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# On the Critical Domination Numbers

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**Abstract** In this paper, we give a series of upper bounds of the critical domination number and show some characterizations of a graph with critical domination number $\triangle+1$.

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We use [1] for terminology and notation not defined here. Let $G=(V(G),E(G))$ be a graph. The closed neighborhood of $x\in V(G)$ is denoted $N[x]$, If $A,B\subseteq V(G)$, Let $E(A,B)=\{uv|u\in A,v\in B,uv\in E(G)\}$ and $e(A,B)=|E(A,B)|$, $e(\{v\},B)$ is in brief as $e(v,B)$. We say $A$ dominate $B$, denoted $A\Rightarrow B$, if for each $b\in B$, $N[b]\cap A\neq\varnothing$. When $B=V(G)$ we say $A$ is a dominating set of $G$. The domination number $r(G)$ of $G$ is the size of a smallest dominating set D. [3] first presented the bondage number $b(G)$ of a graph $G$ to be the cardinality of a smallest set $E$ of edges for which $r(G-E)>r(G)$, and obtained some bounds for them. Replacing the deletions of edges in the concept of bondage number with additions of edges, [8] introduces the concept of critical domination number as follows: for $r(G)\geqslant 2$, the critical domination number $r_c(G)$ is defined as $r_c(G)=\min\{|S|\,|r(G+S)<r(G),S\subseteq E(\bar{G})\}$. [2]defined a whip in $G$ as any spanning subgraph $F$ of $G$ such that each component of $F$ is a star and $F$ has precisely $r(G)$ components. Let $W(G)$ denote the set of all whips in $G$. For $F\in W(G)$, we denote the $r=r(G)$ stars as $S_1,S_2,\cdots,S_r$, the center $x_i$ of $S_i(i=1,2,\cdots,r)$. Without loss of generality, we assume $|S_1|\leqslant|S_2|\leqslant\cdots\leqslant|S_r|$ ($|V(S_i)|$ is in brief as $|S_i|$). In this paper, we give some upper bounds for $r_c(G)$ and characterize the properties of $G$ with $r_c(G)=\triangle(G)+1$.

**Lemma**[8] If $S\subseteq E(\bar{G})$, $r(G+S)<r(G)$ and $r_c(G)=|S|$, then $r(G+S)=r(G)-1$.

**Theorem 1** $r_c(G)=\min\{|X|\,|r(G-X)=r(G)-1,X\subseteq V(G)\}$.

**Proof** Let $X_0\subseteq V(G)$, $r(G-X_0)=r(G)-1$ and $|X_0|=\min\{|X|r(G-X)=r(G)-1,X\subseteq V(G)\}$, $\{x_1,x_2,\cdots,x_{r-1}\}$ is a smallest dominating set of $G-X_0$, $E_1=\{x_1x|x\in X_0,x_1x\in E(\bar{G})\}$, then $r(G+E_1)=r(G)-1$, $r_c(G)\leqslant|E_1|=|X_0|$. Let $S_0\subseteq E(\bar{G})$, $r(G+S_0)<r(G)$, $r_c(G)=|S_0|$. Suppose $D_1=\{y_1,y_2,\cdots,y_t\}$ is a smallest dominating set of $G+S_0$. Ac-

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cording to Lemma, $t = r(G) - 1$. For every $e = ab \in S_0$, $|\{a,b\} \cap D_1| = 1$ by the definition of $r_c(G)$, Let $X = \{x | xy \in S_0, x \in V(G), y \in V(G)\}$, $X_0 = X - D_1$, then $|X_0| = |S_0| = r_c(G)$, $r(G - X_0) = r - 1$. Thus $\min\{|X| \,|\, r(G-X)=r(G)-1, X \subseteq V(G)\} \leqslant r_c(G)$.

Let $K_{n_1,n_2,\cdots,n_t}$ be a complete $t$-partite graph, by Theorem 1, we have following

**Corollary** 1[8] $r_c(K_{n_1,n_2,\cdots,n_t}) = \min\{n_2 - 1 | i = 1,2,\cdots,t\}$.

Let $(d_1, d_2, \cdots, d_n)$ be a nondecrease degree sequence of $G$. If $S_1, S_2 \subseteq V(G)$, $|S_1| = r(G) - 1 = r - 1$ and $S_1 \Rightarrow S_2$, obviously we have $|S_2| \leqslant d_n + d_{n-1} + \cdots + d_{n-r+2} + r - 1$.

**Corollary** 2 Let $(d_1, d_2, \cdots, d_n)$ be a nondecrease degree sequence of $G$, then $r_c(G) \geqslant n - (d_n + d_{n-1} + \cdots + d_{n-r+2} + r - 1)$.

For $F \in W(G)$, let $F = S_1 \cup S_2 \cup \cdots \cup S_r$ and the center of $S_i$ is $x_i$, $i = 1,2,\cdots,r = r(G)$. And let $X = \{x_1, x_2, \cdots, x_r\}$, $C(x_i) = N(X - \{x_i\}) \cap V(S_i)$, $(i = 1,2,\cdots,r(G))$. By the definitions of $r_c(G)$ and $W(G)$ we can obtain the following

**Theorem** 2 $r_c(G) \leqslant \min\limits_{F \in W(G)} \{|S_1|\}$.

**Proof** $r_c(G) \leqslant \min\limits_{F \in W(G)} (\min\{|V(S_i)| - |c(x_i)| \,|\, i = 1,2,\cdots,r(G)\})$

$\leqslant \min\limits_{F \in W(G)} (\min\{|V(S_i)| \,|\, i = 1,2,\cdots,r(G)\}) \leqslant \min\limits_{F \in W(G)} \{|S_1|\}$.

Let $(d_1, d_2, \cdots, d_n)$ be a nondecrease degree sequence of $G$ as above. since $F \in W(G)$, $F = S_1 \cup S_2 \cup \cdots, \cup S_r$, $|S_1| \leqslant d_{n-r+1} + 1$,

$|S_2| \leqslant d_{n-r+2} + 1, \cdots, |S_r| \leqslant d_n + 1$. So $r_c(G) \leqslant d_{n-r+1} + 1$.

On the other hand, if $d(x_0) = \triangle(G)$, and $S = \{vx_0 | vx_0 \in E(\bar{G}), v \in V(G)\}$, then $r(G + S) = 1 < r(G)$. Hence $r_c(G) \leqslant n - \triangle - 1$. Thus $r_c(G) \leqslant n/2$. Therefore we have the following

**Corollary** 3 $r_c(G) \leqslant \min\{d_{n-r+1} + 1, n/2\} \leqslant \min\{\triangle + 1, n/2\}$.

If $G$ there exists a vertex of with degree 1, by Theorem 2 we have following.

**Corollary** 4 $r_c(G) \leqslant \min\{d(v) + 1 | uv \in E(G), d(u) = 1\}$.

If $\mathrm{diam}(G) \leqslant 2$, then for every $F \in W(G)$, $F = S_1 \cup S_2 \cup \cdots, S \cup r$, $N[x_i] \cap N[x_j] \neq \varnothing$, $i \neq j$. By Theorem 2, we have following

**Corollary** 5[8] If $\mathrm{diam}(G) \leqslant 2$, then $r_c(G) \leqslant \triangle(G)$.

**Corollary** 6[8] $r_c(P_n) = r_c(C_n) = \begin{cases} 1, n \equiv 1(\mathrm{mod}3), \\ 2, n \equiv 2(\mathrm{mod}3), \\ 3, n \equiv 0(\mathrm{mod}3) \end{cases}$

where $P_n$ and $C_n$ denote the path and cycle with $n$ vertices respectively.

**Remark** 1 The first inequality of the proof of Theorem 2 can't be changed to equality. We consider $G = (V(G), E(G))$, $V(G) = \{x_1, x_2, x_3, x_4, x_5\} \cup V_1 \cup V_2$, $|V_1| = |V_2| = k \geqslant 2$, $E(G) = E_1 \cup E_2 \cup E_3$, $E_1 = \{x_1x_4, x_2x_5\}$, $E_2 = \{uv | u,v \in V_1 \cup \{x_1, x_3\}\}$, $E_3 = \{uv | u,v \in V_2 \cup \{x_2, x_3\}\}$. The smallest dominating set of $G$ is only $\{x_1, x_2\}$. i. e. $r(G) = 2$. $r_c(G) = 2$, since $S = \{x_3x_4, x_3x_5\}$, $r(G + S) = 1$. $F = S_1 \cup S_2$, $|s_i| = k + 3$, $c(x_i) = x_3$, $i = 1,2$.

**Theorem** 3 $r_c(G) = \Delta + 1$ if and only if $G$ have following properties: (i) Every smallest dominating set $S$ is an independent set; (ii) If $S$ is a smallest dominating set, then for each $x \in S$, $d(x) = \Delta$; (iii) If $S$ is a smallest dominating set, then for each $v \in V(G) - S$, $e(v,S) = 1$; (iv) $r_c(G) * r(G) = n$.

**Proof** For every $F \in W(G), F = S_1 \cup S_2 \cup \cdots \cup S_r, \Delta + 1 = r_c(G) \leqslant |S_1| \leqslant |S_2| \leqslant \cdots \leqslant |S_r| \leqslant \Delta + 1$, then $|S_i| = \Delta + 1$, for all $i$. Furthermore, $c(x_i) = \emptyset$, otherwise $r_c(G) < |S_i|$. This is to say $S = \{x_1, x_2, \cdots, x_r\}$ is an independent set and $N(x_i) \cap N(x_j) = \emptyset, i \neq j$. $c(x_i) = \emptyset$ also implies $e(v,S) = 1$ for every $v \in V(G) - S$, since $V(G) = V(S_1) \cup V(S_2) \cup \cdots \cup V(S_r)$. (iv) is obvious from above.

Following, we assume that $G$ have properties (i)～(iv) and $r_c(G) < \Delta + 1$. Let $(d_1, d_2, \cdots, d_n)$ be a nondecrease sequence of $G$, then $\Delta \geqslant r_c(G) = n/r(G) \geqslant n - (d_n + d_{n-1} + d_{n-r+2} + r - 1) \geqslant n - (r-1)(\Delta + 1)$. i.e. $n \leqslant (r-1)(\Delta+1) + \Delta (*)$ by (iv) and Corollary 2. On the other hand, for every $F \in W(G)$, $F = S_1 \cup S_2 \cup \cdots \cup S_r$, we have $V(G) = V(S_1) \cup V(S_2) \cup \cdots \cup V(S_r)$ and $|V(S_i)| = \Delta + 1, V(S_i) \cap V(S_j) = \emptyset$ by (i)～(iv). Thus $n = r * (\Delta + 1)$. this contradicts ( * ). Therefore $r_c(G) \geqslant \Delta + 1$. By Corollary 3, we have $r_c(G) = \Delta + 1$.

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# 关于控制临界数


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## 摘 要


本文给出控制临界数的一系列上界，且刻划了控制临界数为△+1的图的特征.



**关键词**：星；控星族；控制临界数