2005,**25B**(4):672-680

Acta Mathematica Scientia
数学物理学报

ENTIRE CHROMATIC NUMBER AND Δ-MATCHING OF OUTERPLANE GRAPHS [1]

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Abstract Let G be an outerplane graph with maximum degree Δ and the entire chromatic number $\chi_{\rm vef}(G)$. This paper proves that if $\Delta \geq 6$, then $\Delta+1 \leq \chi_{\rm vef}(G) \leq \Delta+2$, and $\chi_{\rm vef}(G) = \Delta+1$ if and only if G has a matching M consisting of some inner edges which covers all its vertices of maximum degree.

Key words Outerplane graph, matching, entire chromatic number

2000 MR Subject Classification 05C15

1 Introduction

We only consider simple graphs in this paper unless otherwise stated. For a plane graph G, we denote its vertex set, edge set, face set, minimum degree, and maximum degree by $V(G)$, $E(G)$, $F(G)$, $\delta(G)$, and $\Delta(G)$, respectively. For $v \in V(G)$, let $d_G(v)$ denote the degree of v in G, and $N_G(v)$ the neighbor set of v in G. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. A vertex (or face) of degree k is called a k-vertex (or k-face). Other statements and notations can be found in [2].

A plane graph G is k- entire colorable if the elements of $V(G) \cup E(G) \cup F(G)$ can be colored with k colors such that any two adjacent or incident elements receive different colors. The entire chromatic number $\chi_{\rm vef}(G)$ of G is the minimum number k such that G is k-entire colorable.

By the definition, $\chi_{\rm vef}(G) \geq \Delta(G)+1$. Kronk and Mitchem[4] conjectured that $\chi_{\rm vef}(G) \leq \Delta(G)+4$ for any plane graph G and they confirmed the conjecture for the case $\Delta(G) \leq 3$. In 1996, Borodin[3] established the conjecture for all plane graphs G with $\Delta(G) \geq 7$. More recently, Sanders and Zhao[5] further settled the case $\Delta(G)=6$. Thus the conjecture remains open only for the case $\Delta(G)=4,5$. Wang[7] recently proved that every plane graph G is $(\chi'(G)+4)$-entire colorable, where $\chi'(G)$ is the chromatic index of G. This implies that Kronk and Mitchem's conjecture holds for bipartite plane graphs. It is proved in [6] that every outerplane graph G

[1]Received March 28, 2003; revised May 30, 2004. Research supported partially by NSFC(10471131) and ZJNSF(M103094)

with $\Delta(G) \geq 6$ satisfies $\Delta(G)+1 \leq \chi_{\text{vef}}(G) \leq \Delta(G)+2$ and $\chi_{\text{vef}}(G) = \Delta(G)+1$ if G is 2-connected. Both lower and upper bounds of this result are tight. A fan F_n of order $n \geq 6$ has $\chi_{\text{vef}}(F_n) = \Delta(F_n)+1$ and a star $K_{1,n-1}$ of order $n \geq 6$ has $\chi_{\text{vef}}(K_{1,n-1}) = \Delta(K_{1,n-1})+2$. Thus it seems very interesting to give a complete classification of all outerplane graphs G with $\Delta(G) \geq 6$ according to their entire chromatic numbers. This paper presents a perfect solution for the problem.

2 Structural Properties

A plane graph G is called an outerplane graph if all the vertices of G lie on the boundary of some face. This face is called outer face, denoted by $f_0(G)$, and other faces inner faces. The edges on the boundary of outer face are called outer edges, and other edges inner edges. Let $E_{\text{in}}(G)$ and $E_{\text{out}}(G)$ denote the sets of inner edges and outer edges of G, respectively. In the sequel, we use $[xyz]$ to denote a 3-face with boundary vertices x, y, and z. Moreover, for $k = 0, 1, \cdots, \Delta = \Delta(G)$, let $V_k(G)$ denote the set of all k-vertices in G.

Lemma 2.1 Every outerplane graph of order ≥ 2 contains two vertices of degree at most 2.

Lemma 2.2 If G is a 2-connected outerplane graph with $|V(G)| \geq 5$, then

(1) $|N_G(u) \cap V_2(G)| \leq 2$ for each vertex $u \in V(G)$; and

(2) $N_G(u) \neq N_G(v)$ for any two distinct vertices $u, v \in V_2(G)$.

Lemmas 2.1 and 2.2 are straightforward and thus we omit their proofs.

Lemma 2.3 Let G be a 2-connected outerplane graph and $s^* \in V(G)$. Then G contains one of the following configurations:

(1) Two adjacent 2-vertices u and v such that $s^* \notin \{u, v\}$.

(2) A 3-face $[uxy]$ with $d_G(u) = 2$, $d_G(x) = 3$, and $xy \in E_{\text{in}}(G)$ such that $s^* \notin \{u, x\}$.

(3) A 3-face $[uxy]$ with $d_G(u) = 2$, $d_G(x) = d_G(y) = 4$, and $xy \in E_{\text{in}}(G)$ such that $s^* \notin \{u, x, y\}$.

(4) Three 3-faces $[xu_1v_1], [xu_2v_2]$, and $[xv_1v_2]$ with $d_G(u_1) = d_G(u_2) = 2$, $d_G(x) = 4$, and $v_1v_2 \in E_{\text{in}}(G)$ such that $s^* \notin \{x, u_1, u_2\}$.

Proof If $|V(G)| \leq 4$ or $\Delta(G) = 2$, G contains obviously either (1) or (2). Thus assume that $|V(G)| \geq 5$ and $\Delta(G) \geq 3$. Suppose to the contrary that the lemma is false, i.e., G contains none of the configurations (1) to (4). Let

$$\begin{aligned}
&\widetilde{V} = \{v \in V_2(G) \setminus \{s^*\} | xy \notin E(G) \text{ with } N_G(v) = \{x, y\}\}, \\
&\widetilde{E} = \{xy | v \in \widetilde{V} \text{ with } N_G(v) = \{x, y\}\}, \\
&H = G - \widetilde{V} + \widetilde{E}.
\end{aligned}$$

Thus H is a 2-connected outerplane graph with $s^* \in V(H)$. It is easy to see that H also doesn't contain any of (1) to (4). For every 2-vertex $v \in V_2(H) \setminus \{s^*\}$ with $N_H(v) = \{x, y\}$, we have $xy \in E(G)$, $d_H(x) \geq 4$, $d_H(y) \geq 4$, and $\max\{d_H(x), d_H(y)\} \geq 5$ (when $s^* \in \{x, y\}$, $d_H(s^*) \geq 3$ since H is 2-connected). Without loss of generality, we assume $x \neq s^*$ (otherwise, we take $y \neq s^*$).

If $y \neq s^*$, we furthermore assume that $d_H(y) \geq 5$. Hence one of the following cases holds by the assumption:

(a) $d_H(x) \geq 5$;

(b) $d_H(x) = 4$ and $N_H(x) \cap (V_2(H) \setminus \{s^*, v\}) = \emptyset$;

(c) $d_H(x) = 4$ and there exists $x_1 \in N_H(x) \cap (V_2(H) \setminus \{s^*, v\})$. Let y_1 denote the neighbor of x_1 in H different from x. Then $yy_1 \notin E(H)$ because H contains no (4).

If $y = s^*$, we can similarly prove that one of (a), (b), and (c) holds.

Let H_1 denote the graph obtained from H by handling all 2-vertices v in $V_2(H) \setminus \{s^*\}$ in this way: if either (a) or (b) holds, we remove the vertex v; if (c) holds, we add the edge yy_1 after removing the vertices v, x, and x_1. Clearly, H_1 is a 2-connected outerplane graph. Using Lemma 2.2, we may prove that $d_{H_1}(s^*) \geq 2$ and $d_{H_1}(t) \geq 3$ for all $t \in V(H_1) \setminus \{s^*\}$. This implies that $V_2(H_1) \setminus \{s^*\} = \emptyset$, which contradicts Lemma 2.1.

Theorem 2.4 Let G be an outerplane graph with $\delta(G) = 2$. Then G contains one of the following configurations:

(1) Two adjacent 2-vertices u and v.

(2) A 3-face $[uxy]$ with $d_G(u) = 2$, $d_G(x) = 3$, and $xy \in E_{\text{in}}(G)$.

(3) A 3-face $[uxy]$ with $d_G(u) = 2$, $d_G(x) = d_G(y) = 4$, and $xy \in E_{\text{in}}(G)$.

(4) Three 3-faces $[xu_1v_1], [xu_2v_2]$, and $[xv_1v_2]$ with $d_G(u_1) = d_G(u_2) = 2$, $d_G(x) = 4$, and $v_1v_2 \in E_{\text{in}}(G)$.

Proof If G is 2-connected, the result follows immediately from Lemma 2.3. In fact, we may choose any vertex of G as the specific vertex s^*. Otherwise, let B be a block of G that contains a unique cut vertex, say s^*, of G. Since B is 2-connected and $s^* \in V(B)$, B contains one of the following configurations by Lemma 2.3.

(1′) Two adjacent 2-vertices u and v such that $s^* \notin \{u, v\}$.

(2′) A 3-face $[uxy]$ with $d_B(u) = 2$, $d_B(x) = 3$, and $xy \in E_{\text{in}}(B)$ such that $s^* \notin \{u, x\}$.

(3′) A 3-face $[uxy]$ with $d_B(u) = 2$, $d_B(x) = d_B(y) = 4$, and $xy \in E_{\text{in}}(B)$ such that $s^* \notin \{u, x, y\}$.

(4′) Three 3-faces $[xu_1v_1], [xu_2v_2]$, and $[xv_1v_2]$ with $d_B(u_1) = d_B(u_2) = 2$, $d_B(x) = 4$, and $v_1v_2 \in E_{\text{in}}(B)$ such that $s^* \notin \{x, u_1, u_2\}$.

Note that $E_{\text{in}}(B) \subseteq E_{\text{in}}(G)$ and $d_B(t) = d_G(t)$ for all $t \in \{u, v, x, y, u_1, u_2\}$. Thus (1′) to (4′) are the desired subgraphs of G.

Corollary 2.5 Let G be an outerplane graph with $\delta(G) = 2$. Then G contains one of the following configurations:

(1) Two adjacent 2-vertices u and v.

(2) A 3-face $[uxy]$ with $d_G(u) = 2$ and $d_G(x) = 3$.

(3) Two 3-faces $[xu_1v_1]$ and $[xu_2v_2]$ with $d_G(u_1) = d_G(u_2) = 2$ and $d_G(x) = 4$.

3 Entire Chromatic Number

A matching M of an ouerplane graph G is called a Δ-matching if it consists of some inner edges and covers all its vertices of maximum degree.

Theorem 3.1 Let G be an outerplane graph and $t(G) = \max\{\Delta(G) + 2, 7\}$. Then G admits a $t(G)$-entire coloring satisfying the following Property (P1):

(P1): some color is only used to color the outer face $f_0(G)$.

Proof By induction on the vertex number $|V(G)|$. When $|V(G)| \le 5$, the theorem holds trivially. Suppose that it is true for all the outerplane graphs with less than n vertices, and let G be an outerplane graph with $|V(G)| = n \ge 6$. Note that the proof is easy for the case $\delta(G) \le 1$. Thus we suppose that $\delta(G) = 2$. By Corollary 2.5, we need to handle three cases. For each case, we define an outerplane graph H with $|V(H)| < n$ and let ϕ denote a desired $t(G)$-entire coloring of H with the color set $C = \{1, 2, \cdots, t(G)\}$ by the induction hypothesis. Then we extend ϕ to a $t(G)$-entire coloring of G satisfying (P1). Since every 2-vertex $y \in V(G) \setminus V(H)$ is adjacent or incident to at most six colored elements (namely, at most two vertices, two edges, and two faces) whereas $|C| \ge 7$, y can always be colored properly whatever its adjacent or incident elements have been colored. Thus we may omit the coloring for all 2-vertices in the following proof.

Case 1 G contains two adjacent 2-vertices u and v. Let $u_1 \in N_G(u) \setminus \{v\}$ and $v_1 \in N_G(v) \setminus \{u\}$. Let f' denote the face of G whose boundary contains the edge uv and $f' \ne f_0(G)$.

First suppose that $u_1 \ne v_1$. Define the graph $H = G - u + vu_1$ and let f'', different from the outer face $f_0(H)$, denote the face of H whose boundary contains the edge vu_1. Obviously, $|V(H)| < n$ and $\Delta(H) \le \Delta(G)$, so $t(H) \le t(G)$. Suppose that $\phi(f_0(H)) = c_0$. Thus c_0 is used only once in the coloring ϕ by (P1). In G, we color $f_0(G)$ with c_0, f' with $\phi(f'')$, uu_1 with $\phi(vu_1)$, and uv with a color $c_1 \in C \setminus \{\phi(vv_1), \phi(vu_1), \phi(f''), c_0\}$. Since uv has at most four forbidden colors when it will be colored whereas $t(G) \ge 7$, the coloring is admissible.

Next suppose that $u_1 = v_1$. This implies that $[uvu_1]$ is a 3-face of G. Let $H = G - u - v$ and suppose $\phi(f_0(H)) = c_0$ is used only once. We use $S(x)$ to denote the set of colors assigned to a vertex $x \in V(H)$ and those edges incident to x in H under the coloring ϕ. In G, we further color $f_0(G)$ with c_0, uu_1 with $c_1 \in C \setminus (S(u_1) \cup \{c_0\})$, vu_1 with $c_2 \in C \setminus (S(u_1) \cup \{c_1, c_0\})$, uv with $c_3 \in C \setminus \{c_2, c_1, c_0\}$, and $[uvu_1]$ with $c_4 \in C \setminus \{\phi(u_1), c_3, c_2, c_1, c_0\}$. Since $|S(u_1)| = d_H(u_1) + 1 \le d_G(u_1) - 2 + 1 = d_G(u_1) - 1 \le \Delta(G) - 1$, each element x has at most six or $\Delta(G) + 1$ forbidden colors when we consider to color it. By $t(G) = \max\{\Delta(G) + 2, 7\}$, the coloring is available.

Case 2 G contains a 3-face $[uxy]$ with $d_G(u) = 2$ and $d_G(x) = 3$. Let f^*, different from $[uxy]$, denote the face of G whose boundary contains the edge xy. Define $H = G - u$ and suppose $\phi(f_0(H)) = c_0$. In G, we color $f_0(G)$ with c_0, uy with a color $c_1 \in C \setminus (S(y) \cup \{c_0\})$, $[uxy]$ with a color $c_2 \in C \setminus \{\phi(x), \phi(y), \phi(f^*), \phi(xy), c_1, c_0\}$, and ux with a color $c_3 \in C \setminus (S(x) \cup \{c_2, c_1, c_0\})$. Since $|S(y)| \le d_G(y) - 1 + 1 \le \Delta(G)$ and $|S(x)| \le 3$, the coloring is available.

Case 3 G contains two 3-faces $[xu_1v_1]$ and $[xu_2v_2]$ with $d_G(u_1) = d_G(u_2) = 2$ and $d_G(x) = 4$. Let f^* denote the face of G with xv_1 and xv_2 as boundary edges. Consider the graph $H = G - u_1 - u_2$ and assume $\phi(f_0(H)) = c_0$. In G, we first color $f_0(G)$ with c_0, u_iv_i with a color $c_i \in C \setminus (S(v_i) \cup \{c_0\})$ for $i = 1, 2$.

If $c_1 \in \{\phi(f^*), \phi(x)\}$, we color $[xu_2v_2]$ with $c_3 \in C \setminus \{\phi(x), \phi(v_2), \phi(xv_2), \phi(f^*), c_2, c_0\}$, xu_2 with $c_4 \in C \setminus \{\phi(x), \phi(xv_1), \phi(xv_2), c_3, c_2, c_0\}$, xu_1 with $c_5 \in C \setminus \{\phi(x), \phi(xv_1), \phi(xv_2), c_4, c_1, c_0\}$, and $[xu_1v_1]$ with $c_6 \in C \setminus \{\phi(x), \phi(v_1), \phi(xv_1), \phi(f^*), c_5, c_1, c_0\}$. If $c_2 \in \{\phi(f^*), \phi(x)\}$, we have a similar proof. So assume $c_1, c_2 \notin \{\phi(f^*), \phi(x)\}$. If $\phi(v_2) \ne \phi(xv_1)$, we color xu_2 with $\phi(v_2)$, xu_1 with $\phi(f^*)$, $[xu_2v_2]$ with $c_3 \in C \setminus \{\phi(x), \phi(v_2), \phi(xv_2), \phi(f^*), c_2, c_0\}$, and $[xu_1v_1]$ with $c_4 \in C \setminus \{\phi(x), \phi(v_1), \phi(xv_1), \phi(f^*), c_1, c_0\}$. If $\phi(v_1) \ne \phi(xv_2)$, a similar argument can be established. If $\phi(v_2) = \phi(xv_1)$ and $\phi(v_1) = \phi(xv_2)$, we interchange the colors of u_1v_1 and

xv_1, and color xu_2 with $\phi(v_2)$ and xu_1 with $\phi(f^*)$. Afterwards we color similarly $[xu_1v_1]$ and $[xu_2v_2]$.

It is easy to check that the above coloring is available. Thus the proof of Theorem 3.1 is completed.

Corollary 3.2 If G is an outerplane graph with $\Delta(G) \geq 5$, then $\Delta(G)+1 \leq \chi_{\text{vef}}(G) \leq \Delta(G)+2$; and moreover G admits a $(\Delta(G)+2)$-entire coloring satisfying Property (P1).

Theorem 3.3 If G is an outerplane graph with $\Delta(G) \geq 6$, then $\chi_{\text{vef}}(G) = \Delta(G)+1$ if and only if G has a Δ-matching.

Proof Suppose that $\chi_{\text{vef}}(G) = \Delta(G)+1$. Let ϕ be an arbitrary $(\Delta(G)+1)$-entire coloring of G. Then, for each $u \in V_\Delta(G)$, all the $\Delta(G)+1$ colors used by ϕ must, at the same time, occur on the vertex u and those edges incident to u. This means that there is an inner edge e_u incident to u which receives the same color as $f_0(G)$. It is easy to see that, for any two distinct vertices $u, v \in V_\Delta(G)$, either $e_u = e_v = uv \in E_{\text{in}}(G)$, or $e_u \neq e_v$ and e_u is non-adjacent to e_v in G. We put

$$M_\Delta = \{e \in E_{\text{in}}(G) | \phi(e) = \phi(f_0(G)\}.$$

Then $M_\Delta \subseteq E_{\text{in}}(G)$ and M_Δ covers all the vertices in $V_\Delta(G)$. Hence M_Δ is a Δ- matching of G.

Conversely, if G contains a Δ-matching M_Δ, let us prove that G admits a $(\Delta(G)+1)$-entire coloring ϕ satisfying the following Property (P2):

(P2): all the edges in M_Δ are assigned to the same color as $\phi(f_0(G))$.

We make use of induction on $|V(G)|$. If $|V(G)| = \Delta(G)+1$, G is either a fan F_n or a subgraph $\overline{F_n}$ of F_n with $\Delta(\overline{F_n}) = \Delta(F_n)$, where $n = |V(G)|$. It is easy to check that given a Δ-matching of G, there exists a $(\Delta(G)+1)$-entire coloring of G satisfying Property (P2). Suppose that G is an outerplane graph with a Δ-matching M_Δ and $|V(G)| \geq \Delta(G)+2$. Obviously, we may assume that G is connected and M_Δ is a maximal Δ-matching of G (namely, it contains as many edges as possible). If G contains a 1-vertex u, let v be the neighbor of u and let $H = G - u$. We consider two cases below.

(i) $\Delta(H) = \Delta(G)$. We note that M_Δ is also a Δ-matching of H. By the induction hypothesis, H has a $(\Delta(G)+1)$-entire coloring ϕ with the color set C satisfying Property (P2). Suppose $\phi(f_0(H)) = c_0$. In G, we color $f_0(G)$ with c_0, uv with a color $c_1 \in C \setminus (S(v) \cup \{c_0\})$, and u with a color $c_2 \in C \setminus \{\phi(v), c_1, c_0\}$. When $d_G(v) < \Delta(G)$, $|S(v)| = d_G(v) - 1 + 1 \leq \Delta(G) - 1$ and hence at most $\Delta(G)$ colors are forbidden to color uv; when $d_G(v) = \Delta(G)$, there exists some edge $e \in M_\Delta$ that covers v in G. Since e also covers v in H, $\phi(e) = c_0$ by Property (P2). Again, at most $\Delta(G)$ colors are forbidden to color uv in G. Therefore the above coloring is available.

(ii) $\Delta(H) < \Delta(G)$. In this case, $\Delta(H) = \Delta(G) - 1 \geq 5$. By Corollary 3.2, H admits a $(\Delta(G)+1)$-entire coloring ϕ satisfying Property (P1). First, all the edges in M_Δ are recolored by the same color $\phi(f_0(H))$, then (ii) is reduced to the case (i).

Suppose now $\delta(G) = 2$. By Theorem 2.4, we need to consider the following cases.

Case 1 G contains two adjacent 2-vertices u and v. Let $x \in N_G(u) \setminus \{v\}$ and $y \in N_G(v) \setminus \{u\}$. Consider the graph $H = G - u + xv$ if $x \neq y$ and $H = G - u$ if $x = y$. It suffices to note that M_Δ is a Δ-matching of H. Similarly to the proof of Case 1 in Theorem 3.1, we can extend any $(\Delta(G)+1)$-entire coloring of H satisfying Property (P2) into a $(\Delta(G)+1)$-entire coloring of G satisfying Property (P2).

Case 2 G contains a 3-face $[uxy]$ with $d_G(u)=2$, $d_G(x)=3$, and $xy\in E_{\rm in}(G)$. We denote by f^* the face of G, distinct from $[uxy]$, whose boundary contains the edge xy. Let $H=G-u$. Clearly, M_Δ (or its subset) forms a Δ-matching of H. Let ϕ be a $(\Delta(G)+1)$-entire coloring of H satisfying Property (P2). In G, we first color $f_0(G)$ with $\phi(f_0(H))=c_0$.

If $xy\notin M_\Delta$, there exists an edge $e\in M_\Delta$ that is incident to y in H since, as otherwise, $M_\Delta\cup\{xy\}$ is a Δ-matching of G with more edges than M_Δ, contradict to the maximality of M_Δ. Thus $\phi(e)=c_0$ by Property (P2). We color uy with $c_1\in C\setminus S(y)$, $[uxy]$ with $c_2\in C\setminus\{\phi(x),\phi(y),\phi(xy),\phi(f^*),c_1,c_0\}$, and ux with $c_3\in C\setminus(S(x)\cup\{c_2,c_1,c_0\})$.

If $xy\in M_\Delta$, then no edge incident to the vertex y in H is assigned to the color c_0. We color uy with $\phi(xy)$ and recolor xy with c_0. Then we color $[uxy]$ with $c_1\in C\setminus\{\phi(x),\phi(y),\phi(xy),\phi(f^*),c_0\}$, and ux with $c_2\in C\setminus(S(x)\cup\{c_1,c_0\})$.

Noting that $|S(y)|\le\Delta(G)$, $|S(x)|\le 3$, and $|C|\ge\Delta(G)+1\ge 7$, the above coloring is available.

Case 3 G contains a 3-face $[uxy]$ with $d_G(u)=2$, $d_G(x)=d_G(y)=4$, and $xy\in E_{\rm in}(G)$. Similarly, let f^* denote the face of G, distinct from $[uxy]$, whose boundary contains the edge xy. Let $H=G-u$. Let ϕ be a $(\Delta(G)+1)$-entire coloring of H with $c_0=\phi(f_0(H))$ satisfying (P2). In G, we first color $f_0(G)$ with c_0. If $xy\in M_\Delta$, we furthermore color uy with $\phi(xy)$ and recolor xy with c_0, then color $[uxy]$ with $c_1\in C\setminus\{\phi(x),\phi(y),\phi(xy),\phi(f^*),c_0\}$, and ux with $c_2\in C\setminus(S(x)\cup\{c_1,c_0\})$. So suppose $xy\notin M_\Delta$. By the maximality of M_Δ, there exists some edge $e\in M_\Delta$ that is incident to one of x and y in H. Without loss of generality, we suppose that e is incident to x, so $\phi(e)=c_0$ by Property (P2). We color uy with $c_1\in C\setminus(S(y)\cup\{c_0\})$, $[uxy]$ with $c_2\in C\setminus\{\phi(x),\phi(y),\phi(xy),\phi(f^*),c_1,c_0\}$, and ux with $c_3\in C\setminus(S(x)\cup\{c_2,c_1\})$. Since $|S(x)|\le 4$ and $|S(y)|\le 4$, the coloring is feasible.

Case 4 G contains three 3-faces $[xu_1v_1]$, $[xu_2v_2]$, and $[xv_1v_2]$ such that $d_G(u_1)=d_G(u_2)=2$, $d_G(x)=4$, and $v_1v_2\in E_{\rm in}(G)$. Suppose that f^* is the face of G whose boundary contains v_1v_2 and $f^*\ne[xv_1v_2]$. Let $H=G-u_1-u_2-x$. It is easy to see that M_Δ (or its subset) forms a Δ-matching of H. Suppose that ϕ is a $(\Delta(G)+1)$-entire coloring of H satisfying Property (P2). In order to construct a desired $(\Delta(G)+1)$-entire coloring of G, we first color $f_0(G)$ with $c_0=\phi(f_0(H))$. Then the proof is divided into the following subcases.

Subcase 4.1 $v_1v_2\in M_\Delta$. It follows that c_0 can not occur on those edges in H each of which is incident to v_1 or v_2. We color both u_1v_1 and xv_2 with $\phi(v_1v_2)$, xu_1 with $\phi(v_1)$, and recolor v_1v_2 with c_0. Furthermore, we color xv_1 with $c_1\in C\setminus(S(v_1)\cup\{c_0\})$, u_2v_2 with $c_2\in C\setminus(S(v_2)\cup\{c_0\})$, $[xv_1v_2]$ with $c_3\in C\setminus\{\phi(v_1),\phi(v_2),\phi(v_1v_2),\phi(f^*),c_1,c_0\}$, x with $c_4\in C\setminus\{\phi(v_1),\phi(v_2),\phi(v_1v_2),c_3,c_1,c_0\}$, and $[xu_1v_1]$ with $c_5\in C\setminus\{\phi(v_1),\phi(v_1v_2),c_4,c_3,c_1,c_0\}$.

If $c_1\ne\phi(v_2)$, we color xu_2 with $\phi(v_2)$. So assume $c_1=\phi(v_2)$. When $c_3=c_2$, we color xu_2 properly. When $c_3\ne c_2$, we color xu_2 with c_3. Finally, we can color $[xu_2v_2]$ properly in both cases.

Subcase 4.2 $xv_2\in M_\Delta$. In this case, either c_0 occurs on some incident edge of v_1 in H, or $d_G(v_1)<\Delta(G)$ and c_0 does not occur on any incident edge of v_1 in H. We color xv_2 with c_0, xu_1 with $\phi(v_1)$, xu_2 with $\phi(v_2)$, xv_1 with $c_1\in C\setminus(S(v_1)\cup\{c_0\})$, u_1v_1 with $c_2\in C\setminus(S(v_1)\cup\{c_1,c_0\})$, u_2v_2 with $c_3\in C\setminus(S(v_2)\cup\{c_0\})$, $[xv_1v_2]$ with $c_4\in C\setminus\{\phi(v_1),\phi(v_2),\phi(v_1v_2),\phi(f^*),c_1,c_0\}$, x

with $c_5 \in C \setminus \{\phi(v_1), \phi(v_2), c_4, c_1, c_0\}$, $[xu_1v_1]$ with $c_6 \in C \setminus \{\phi(v_1), c_5, c_4, c_2, c_1, c_0\}$, and $[xu_2v_2]$ with $c_7 \in C \setminus \{\phi(v_2), c_5, c_4, c_3, c_0\}$.

If $xv_1 \in M_\Delta$, we have a similar argument.

Subcase 4.3 $xv_1, xv_2, v_1v_2 \notin M_\Delta$. The maximality of M_Δ implies that, for $i = 1, 2$, there exists some edge e_i incident to v_i in H such that $\phi(e_i) = c_0$. We color $[xv_1v_2]$ with c_0, u_1v_1 with $c_1 \in C \setminus S(v_1)$, xv_1 with $c_2 \in C \setminus (S(v_1) \cup \{c_1\})$, xv_2 with $c_3 \in C \setminus (S(v_2) \cup \{c_2\})$, u_2v_2 with $c_4 \in C \setminus (S(v_2) \cup \{c_3\})$, x with $c_5 \in C \setminus \{\phi(v_1), \phi(v_2), c_3, c_2, c_0\}$, xu_1 with $c_6 \in C \setminus \{c_5, c_3, c_2, c_1, c_0\}$, xu_2 with $c_7 \in C \setminus \{c_6, c_5, c_4, c_3, c_2, c_0\}$, $[xu_1v_1]$ with $c_8 \in C \setminus \{\phi(v_1), c_6, c_5, c_2, c_1, c_0\}$, and $[xu_2v_2]$ with $c_9 \in C \setminus \{\phi(v_2), c_7, c_5, c_4, c_3, c_0\}$. Since $|S(v_i)| \leq \Delta(G) - 1$ for $i = 1, 2$, the coloring is feasible.

Finally, we can color properly all the 2-vertices in $V(G) \setminus V(H)$, similarly to the proof of Theorem 3.1. This completes the proof of the theorem.

We conjecture that Theorem 3.3 holds for an outerplane graph of maximum degree 5. It was proved in [8] that every outerplane graph G with $\Delta(G) = 4$ has $5 \leq \chi_{\rm vef}(G) \leq 6$, and a sufficient and necessary condition for $\chi_{\rm vef}(G) = 5$ was established. Using Corollary 3.2 and repeating the proof of the necessity for Theorem 3.3, we can derive the following

Theorem 3.4 If G is an outerplane graph with $\Delta(G) = 5$ and without a Δ-matching, then $\chi_{\rm vef}(G) = \Delta(G) + 2$.

4 Δ-Matching

Theorem 3.3 shows that the problem of determining the entire chromatic number of an outerplane graph G with $\Delta(G) \geq 6$ is equivalent to search for a Δ-matching in G. In this section, we investigate the existence of Δ-matchings in an outerplane graph. We first prove a useful lemma.

Lemma 4.1 Every outerplane graph G contains a matching M that covers all the vertices of degree at least 3.

Proof The result holds obviously if $|V(G)| \leq 4$. Suppose that G is a connected outerplane graph with $|V(G)| \geq 5$. If G contains a 1-vertex u, let $uv \in E(G)$. By the induction hypothesis, $G - u$ admits a matching M' that covers every vertex of degree at least 3 in $G - u$. Let $M = M' \cup \{uv\}$ if M' does not cover v and $M = M'$ otherwise. It is easy to see that M is a required matching of G.

Now suppose that $\delta(G) = 2$. By Corollary 2.5, we need to consider the following two cases.

Case 1 G contains two adjacent 2-vertices u and v. Let $u_1 \in N_G(u) \setminus \{v\}$ and $v_1 \in N_G(v) \setminus \{u\}$. By the induction hypothesis, $G - u - v$ admits a matching M that covers every vertex of degree at least 3. If M' covers both u_1 and v_1, let $M = M'$. If M' covers exactly one of u_1 and v_1, say u_1, let $M = M' \cup \{vv_1\}$. If M' covers neither u_1 nor v_1, let $M = M' \cup \{uu_1, vv_1\}$ when $u_1 \neq v_1$, and $M = M' \cup \{uu_1\}$ when $u_1 = v_1$. Thus M is a matching of G covering each vertex of degree at least 3.

Case 2 G contains a 3-face $[uxy]$ with $d(u) = 2$. Suppose that M' is a matching of $G - u$ that covers every vertex of degree at least 3 by the induction hypothesis. Similarly, if M' covers both x and y, we take $M = M'$. If M' covers exactly one of x and y, say x, we take $M = M' \cup \{uy\}$. If M' covers neither x nor y, we take $M = M' \cup \{xy\}$. It is easy to show that M is a matching of G covering every vertex of degree at least 3. The proof of the lemma is completed.

Lemma 4.1 is the best possible in the sense that there exist outerplane graphs without a matching covering all vertices of degree at least 2. For example, an odd cycle is an outerplane graph without a perfect matching.

Suppose that G is an outerplane graph and $u \in V(G)$. The inner degree, denoted by $d_{in}(u)$, of u is defined to be the number of inner edges incident to u in G. Set

$$
\begin{aligned}
&A(G) = V_\Delta(G),\\
&\delta_{\rm in}(G) = \min\{d_{\rm in}(u)|u \in A(G)\},\\
&B(G) = \{x \in V(G) \setminus A(G)|, \quad \text{there is } \ y \in A(G) \ \text{ such that } \ xy \in E_{\rm in}(G)\},\\
&G^* = G[A(G) \cup B(G)] - E_{\rm out}(G) - E(G[B(G)]).
\end{aligned}
$$

Then G contains a Δ-matching if and only if G^* has a matching that covers all vertices in $A(G)$. Moreover, if $\delta_{\rm in}(G) = 0$, i.e., G has a vertex of maximum degree not incident to any inner edge, then G has not a Δ-matching. However, we have the following.

Theorem 4.2 If G is an outerplane graph with $\delta_{\rm in}(G) \geq 3$, then G contains a Δ-matching.

Proof Let G^* be defined as above. By Lemma 4.1, G^* admits a matching M^* that covers every vertex of degree at least 3 in G^*. We note that $M^* \subseteq E(G^*) \subseteq E_{\rm in}(G)$ and $A(G) \subseteq V(G^*)$. For every vertex $v \in A(G)$, $d_{G^*}(v) \geq \delta_{\rm in}(G) \geq 3$. It follows that M^* is a matching of G that covers every vertex of maximum degree. Thus M^* is a Δ-matching of G.

The condition that $\delta_{\rm in}(G) \geq 3$ in Theorem 4.2 can not be weaken to the case $1 \leq \delta_{\rm in}(G) \leq 2$. Let G be an outerplane graph obtained by adding n chords $x_1x_3, x_3x_5, x_5x_7, \cdots, x_{2n-1}x_1$ to a $2n$-cycle $x_1x_2x_3 \cdots x_{2n}x_1$, where $n \geq 3$ is odd. It is easy to see that $\delta_{\rm in}(G) = 2$ and G has not a Δ-matching.

Corollary 4.3 Every 2-connected outerplane graph G with $\Delta(G) \geq 5$ contains a Δ-matching.

Proof Since G is 2-connected, every vertex is incident to exactly two outer edges. Thus, for each $u \in A(G)$, $d_{\rm in}(u) = d_G(u) - 2 \geq 5 - 2 = 3$. Therefore $\delta_{\rm in}(G) \geq 3$, and hence the result follows from Theorem 4.2.

Combining Theorem 3.3 and Corollary 4.3, we have the following.

Corollary 4.4 If G is a 2-connected outerplane graph with $\Delta(G) \geq 6$, then $\chi_{\rm vef}(G) = \Delta(G) + 1$.

Theorem 4.5 Let G be an outerplane graph with $\Delta(G) \geq 5$. If one of the following conditions holds, then G has a Δ-matching:

(1) $\delta_{\rm in}(G) \geq |A(G)|$;

(2) G^* contains no cut vertex and each odd component of G^* contains at least one vertex in $B(G)$.

Proof First suppose that (1) holds. Let $A(G) = \{x_1, x_2, \cdots, x_k\}$. Thus $\delta_{\rm in}(G) \geq k$. If $k \geq 3$, G contains a Δ-matching by Theorem 4.2. Assume that $1 \leq k \leq 2$. It is easy to observe that there exists a subset $M = \{e_1, \cdots, e_k\} \subseteq E(G^*)$ such that x_i is incident to e_i for $i = 1, 2, \cdots, k$, and for $i \neq j$, either $e_i = e_j$ or e_i is not adjacent to e_j. Hence M is a Δ-matching of G. Next suppose that (2) holds. It suffices to note that each component of G^* is either a K_2 or a Hamiltonian graph. In every case, G contains obviously a Δ-matching.

Lemma 4.6 (Bollobás [1]) Let G be a graph and $T \subseteq V(G)$. Then G has a matching covering T if and only if

$$o(G - S|T) \leq |S| \quad \text{for all } S \subset V(G),$$

where $o(H|T)$ denotes the number of odd components of H whose vertex set is contained in T.

Applying Lemma 4.6, we have the following result.

Theorem 4.7 An outerplane graph G has a Δ-matching if and only if

$$o(G^* - S|A(G)) \leq |S| \quad \text{for all} \quad S \subset A(G) \cup B(G).$$

Finally, we would like to provide an effective procedure to determine if an outerplane graph has a Δ-matching. It is described as follows:

Step 1 For a given outerplane graph G, define $A(G), B(G), G^*$ and $n = |V(G^*)|$.

Step 2 Construct a graph H with $2n$ vertices obtained from $G^* + K_n$ by joining every vertex of K_n to every vertex of $B(G)$. Thus G has a Δ-matching iff H has a perfect matching.

Step 3 Find a perfect matching of H by means of some known algorithms. Particularly, if $G^*[A(G)]$ is an empty graph, then G^* is a bipartite graph with a vertex bipartition $(A(G), B(G))$. In this case, we can solve it by the well-known Hungarian method (see [2]). If $B(G) = \emptyset$, the problem of solving a Δ-matching of G is equivalent to find a perfect matching in G^*.

Actually, the above procedure provides a polynomial-time algorithm, so it is a good algorithm. Thus all outerplane graphs of maximum degree at least 6 have been completely classified according to their entire chromatic numbers.

References

1 Bollobás B. Extremal Graph Theory. London: Academic Press Inc, 1978

2 Bondy J A, Murty U S R. Graph Theory with Applications. New York: Macmillan, 1976

3 Borodin O V. Structural theorem on plane graphs with application to the entire coloring number. J Graph Theory, 1996, **23**: 233-239

4 Kronk H, Mitchem J. A seven-color theorem on the sphere. Discrete Math, 1973, **5**: 253-260

5 Sanders D P, Zhao Y. On the entire coloring conjecture. Canad Math Bull, 2000, **43**(1): 108-114

6 Wang Weifan. On the colorings of outerplanr graphs. Discrete Math, 1995, **147**: 257-269

7 Wang Weifan. Upper bounds of entire chromatic number of plane graphs. European J Combin, 1999, **20**: 303-305

8 Wang Weifan. Entire chromatic number of outerplane graphs with maximum degree four. Chinese J Engry Math, 2000, **17**(4): 19-24