应 用 数 学
MATHEMATICA APPLICATA
1999, 12(4): 78～81

Edge-face Coloring of 1-outerplane Graphs*

Wang Weifan (王维凡)
(Department of Mathematics Liaoning University Shenyang 110036)
Zhang Kemin (张克民)
(Department of Mathematics Nanjing University Nanjing 210093)

Abstract In this paper, Melnikov conjecture on the edge-face coloring is proved affirmatively.

Keywords Edge-face chromatic number; 1-outerplane graph.
AMS(1991) Subject Classification O5C15

Throughout this paper, we shall restrict ourselves to finite simple plane graphs. Let G be a plane graph, whose vertex set, edge set, face set, vertex number, edge number, maximum degree and minimum degree of vertices are denoted by $V(G)$, $E(G)$, $F(G)$, $p(G)$, $q(G)$, $\Delta(G)$ and $\delta(G)$, respectively. Let $N_G(u)$ denote the neighboor set of a vertex u in G. A plane graph G is k-edge face colorable if the elements of $E(G) \cup F(G)$ can be colored with k colors such that any two distinct adjacent or incident elements receive different colors. The edge-face chromatic number $\chi_{ef}(G)$ of G is defined as the minimum number k for which G is k-edge face colorable. Clearly $\chi_{ef}(G) \geq \Delta(G)$. On the other hand, Melnikov put forward in [3] that

Conjecture 1 For each plane graph G, $\chi_{ef}(G) \leq \Delta(G) + 3$.

Recently, [4] gives a affirmative answer for Conjecture 1 by means of Four-Color Theorem. However, because of the length of machine proof of Four-Color Problem, one expect a new proof for Conjecture 1 without the aid of Four-Color Theorem. Moreover, note that the edgeface chromatic number of an odd cycle is five, the upper bound $\Delta + 3$ of Conjecture 1 is sharp. But we so far have not found other examples to illustrate this fact. Thus we raise that

Conjecture 2 For each plane graph G with $\Delta(G) \geq 3$, $\chi_{ef}(G) \leq \Delta(G) + 2$.

Therefore another research subject in this area is to find precise upper bounds of χ_{ef} for $\Delta \geq 3$. In this paper, we consider the situation of 1-outerplane graphs. A plane graph G is called a 1-outerplane graph if there is a vertex $u \in V(G)$ such that $G-u$ is an outerplane graph, where u is called a base of G. A vertex with degree k in G is called a k-vertex and let $V_k(G)$ denote the set of all k-vertices in G, $k = 0, 1, \cdots, \Delta(G)$. For an edge $e = xy$ in G, we define $w_G(e) = d_G(x) + d_G(y) - 2$ as a weight of e in G and call e a k-edge of G if $w_G(e) = k$. Let $b(f)$ denote the boundary of a face f of G. A face f is said to be a k-face of G if $|V(b(f))| = k$. Sometimes, f is denoted by a sequence of all vertices in $b(f)$. A k-edge face coloring of a plane graph G is simply writen as a k-EF coloring. Let $\sigma(y)$ denote the color assigned to a el-

* Received: November 24, 1998
The project is supported by NSFC and SFECLN.

ement $y \in E(G) \cup F(G)$ under a given coloring σ, and $C_\sigma(u)$ denote the set of colors which are colored on the edges incident to a vertex u under σ. Moreover, $y[m]$ is denoted that at most m colors can not be used when coloring the element y. Let H be a block of G. If H contains at most one cut vertex of G, say x, then H is said to be a suspending block of G at x and x a suspending cut vertex of G. Obviously, every plane graph with cut vertices has at least two suspending blocks. When G contains no cut vertex, each component of G is considered to be a suspending block at any vertex.

Lemma 1([5]) If G is an outerplane graph, then $\delta(G) \leq 2$.

Lemma 2([7]) Let G be an outerplane graph with $\delta(G) = 2$ and H a suspending block of G. Then (i) There are two vertices $u, v \in V_2(H) \cap V_2(G)$ such that $uv \in E(G)$; or (ii) G contains a 3-face xyz such that $x \in V_2(H) \cap V_2(G)$ and $y \in V_k(H) \cap V_k(G)$ with $2 \leq k \leq 4$.

Lemma 3 If G is a 1-outerplane graph, then $\delta(G) \leq 3$.

Lemma 4 Let G be a 2-edge connected 1-outerplane graph with $\Delta(G) \geq 5$, then at least one of the following is true: (i) A 2-vertex u is adjacent to a k-vertex v, $k \leq \Delta(G) - 2$, where u is not on any triangle of G. (ii) A 2-vertex u is on a 3-face f. (iii) An edge e is on a 3-face f with $w_G(e) \leq 5$. (iv) A 6-edge e is on the common boundary of a 3-face f_1 and a k-face f_2 with $3 \leq k \leq 4$; and moreover if $k = 4$, $b(f_2)$ contains a 2-vertex v. (v) Two 2-vertices u and v are on a 4-face $uxvy$.

Proof Let t be a base of G. Then $H = G - t$ is an outerplane graph. Since G has no cut edge, $\delta(G) \geq 2$. Further, by Lemmas 1 and 3, we obtain $1 \leq \delta(H) \leq 2$.

Case 1 $\delta(H) = 2$. By Lemma 2, we have (a) H contains two adjacent 2-vertices u and v; or (b) H contains a 3-face xyz with $d_H(x) = 2$ and $2 < d_H(y) \leq 4$. Suppose that (a) holds for H. Let $u_1 \in N_G(u) \setminus \{v\}$ and $v_1 \in N_G(v) \setminus \{u\}$. If $u_1 = v_1$, i.e. uvu_1 is a 3-face of H, then (iii) holds for G since $w_G(uv) = d_G(u) + d_G(v) - 2 \leq (d_H(u) + 1) + (d_H(v) + 1) - 2 = 4$. If $u_1 \neq v_1$, then when $ut, vt \in E(G)$, (iii) holds for G and otherwise (i) follows. If (b) is true for H, then, $w_G(xy) \leq w_H(xy) + 2 = d_H(x) + d_H(y) - 2 + 2 \leq 2 + 4 = 6$, and $w_G(xy) = 6$ if and only if $d_H(y) = 4$ and $xt, yt \in E(T)$. This implies that xy is either on a 3-face xyz with $w_G(xy) \leq 5$, or on two 3-faces xyz and txy with $w_G(e) = 6$. Hence either (iii) or (iv) holds for G.

Case 2 $\delta(H) = 1$. We first, by $\delta(G) \geq 2$, claim that $V_1(H) \subseteq N_G(t) \cap V_2(G)$. Next note that each component of H contains at least two vertices since G is 2-edge connected. Let B be a suspending block of H with as many vertices as possible. If $|V(B)| \geq 3$, then B is a 2-connected outerplane graph. By Lemma 2, we may reduce the problem to Case 1. Now assume that $|V(B)| = 2$, i.e. $B = K_2$. This implies that B is a pendent edge of H. If some suspending cut vertex of H is adjacent to at least two 1-vertices of H, then (ii) or (v) holds for G. Otherwise, each vertex of H is adjacent to at most one 1-vertex in H. Set $H_1 = H - V_1(H)$. Obviously, $1 \leq \delta(H_1) \leq 2$. If $\delta(H_1) = 1$, then H contains a 1-vertex adjacet to a 2-vertex, and thus either (i) or (ii) follows easily. So suppose $\delta(H_1) = 2$. By Lemma 2, we have

(a1) H_1 contains two adjacent 2-vertices u_1 and v_1; or

(b1) H_1 contains a 3-face $x_1y_1z_1$ with $d_{H_1}(x_1) = 2$ and $2 \leq d_{H_1}(y_1) \leq 4$.

Suppose that (a1) holds. If neither u_1 nor v_1 are adjacent to 1-vertices of H, the problem can be reduced to Case 1. If either u_1 or v_1 is adjacent to some 1-vertex of H, it is easily checked that (i) or (ii) holds for G because $\Delta(G) \geq 5$ and $\max\{d_G(u_G(u_1)), d_G(v_1)\} \leq 4$. Second suppose that (b_1) holds. If x_1 is adjacent to some 1-vertex of H, a similar discussion can yield (i) or (ii). Otherwise, suppose that x_1 is not adjacent to any 1-vertex of H. When y_1 also is not adjacent to any 1-vertex of H, the proof is similar to Case 1. Hence let y_1 be adjacent to some 1-vertex in H, say w. Then obviously $tw \in E(G)$. If $tx_1 \notin E(G)$ or $ty_1 \in E(G)$, we have (ii). If $tx_1 \in E(G)$ but $ty_1 \notin E(G)$, we have either (iii) when $d_G(y_1) \leq 4$, or (iv) when $d_G(y_1) = 5$.

Theorem 1 If G is 1-outerplane graph with $\Delta(G) \geq 4$, then $\chi_{ef}(G) \leq \max\{\Delta(G) + 1, 7\}$.

Proof First note that the cases $\Delta = 4, 5$ are the relaxations of the case $\Delta = 6$. Thus it suffices to prove the theorem for $\Delta \geq 6$. We use induction on $q(G)$. When $q(G) \leq 6$, the theorem holds trivially. Suppose that the theorem holds for $m - 1$, let G be a 1-outerplane graph with $\Delta(G) \geq 6$ and $|E(G)| = m \geq 7$. If G contains a cut edge e, we set $G - e = G_1 \cup G_2$. By the induction assumption, G_1 and G_2 are $(\Delta + 1)$-EF colorable. Based on the colorings of G_1 and G_2, we form easily a $(\Delta + 1)$-EF coloring of G. Thus we may assume that G is 2-edge connected. According to Lemma 4, we consider five cases:

Case 1 There are a 2-vertex u and a k-vertex v with $k \leq \Delta(G) - 2$ such that $uv \in E(G)$ and u is not on any triangle of G. Let $w \in N_G(u) \setminus \{v\}$, and set $H = G - u + vw$. Thus $\Delta(H) = \Delta(G) \geq 6$. By the induction assumption, we can color H with $\Delta + 1 (\geq 7)$ colors and then color the remaining edges of G: $uv[\Delta]$.

Case 2 There is a 2-vertex u on a 3-face $f = uyz$. Let f_0 and f_1 be two neighbour faces of f in G with $u \in b(f_0)$ and $yz \in b(f_1)$. If $f_0 = f_1$, without loss of generality, we assume that $\{yz, yu\}$ is a 2-edge cut of G and so y is a cut vertex of G. Let $G = G_1 \cup G_2$ such that $G_1 \cap G_2 = \{y\}$ and $d_{G_1}(y) = 2$. By the induction assumption, G_1 and G_2 are $(\Delta + 1)$-EF colorable. Selecting suitable colorings of G_1 and G_2, we can get a $(\Delta + 1)$-EF coloring of G. Now suppose $f_0 \neq f_1$. Set $H = G - u$. Let f_0^* denote the face of H which is divided into the union of f and f_0 in G. By the induction assumption, H has a $(\Delta + 1)$-EF coloring λ with a color set C. Based on λ, we form a $(\Delta + 1)$-EF coloring σ of G as follows: If $d_G(y) \leq \Delta(G) - 1$ or $\lambda(f_0^*) \in C_\lambda(y)$, we put: $\sigma(f_0) = \lambda(f_0^*)$, $uz[\Delta]$, $uy[\Delta]$, $f[5]$. If $d_G(z) \leq \Delta(G) - 1$ or $\lambda(f_0^*) \in C_\lambda(z)$, we put: $\sigma(f_0) = \lambda(f_0^*)$, $uy[\Delta]$, $uz[\Delta]$, $f[5]$. If $d_G(y) = d_G(z) = \Delta(G)$ and $\lambda(f_0^*) \notin C_\lambda(y) \cup C_\lambda(z)$, we put: $\sigma(uy) = \lambda(yz)$, $\sigma(yz) = \sigma(f_0) = \lambda(f_0^*)$, $uz[\Delta]$, $f[4]$.

Case 3 There is an edge e on a 3-faces f with $w_G(e) \leq 5$. Color $G - e$ with $\Delta + 1$ colors and then put: $e[6]$, $f[6]$.

Case 4 First let a 6-edge e be on two 3-face f_1 and f_2. Color $G - e$ with $\Delta + 1$ colors and then put: $e[6]$, $f_1[5]$, $f_2[6]$. Second let a 6-edge e be on a 3-face f_1 and a 4-face f_2, where $b(f_2)$ contains a 2-vertex. It is easily seen that f_2 is adjacent to at most three faces. Thus color $G - e$ with $\Delta + 1$ colors and then put: $e[6]$, $f_2[6]$, $f_1[6]$.

Case 5 There is a 4-face $f = uxvy$ with $d_G(u) = d_G(v) = 2$. Set $H = G - u$ and form a $(\Delta + 1)$-EF coloring λ of H with a color set C. Let f_u and f_v denote two neighbour faces of f in G with $u \in b(f_u)$ and $v \in b(f_v)$. Let f_u^* be the face of H which is divided into the union of f_u and f in G. First suppose $f_u \neq f_v$. Note that the colors $\lambda(f_u^*)$, $\lambda(f_v)$, $\lambda(vx)$ and $\lambda(vy)$ are pairwise distinct under λ. Put $\sigma(f_u) = \lambda(f_u^*)$. Then, by the symmetry, it is enough to consider several cases as follows: If $\lambda(f_u^*) \in C_\lambda(x)$, then we put: $uy[\Delta], ux[\Delta], f[6]$. If $\lambda(f_v) \in C_\lambda(x)$, then we put: $\sigma(ux) = \lambda(vx), \sigma(uy) = \lambda(vy), vy[\Delta], vx[\Delta], f[6]$.

If $(C_\lambda(x) \cup C_\lambda(y)) \cap \{\lambda(f_u^*), \lambda(f_v)\} = \emptyset$, first suppose that $C_\lambda(x) \setminus \{\lambda(vx)\} = C_\lambda(y) \setminus \{\lambda(vy)\}$. Since $|C_\lambda(x) \setminus \{\lambda(vx)\}| \leq \Delta(G) - 2$, there must exist three different colors α, β and γ in $C \setminus (C_\lambda(x) \setminus \{\lambda(vx)\})$. Let $\alpha \notin \{\lambda(f_u^*), \lambda(f_v)\}$. Hence we put: $\sigma(ux) = \sigma(vy) = \alpha, \sigma(uy) = \lambda(f_v), \sigma(vx) = \lambda(f_u^*), f[6]$. Next let $C_\lambda(x) \setminus \{\lambda(vx)\} \neq C_\lambda(y) \setminus \{\lambda(vy)\}$. If $C_\lambda(x) \setminus \{\lambda(vx)\} \subset C_\lambda(y) \setminus \{\lambda(vy)\}$, it follows that $|C_\lambda(x) \setminus \{\lambda(vx)\}| \leq \Delta(G) - 3$ and so $d_G(x) = d_H(x) + 1 = |C_\lambda(x)| + 1 = |C_\lambda(x) \setminus \{\lambda(vx)\}| + 2 \leq \Delta(G) - 1$. In this case, we put: $uy[\Delta], ux[\Delta], f[6]$. Otherwise, we can take a $\alpha \in (C_\lambda(x) \setminus \{\lambda(vx)\}) \setminus (C_\lambda(y) \setminus \{\lambda(vy)\})$ and $\beta \in (C_\lambda(y) \setminus \{\lambda(vy)\}) \setminus (C_\lambda(x) \setminus \{\lambda(vx)\})$ such that $\alpha \neq \beta$. Then we put: $\sigma(uy) = \alpha, \sigma(vx) = \beta, \sigma(ux) = \lambda(f_v), \sigma(vy) = \lambda(f_u^*), f[6]$. If $f_u = f_v$, the proof is similar and simpler.

Corollary 1 Melnikov's conjecture is true for all 1-outerplane graphs.

Corollary 2 If G is a 1-outerplane graph with $\Delta(G) \geq 6$, then $\Delta(G) \leq \chi_{ef}(G) \leq \Delta(G) + 1$.

References

1 Borodin O V. Simultaneous coloring of edges and faces of planar graphs. Discrete Math., 1994, 128: 21～33

2 Lin Cuiqin, Hu Guanzhang and Zhang Zhongfu. A six-color theorem for the edge-face coloring of plane graphs. Discrete Math., 1995, 141: 291～297

3 Melnikov L S. Recent advance in graph theory. Proc. Internat. Symp. Prague, Problem Section: 1975, 543

4 Waller A O. Simultaneously coloring the edges and faces of plane graphs. J. Combin. Theory, 1997, B69: 219～221

5 Wang Weifan. On the colorings of outerplanar graphs. Discrete Math., 1995, 147: 257～269

6 Wang Weifan and Zhang Kemin. A proof of Melnikov conjecture for the case Δ= 4. Chinese Science Bulletin, 1998, 3(43): 330～331

7 Wang Weifan and Zhang Kemin. The total chromatic number of pseudo-outerplane graphs. Applied Math. - A J. of Chinese Universities, 1997, B(4): 455～462

1-外平面图的边面全色数

王维凡
(**辽宁大学数学系 沈阳** 110036)

张克民
(**南京大学数学系 南京** 210093)

摘　要

一个平面图 G **被称为** 1-**外平面图如果存在一个顶点** u **使得** $G - u$ **是一个外平面图. 本文证明了**Melnikov **的边面染色猜想对所有**1-**外平面图成立.**

关键词　1-**外平图; 边面全色数**