geq 2 \), then \( G \) contains a \( T_{8n} \). Thus we may assume \( E(G[V']) = \{v_2v_3, v_4v_5\} \). In this case, we have \( N_U(v_i) = \emptyset \) for \( 2 \leq i \leq 5 \). By Lemmas 1 and 4, \( G[U] \) contains a \( C_7 \) and hence \( N_V(u) = \emptyset \) for any \( u \in U \) since otherwise \( G \) contains a \( T_{8n} \). By Lemma 4, we have \( \delta(G[V]) \geq 5 \) which implies \( v_3w_1, v_5w_2 \in E(G) \) and hence \( G \) contains a \( T_{8n} \). Thus we have \( R(T_{8n}, W_6) \leq 15 \).

**Theorem 6.** \( R(T, W_6) = 15 \) for \( T = T_{8f}, T_{8h}, T_{8j}, T_{8l} \).

**Proof.** By Lemma 7, \( G \) contains an \( S_8(1, 3) \). Let \( T \) be an \( S_8(1, 3) \) in \( G \) with \( V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\} \) and \( E(T) = \{v_0v_1 \mid 1 \leq i \leq 4\} \cup \{v_1w_1, w_1w_2, w_2w_3\} \). Set \( U = V(G) - V \). Obviously, \( |U| = 7 \).

If \( G \) contains no \( T_{8f} \), then \( w_3v_i \notin E(G) \) for \( 2 \leq i \leq 4 \) and \( N_U(w_3) = \emptyset \) and \( d_U(v_i) \leq 1 \) for \( 2 \leq i \leq 4 \). Thus, since \( |U| = 7 \), we can choose three vertices \( u_1, u_2, u_3 \in U \) such that \( v_iu_j \notin E(G) \) for \( 2 \leq i \leq 4 \) and \( 1 \leq j \leq 3 \) and hence \( \overline{G}[\{w_3, v_2, v_3, v_4, u_1, u_2, u_3\}] \) contains a \( W_6 \) with the hub \( w_2 \), a contradiction. Thus we have \( R(T_{8f}, W_6) \leq 15 \).

If \( G \) contains no \( T_{8h} \), then \( w_2v_i \notin E(G) \) for \( 2 \leq i \leq 4 \) and \( N_U(w_2) = \emptyset \). If \( \Delta(G[U]) \leq 2 \), then since \( |U| = 7 \), \( \overline{G}[U] \) contains a \( C_6 \) by Lemma 1 and hence \( \overline{G} \) contains a \( W_6 \) with the hub \( w_2 \), a contradiction. Hence we may assume \( u \in U \) and \( u_1, u_2, u_3 \in N_U(u) \). In this case, we have \( v_iu_j \notin E(G) \) for \( 2 \leq i \leq 4 \) and \( 1 \leq j \leq 3 \) for otherwise \( G \) contains a \( T_{8h} \) and hence \( \overline{G}[\{w_2, v_2, v_3, v_4, u_1, u_2, u_3\}] \) contains a \( W_6 \) with the hub \( w_2 \), again a contradiction. Thus we have \( R(T_{8h}, W_6) \leq 15 \).

If \( G \) contains no \( T_{8j} \), then \( v_1v_i \notin E(G) \) for \( 2 \leq i \leq 4 \), \( N_U(v_1) = \emptyset \) and \( d_U(v_i) \leq 1 \) for \( 2 \leq i \leq 4 \). Thus, since \( |U| = 7 \), we can choose three vertices \( u_1, u_2, u_3 \in U \) such that \( v_iu_j \notin E(G) \) for \( 2 \leq i \leq 4 \) and \( 1 \leq j \leq 3 \) and hence \( \overline{G}[\{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}] \) contains a \( W_6 \) with the hub \( w_2 \), a contradiction. Thus we have \( R(T_{8j}, W_6) \leq 15 \).

If \( G \) contains no \( T_{8l} \), then \( N_U(v_1) = \emptyset \) for \( 2 \leq i \leq 4 \) and \( \{v_2, v_3, v_4\} \) is an independent set. By Lemma 4, we have \( \delta(G[U]) \geq 4 \)
and hence $G[U]$ contains a $C_7$ by Lemma 1. In this case, we have $d_U(v) = 0$ for each $v \in V(T)$ which implies $\delta(G[V]) \geq 5$ by Lemma 4. Noting that $\{v_2, v_3, v_4\}$ is an independent set, we have $v_2v_1, v_3w_3 \in E(G)$ and hence $G$ contains a $T_{8l}$. Thus we have $R(T_{8l}, W_6) \leq 15.$

Theorem 7. $R(T_{8g}, W_6) = 15.$

Proof. By Lemma 6, $G$ contains an $S_8[4]$. Let $T$ be an $S_8[4]$ with $V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0, v_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, w_1w_2, w_1w_3\}$. Set $U = V(G) - V(T)$. If $G$ contains no $T_{8g}$, then $w_2v_i \notin E(G)$ for $2 \leq i \leq 4$, $N_U(w_2) = \emptyset$ and $d_U(v_i) = 0$ for $2 \leq i \leq 4$. Thus, for any three vertices $u_1, u_2, u_3 \in U$, $\overline{G}\{w_2, v_2, v_3, u_1, u_2, u_3\}$ contains a $W_6$ with the hub $w_2$, a contradiction. Hence we have $R(T_{8g}, W_6) \leq 15.$

Theorem 8. $R(T_{8l}, W_6) = 15.$

Proof. By Lemma 8, $G$ contains an $S_8(3,1)$. Let $T$ be an $S_8(3,1)$ in $G$ with $V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\}$ and $E(T) = \{v_0, v_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, v_2w_2, v_3w_3\}$. Set $U = V(G) - V(T)$. Obviously, $|U| = 7$. If $G$ contains no $T_{8l}$, then we have $v_4v_i \notin E(G)$ for $1 \leq i \leq 3$, $N_U(v_i) = \emptyset$ for $1 \leq i \leq 3$ and $d_U(v_4) \leq 1$. Thus, since $|U| = 7$, there is three vertices $u_1, u_2, u_3 \in U$ such that $u_1, u_2, u_3 \notin N_U(v_4)$ and hence $\overline{G}\{v_4, v_1, v_2, v_3, u_1, u_2, u_3\}$ contains a $W_6$ with the hub $v_4$, a contradiction. Hence we have $R(T_{8l}, W_6) \leq 15.$

Theorem 9. $R(T_{8o}, W_6) = 15.$

Proof. By Theorem 8, $G$ contains a $T_{8l}$. Let $T$ be a $T_{8l}$ with $V(T) = \{v_0, \ldots, v_4, w_1, \ldots, w_4\}$ and $E(T) = \{v_0v_i \mid 1 \leq i \leq 3\} \cup \{v_1w_1, v_2w_2, v_3w_3, v_4w_4\}$. Set $U = V(G) - V(T)$. If $G$ contains no $T_{8o}$, then we have $w_4w_i \notin E(G)$ for $1 \leq i \leq 3$ and $N_U(w_i) = \emptyset$ for $1 \leq i \leq 4$. Thus, for any three vertices $u_1, u_2, u_3 \in U$, $\overline{G}\{w_4, w_1, w_2, w_3, u_1, u_2, u_3\}$ contains a $W_6$ with the hub $w_4$, a contradiction. Hence we have $R(T_{8o}, W_6) \leq 15.$

Theorem 10. $R(T_{8s}, W_6) = 15.$
Proof. By Theorem 4, \( G \) contains a \( T_{8a} \). Let \( T \) be a \( T_{8a} \) with 

\[
V(T) = \{v_0, \ldots, v_4, w_1, w_2, w_3\} \quad \text{and} \quad E(T) = \{v_0v_i \mid 1 \leq i \leq 4\} \cup \{v_1w_1, v_1w_2, w_2w_3\}.
\]

Set \( U = V(G) - V(T) \). Obviously, \( |U| = 7 \).

If \( G \) contains no \( T_{8m} \), then \( \{v_2, v_3, v_4\} \) is an independent set and 

\[
N_U(v_i) = \emptyset \quad \text{for} \quad 2 \leq i \leq 4.
\]

By Lemmas 1 and 4, \( G[U] \) contains a \( C_7 \). This implies \( N_U(w_2) = \emptyset \) for otherwise \( G \) contains a \( T_{8m} \). If \( \{v_2, v_3, v_4\} \subseteq N(w_2) \), then \( G \) contains a \( T_{8m} \). Hence we may assume \( v_2w_2 \not\in E(G) \). Thus, for any three vertices \( u_1, u_2, u_3 \in U \), 

\[
\overline{G}[\{v_2, v_3, v_4, w_2, u_1, u_2, u_3\}]
\]

contains a \( W_6 \) with the hub \( v_2 \), a contradiction. Hence we have \( R(T_{8a}, W_6) \leq 15 \).

The proof of Theorem 3 is completed.

5. Proof of Theorem 1

Proof of Theorem 1. If \( \Delta(T_n) = 2 \), then Theorem 1 holds by Lemma 2. Hence we may assume \( \Delta(T_n) \geq 3 \). If \( n = 5 \), then \( T_5 = S_5(1,1) \) and hence Theorem 1 holds. If \( n \geq 6 \) and \( \Delta(T_n) \geq n - 3 \), then Theorem 1 holds by Theorems D and E. Thus we may assume \( 3 \leq \Delta(T_n) \leq n - 4 \). In this case, we have \( n \geq 7 \). If \( n = 7 \), then Theorem 1 holds by Theorem 2. If \( n = 8 \), then Theorem 1 holds by Theorem 3.

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