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Hamiltonian Multipartite Tournaments*

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Abstract

In this paper we show that any almost regular n-partite ($n \geq 7$) tournament is Hamiltonian.

Key words: Multipartite Tournaments, Hamiltonian Cycle.

哈密尔顿多部竞赛图

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摘 要: 本文证明了如下结果: 设 T 为几乎正则 n- 部竞赛图 ($n \geq 7$), 则 T 必含哈密尔顿圈.

关键词: 多部竞赛图, 哈密尔顿圈.

1. Introduction

We use the terminology and notations of [1]. Let $D = (V(D), A(D))$ be a digraph. If xy is an arc of D, then we say that x dominates y, denoted by $x \rightarrow y$. More generally, if A and B are two disjoint vertex set of D such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \Rightarrow B$. The outset $N^+(x)$ of a vertex x is the set of vertices dominated by x in D, and the inset $N^-(x)$ is the set of vertices dominating x in D. The outdegree (*resp.* indegree) $d^+(x)$ (*resp.* $d^-(x)$) of a vertex x is defined by $|N^+(x)|$ (*resp.* $|N^-(x)|$). Moreover, we denote by Δ^+ the maximum outdegree in D and δ^+ the minimum indegree in D. For a vertex set A of D, we define $N^+(A) = \bigcup\limits_{x \in A} N^+(x) \backslash A, N^-(A) = \bigcup\limits_{x \in A} N^-(x) \backslash A$. The irregularity $i(D)$ is $\max|d^+(x) - d^-(y)|$ over all vertices x and y of $D(x = y$ is admissible). If $i(D) = 0$, we say D is regular; If $i(D) = 1$, we say D is almost regular. D is k-connected if for any set A of at most $k - 1$ vertices, the subdigraph $D - A$ is strong. If D is k-connected but not $(k+1)$-connected, we say the connectivity of D is k, denoted by $k = \kappa(D)$. A digraph obtained by replacing each edge of a complete n-partite ($n \geq 2$) graph by exactly one arc with the same end vertices is called an n-partite tournament or multipartite tournament. For surveys on multipartite tournaments, see [3] and [4].

Camion [2] proved that a tournament is Hamiltonian if and only if it is strong. Characterization of Hamiltonian multipartite tournaments seems to be interesting but very difficult. The recent one was given by A.Yeo [5].

Theorem A (Yeo [5]) *If T is a k-connected n-partite tournament with the partite sets $V_1, V_2, \ldots, V_n$ such that $k \geq \max\limits_{1 \leq i \leq n} \{|V_i|\}$, then T is Hamiltonian.*

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Yeo [5] proved that any regular multipartite tournament is Hamiltonian. Volkmann [4] conjectured that if T is an almost regular n-partite ($n \geq 4$) tournament, then T contains a k-cycle for all $k, 3 \leq k \leq |V(T)|$. This is not true for $n = 4$, we can easily construct an almost regular 4-partite tournament with six vertices which contains no Hamiltonian cycle. In this paper we show that any almost regular n-partite ($n \geq 7$) tournament is Hamiltonian.

2. Main Result

Lemma 1. *Let T be an almost regular multipartite tournament with the partite sets $V_1, V_2, \ldots, V_n$, then*

(a) $\Delta^+ - \delta^+ \leq 2$.

(b) $|V_i| - |V_j| \leq 2$, for all $i \neq j$.

Proof. (a) Suppose there exist $u, v \in V(T)$ such that $d^+(u) - d^+(v) \geq 3$. By the almost regularity of T, we have $d^-(u) \geq d^+(u) - 1 \geq d^+(v) + 3 - 1 = d^+(v) + 2$, a contradiction.

(b) Note that $d^+(x) + d^-(x) = |V(T)| - |V(x)|$ for each $x \in V(T)$. Let $x, y \in V(T)$ so that $V(x) = V_i$ and $V(y) = V_j$. Then $||V_i| - |V_j|| = ||V(x)| - |V(y)|| = |d^+(x) - d^+(y) + d^-(x) - d^-(y)| = |(d^+(x) - d^-(y)) + (d^-(x) - d^+(y))| \leq |d^+(x) - d^-(y)| + |d^-(x) - d^+(y)| \leq 1 + 1 = 2$.

Lemma 2. Let T be an almost regular n-partite tournament of order p with the partite sets $V_1, V_2, \ldots, V_n$, $r = \max\limits_{1 \leq i \leq n}\{|V_i|\}$, then $\kappa(T) \geq (p - 2r)/3$.

Proof. Let S be any vertex set of T such that $T - S$ is not strong. Let $T_1, T_2, \ldots, T_l$ be the strong components of $T - S$, then there are T_i, T_j such that $N^+(V(T_i)) \subseteq S$ and $N^-(V(T_j)) \subseteq S$. Suppose, without loss of generality, that $|V(T_j)| \leq |V(T_i)|$ and $j = 1$, then $|V(T_1)| \leq (p - |S|)/2$. Since T is almost regular, we have $\delta^-(T) \geq (p - r - 1)/2$. On the other hand, it is easy to verify that $\delta^-(T_1) \leq (|V(T_1)| - 1)/2$. Hence we have

$$(p - r - 1)/2 \leq \delta^-(T_1) + |S| \leq (|V(T_1)| - 1)/2 + |S| \leq (p + 3|S| - 2)/4.$$

Which yields $\kappa(T) = |S| \geq (p - 2r)/3$.

Theorem. Let T be an almost regular n-partite ($n \geq 7$) tournament, then T is Hamiltonian.

Proof. Let $V_1, V_2, \ldots, V_n$ be the partite sets of T and $r = \max\limits_{1 \leq i \leq n}\{|V_i|\}$. By Theorem A, it is sufficient to prove that $\kappa(T) \geq r$.

If $r = 1$, then T is an almost regular tournament and is Hamiltonian. So we assume $r \geq 2$.

Denote $s = max_{i,j}\{||V_i| - |V_j||\}$, by Lemma 1, we have $s \leq 2$. If $s \in \{0, 1\}$, then

$$\kappa(T) \geq (p - 2r)/3 \geq ((n-1)(r-1) + r - 2r)/3 \geq (5r - 6)/3 \geq (3r - 2)/3 > r - 1,$$

Hence in the following we always assume that $s = 2$ and $r \geq 3$.

Case 1. $n \geq 11$.

By Lemma 2, we have

$$\kappa(T) \geq (p - 2r)/3 \geq ((n-1)(r-s) + r - 2r)/3 = ((n-1)(r-s) - r)/3.$$

Since $r \geq 3$, $\kappa(T) \geq ((n-1)(r-2) - r)/3 \geq (9r - 20)/3 \geq (3r - 2)/3 > r - 1$. Hence in this case we have $\kappa(T) \geq r$.

Case 2. $n = 10$.

By Lemma 2, we have $\kappa(T) \geq (p - 2r)/3 \geq r - 1$. We shall show that this equality does not hold. Otherwise we must have that $r = 3$, $\kappa(T) = (p - 2r)/3 = r - 1$ and that *any partite set has only one vertex except one partite set have three vertices. Note that* $\kappa(T) = (p-2r)/3 = r-1$*, by the proof process of Lemma 2, this equality holds only if*

(1) $|S| = 2$ *since* $r = 3$.

(2) $|V(T_1)| = |V(T_2)| = (p-|S|)/2 = 5$,

(3) $\delta^-(T) = (p-r-1)/2 = 4$,

(4) $\delta^-(T_1) = \delta^-(T_2) = 2$,

(5) T_1, T_2 *are regular tournaments of order 5.*

Let $S = \{x,y\}$. *Since* $\delta^-(T_1) + |S| \geq \delta^-(T)$ *and* $N^-(T_1) \subseteq S$, $S \Rightarrow V(T_1)$. *By symmetry, we have* $V(T_2) \Rightarrow S$. *Since* T_1, T_2 *are regular tournaments, we have* $d^+(v) \geq 6$ *and* $d^-(v) = 4$ *for each* $v \in V(T_1)$*, which contradicts the almost regularity of* T.

Therefore we have $\kappa(T) \geq 3 = r$.

Case 3. $n = 9$.

If $r \geq 4$*, then* $\kappa(T) \geq (p-2r)/3 \geq ((n-1)(r-s)+r-2r)/3 \geq (7r-16)/3 \geq r$. *So in the following we assume that* $r = 3$. *If* $p \geq 13$*, then* $\kappa(T) \geq (p-2r)/3 > 2$*, and so* $\kappa(T) \geq 3 = r$. *Since* $n = 9$ *and* $s = 2$, $p \in \{11, 12\}$.

3.1. $p = 11$.

It is obvious, by Lemma 2, that T *is 2-connected. Since* $n = 9$ *and* $s = 2$*, any partite set has only one vertex except one partite set (say* V_1*) have three vertices. If there exist* $x, y \in V(T)$ *such that* $T - \{x,y\}$ *is not strong. Let* $T_1, T_2, \ldots, T_m$ *be strong components of* $T - \{x,y\}$ *such that there are no arcs from* T_j *to* T_i *if* $i < j$. *Hence we have* $N^-(V(T_1)) \subseteq \{x,y\}$ *and* $N^+(V(T_m)) \subseteq \{x,y\}$. *Without loss of generality we assume* $|V(T_m)| \leq 4$. *Then there exists* $u \in V(T_m)$ *such that* $d^+_{T_m}(u) \leq 1$*, and so* $d^+(u) \leq 3$. *By Lemma 1, we have* $\Delta^+ \leq 5$. *If* $u \in V_1$*, then by the almost regularity of* T*, we have* $d^+(u) = d^-(u) = 4$. *This contradicts* $d^+(u) \leq 3$. *If* $u \notin V_1$*, then we have* $d^+(u) = d^-(u) = 5$*, a contradiction too.*

3.2. $p = 12$.

In this case we have $\kappa(T) \geq (p-2r)/3 \geq 2$*, an analogous argument of Case 2 will deduce that this equality does not hold. Hence we have* $\kappa(T) \geq 3 = r$.

Using an analogous argument as Case 2 and Case 3, we can check that $\kappa(T) \geq r$ *if* $n \in \{7, 8\}$.

This completes the proof of the Theorem. □

Remark. *For* $n \in \{5, 6\}$*, it remains open.*

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