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Colorings of Hypergraphs

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Abstract This is a survey for the colorings of hypergraphs in recent thirty years. It includes basic results, critical colorability, bicolorability, non-bicolorability and some extremal problems concerning vertex coloring, edge coloring and other colorings of hypergraphs.

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1 Introduction

Hypergraphs are an important generalization of ordinary graphs. The colorings for hypergraphs also are the natural extension of colorings for graphs. This field has various applications (timetabling and scheduling problems, planning of experiments, multi-user source coding etc) and offers rich connections with other combinatorial areas (probabilistic methods, extremal set theory, Ramsey theory, discrepancy theory etc). Thus many research results on the field have been obtained in last thirty years. In this paper we attempt to give a chief survey for the related area. We first declare that the concepts and notations used here consistent with that in [18] or [31]. However it is necessary to recall some of which are applied frequently in the following sections.

Let $V = \{v_1, v_2, \cdots, v_n\}$ be a finite set and let $E = \{E_1, E_2, \cdots, E_m\}$ is a family of subsets of V such that $E_i \neq \emptyset$ $(i = 1, 2, \cdots, m)$ and $\cup_{i=1}^m E_i = V$. We call $H = (V, E)$, or simply $H = (E_1, E_2, \cdots, E_m)$, a hypergraph, where V and E are called the vertex set and edge set of H respectively. A simple hypergraph is a hypergraph $H = (E_1, E_2, \cdots, E_m)$ such that $E_i \subseteq E_j$ implies $i = j$. Let $n(H)$, $m(H)$, $\Delta(H)$, $\alpha(H)$ and $\tau(H)$ denote the vertex number, the edge number, the maximum degree of vertices, the stability number and the transversal number of H respectively. The rank $r(H)$ of hypergraph H is defined by $r(H) = \max_{E' \in E(H)} |E'|$. An r-uniform hypergraph H is a hypergraph such that any edge in H contains r vertices. A hypergraph is linear if $|E_i \cap E_j| \leq 1$ for all $i \neq j$. The dual of a hypergraph $H = (E_1, E_2, \cdots, E_m)$ on $V = \{v_1, v_2, \cdots, v_n\}$ is a hypergraph $H^* = (V_1, V_2, \cdots, V_n)$ whose vertices $e_1, e_2, \cdots, e_m$ correspond to the edges of H and with edge $V_i = \{e_j | v_i \in E_j \ in\ H\}$. For a set $J \subset \{1, 2, \cdots, m\}$ we call the

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family $H' = (E_j | j \in J)$ the partial hypergraph generated by the set J. The set of vertices of H' is a nonempty subset of V. For a set $A \subset V$ we call the family $H_A = (E_j \cap A | 1 \le j \le m, E_j \cap A \ne \emptyset)$ the subhypergraph induced by the set A.

As for graphs, the colorings of hypergraphs can be classified into the vertex coloring, the edge coloring, the vertex and edge total coloring and other colorings satisfying certain restrictions. A weak (strong, resp.) k-coloring of a hypergraph H is a partition $(V_1, V_2, \cdots, V_k)$ of $V(H)$ into k classes such that every edge which is not loop is not monochromatic (no color appears twice in the same edge, resp.). The weak chromatic number $\chi(H)$ (strong chromatic number $\chi_s(H)$, resp.) of a hypergraph H is the smallest integer k for which H has a weak (strong, resp.) k-coloring. A hypergraph H is weak (strong, resp.) k-colorable if H has a weak (strong, resp.) k-coloring. H is said to be weak (strong, resp.) k-chromatic if $\chi(H) = k$ ($\chi_s(H) = k$ resp.). For the sake of convenience, we shall omit the word *'weak'* in the terms such as *'weak $k-$coloring'*, *'weak $k-$colorable'*,*' weak $k-$chromatic'* and so on with a exception of particular declaration. An equitable k-coloring of a hypergraph H is a k-partition $(V_1, V_2, \cdots, V_k)$ of $V(H)$ such that in every edge E_i all the colors occur the same number of times (or to within 1). A uniform k-coloring of a hypergraph H is a k-partition $(V_1, V_2, \cdots, V_k)$ of $V(H)$ such that the number of vertices of the same color is always the same (or to within 1).

2 Some Basic Results

In this section we shall describe some fundamental properties and straightforward results concerning the colorings of hypergraphs. Let us start with the case of two kinds of special hypergraphs. A hypergraph is said to be a complete r-uniform hypergraph of order n, denoted by K_n^r, if $E(K_n^r)$ consists of all the subsets of $V(K_n^r)$ of cardinality r, where $1 \le r \le n$. A hypergraph is said to be a complete r-partite hypergraph of order n, denoted by $K^r_{n_1, n_2, \cdots, n_r}$, if $V(K^r_{n_1, n_2, \cdots, n_r}) = V_1 \cup V_2 \cup \cdots \cup V_r$, $V_i \cap V_j = \emptyset$ $(i \ne j)$, $|V_i| = n_i$ $(i = 1, 2, \cdots, r)$, and $E(K^r_{n_1, n_2, \cdots, n_r})$ consists of all distinct r-subsets E' of $\cup_{i=1}^r V_i$ such that $|E' \cap V_i| = 1$ for $i = 1, 2, \cdots, r$. Moreover, for a real number x, let $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the greatest integer no more than x and the smallest integer no less than x respectively. The number of combinations of n things taken r at a time is denoted by C_n^r or $\binom{n}{r}$. By the definitions, the following two results are obvious.

Theorem 2.1 Let n, r be integers with $2 \le r \le n$, then $\chi(K_n^r) = \lceil \frac{n}{r-1} \rceil$ and $\chi_s(K_n^r) = n$.

Theorem 2.2 Let $r \ge 2$ and $n_1, n_2, \cdots, n_r \ge 1$ be integers with $n_1 \le n_2 \le \cdots \le n_r$, then $\chi(K^r_{n_1, n_2, \cdots, n_r}) = 2$, and $\chi_s(K^r_{n_1, n_2, \cdots, n_r}) = r$.

There are close relation between the chromatic number and other parameters of hypergraphs, which are very similar to the corresponding cases of graphs. For instance, two inequalities containing the chromatic number, the stability number and the vertex number can be stated as follows (see [18]).

Theorem 2.3 Let H be a hypergraph of order n, then $\chi(H)\alpha(H) \ge n$.

Theorem 2.4 Let H be a hypergraph of order n, then $\chi(H) + \alpha(H) \le n + 1$.

For $v \in V(H)$, we define the star $H(v)$ with center v to be the partial hypergraph formed by the edges containing v. A β-star of a vertex v is $H^\beta(v)(\subset H(v))$ such that (1) $E_i \in H^\beta(v) \Rightarrow |E_i| \geq 2$, and (2) $E_k, E_j \in H^\beta(v) \Rightarrow E_k \cap E_j = \{v\}$. The β-degree of a vertex v is the largest number of edges of a β-star of v. We denote the β-degree in H of v by $d_H^\beta(v)$ and write $\Delta^\beta(H) = \max_{v\in V(H)} d_H^\beta(v)$ and $\delta^\beta(H) = \min_{v\in V(H)} d_H^\beta(v)$. Then we can obtain upper bounds for the chromatic number with the following assertion (see [18]).

Theorem 2.5 Let H be a hypergraph and H/A denote the family of edges of H contained in A. Then

$$\chi(H) \leq \max_{A\subset V(H)} \delta^\beta(H/A) + 1.$$

From Theorem 2.5 we deduce easily the following result due to Lovász.

Corollary 2.5.1[50] For every hypergraph H with maximum β-degree Δ^β, we have $\chi(H) \leq \Delta^\beta(H) + 1$. Moreover, for each r, this bound is the best possible since $\chi(K_n^r) = \Delta^\beta(K_n^r) + 1$.

The well-known Brook's graph-coloring theorem can be extended to linear hypergraphs: A connected linear hypergraph H forces equality in Corollary 2.5.1 iff H is an odd cycle or a complete graph (Lepp and Gardner 1973). However, no exact analogue is known for general hypergraphs. Two other corollaries of Theorem 2.5 also are given in [18], which establish the relations among the β-degree, the vertex number and the stability number as well as the transversal number.

Corollary 2.5.2 Let H be a hypergraph of order n. Then

$$\alpha(H) \geq \frac{n}{\Delta^\beta(H) + 1}.$$

Corollary 2.5.3 Let H be a hypergraph of order n without loops. Then

$$\tau(H) \leq \frac{n\Delta_\beta(H)}{\Delta^\beta(H) + 1}.$$

Tomescu[67,68] characterized the following properties of k-chromatic hypergraphs.

Theorem 2.6[67] Every hypergraph H with $\chi(H) = k$ contains a path of length at least $k - 1$.

Theorem 2.7[68] Every $(r+1)$-uniform hypergraph H with $\chi(H) = k \geq 2$ contains a cycle of length at least k.

Indeed, the number of all partions of a set X with n elements into k classes is the Stirling number of the second kind with parameters n and k and is denoted by $S(n,k)$. Let $C(n,r,k)$ ($C^*(n,r,k)$, resp.) denote the maximum number of weak k-colorings in the class of $(r+1)$-uniform hypergraphs H of order n (which are connected, resp.) having $\chi(H) = k$. Also let

$$S(n) = \max_k S(n,k), \quad C(n,r) = \max_k C(n,r,k), \quad C^*(n,r) = \max_k C^*(n,r,k).$$

In [68], Tomescu first showed an estimation

$$S(n)^{\frac{1}{n}} \sim \frac{n}{e \ln n}$$

and then by it proved the following result.

Theorem 2.8[68] For every $r \geq 1$, we have $C(n,r)^{\frac{1}{n}} \sim C^*(n,r)^{\frac{1}{n}} \sim \frac{n}{en \ln n}$ as $n \to \infty$.

To obtain lower bound estimate of chromatic numbers of hypergraphs, a powerful topological method was initiated in 1978 by Lovász (see [31]). For any an r-uniform hypergraph H, it associate a simplicial complex $C(H)$ as follows: The vertices of $C(H)$ are all the $n!m(H)$ ordered r-tuples $(v_1, v_2, \cdots, v_r)$ of vertices of H, where $\{v_1, v_2, \cdots, v_r\} \in E(H)$. A set of vertices $(v_1^i, v_2^i, \cdots, v_r^i)_{i \in I}$ of $C(H)$ forms a face if there is a complete r-partite subhypergraph of H on the (pairwise disjoint) sets of vertices $V_1, V_2, \cdots, V_r$ such that $v_j^i \in V_j$ for all $i \in I$ and $1 \leq j \leq r$. Alon, Frank and Lovász[7] proved following result.

Theorem 2.9[7] Let r be a prime and H an r-uniform hypergraph. If $C(H)$ is $(k-1)(r-1)-1$-connected, then $\chi(H) > k$.

This theorem extends a result of Lovász (in case $r = 2$, see [31]) and is conjectured to hold for every positive integer r. Another stronger result was recently obtained by a different use of simplicial complexes.

Theorem 2.10[58] Let n, h, j, k, r be positive integers. If $n(j-1) \geq (k-1)(r-1) + rh$, then for every k-coloring of the h-subsets of an n-set, there is at least one r-tuple of h-sets having the same color such that any j of them have empty insection.

Generally speaking, to prove that a chromatic number is large is by no means easy. The question is widely related to Ramsey theory. Lovász[50] and then Nešetril and Rödl[55], gave a constructive proof of the following important result of Erdö s and Hajnal[34].

Theorem 2.11 For all integers r, k, s with $r \geq 2$, there exists an r-uniform hypergraph H with $\chi(H) \geq k$ in which no cycle is shorter than s.

Now let us observe the chromatic numbers of product and union of hypergraphs. Given two hypergraphs H_1 and H_2, their direct product is a hypergraph $H_1 \times H_2$ with vertex set $V(H_1 \times H_2) = V(H_1) \times V(H_2)$ and with edge set $E(H_1 \times H_2) = \{E_1 \times E_2 | E_1 \in E(H_1), E_2 \in E(H_2)\}$. Given $p, q > 0$, let $f(p,q)$ denote the smallest chromatic number $\chi(H_1 \times H_2)$ of any direct product where $\chi(H_1) = p$ and $\chi(H_2) = q$. Is it true that $f(p,q) \to \infty$ as $p, q \to \infty$? This problem was posed by Berge and Simonovits [15]. They remarked that the problem of finding a good estimate for $f(p,q)$ seems to be difficult. However they proved the maximum chromatic number of a direct product $H_1 \times H_2$, with $\chi(H_1) = p$ and $\chi(H_2) = q$, is attained by the product $K_p^2 \times K_q^2$. Sterboul[62] proved among other similar results

$$\lim_{m \to \infty} \frac{\chi(K_m^2 \times K_m^2)}{\sqrt{m}} = 1.$$

The chromatic number of direct products $K_m^r \times K_n^s$ were also studied by Erdös and Rado[32] and by Chvátal[24] (also see [49]). In particular Zhu[75] proved the following two interesting results which determine the chromatic number of the product $H_1 \times H_2$ of two hypergraphs H_1 and H_2 with large complete subhypergraphs.

Theorem 2.12[75] Let H_1 and H_2 be two $(k+1)$-chromatic hypergraphs such that each of H_1 and H_2 contains a complete subhypergraphs of order k and each of H_1 and H_2 has a vertex-critical $(k+1)$-chromatic subhypergraph which has nonempty intersection with the corresponding complete subhypergraph of order k. Then $H_1 \times H_2$ is of chromatic number $k+1$.

Theorem 2.13[75] Let H_1 be a $(k+1)$-chromatic hypergraph such that each vertex of H_1 is contained in a complete subhypergraph of order k. Then for any $(k+1)$-chromatic hypergraph H_2, $H_1 \times H_2$ is of chromatic number $k+1$.

Problem 2.14 Determine or estimate the value of $f(p,q)$ for given $p,q > 0$ and for all hypergraphs.

Next an excellent sufficient and necessary condition characterizing the chromatic number of union of finite hypergraphs has been obtained by Miller and Müller.

Theorem 2.15[53] Let H be a hypergraph and let $m_1, m_2, \cdots, m_k$ be positive integers. Then H is the union of k hypergraphs H_i $(i = 1, 2, \cdots, k)$ with no edges in common and $\chi(H_i) \leq m_i$ if and only if $\chi(H) \leq m_1 m_2 \cdots m_k$.

In studying the uniform k-coloring of hypergraphs, Berge and Sterboul established the following result.

Theorem 2.16[17] Let H be an r-uniform hypergraph of order $n = kr$ which has no uniform k-coloring and which has the minimum number of vertices for this condition. Then H is a star of K_n^r.

3 Critical Hypergraphs

In hypergraph coloring theory, k-critical hypergraphs have attracted much attention. The purpose of this section is to collect some significant results about edge critical hypergraphs and vertex critical hypergraphs. A hypergraph H is edge k-critical if it is k-chromatic but any proper partial subhypergraph is $(k-1)$-colorable. Similarly, a hypergraph H is verex k-critical if it is k-chromatic but $H - v$ is $(k-1)$-colorable for all vertices v of H. Clearly, every hypergraph which is not 2-colorable has a partial hypergraph which is edge critical, and similarly, every hypergraph which is not 2-colorable contains a set A of vertices such that the hypergraph H/A is vertex critical. Further, an edge critical hypergraph must be vertex critical, but the converse is not true.

As mentioned above with regard to Theorem 2.9, construction of edge χ-critical r-uniform hypergraphs often use r'-uniform hypergraphs with $r' > r$: a consequence of various constructions of this type is the existence of edge χ-critical r-uniform hypergraphs with arbitrarily large minimum β-degree, for $\chi \geq 4$ and $r \geq 2$ (see [66]). Moreover, in contract with the case of graphs, Müller, Rödl and Turzik exhibited, for any integers $r \geq 3$ and h, an edge 3-critical r-uniform hypergraph with all β-degrees at least h and proved the existence of a linear hypergraph with same properties. Notice that an edge k-critical r-uniform hypergraph on n vertices exists for $n \geq (k-1)(r-1)+1$ (Abbott and Hanson for $r = 3$, Toft for $r \geq 4$, see [66]). Seymour[60] gave a characterization of edge 3-critical hypergraphs such that $m(H) = n(H)$ as follows:

Theorem 3.1 Every edge 3-critical hypergraph H satisfies $m(H) \geq n(H)$. Moreover, the bipartite incidence graph admits a vertex-to-edge matching.

Seymour's proof is by linear algebra. Aharoni and Linial[6] extended Theorem 3.1 to the infinite case and gave a purely combinatorial proof. In addition, Lovász found an upper bound for the size of edge 3-critical r-uniform hypergraphs (see [31]):

Theorem 3.2 For any edge 3-critical r-uniform hypergraph H on n vertices, we have $m(H) \leq C_n^{r-1}$.

There are only a finite number of edge 3-critical intersecting r-uniform hypergraphs. Bounds on their maximum size $N(r)$ were given by Erdös and Lovász[36]:

$$\sum_{s=1}^{r} \frac{r!}{s!} = \lfloor (e-1)r! \rfloor \leq N(r) \leq r^r.$$

The lower bound is conjectured to be exact. Furthermore, the other results of edge critical hypergraphs may be found in Toft's work (see [64, 65]). We only present an interesting result in [65], which is very similar to the corresponding case of graphs.

Theorem 3.3[65] Let H_1 and H_2 be two vertex-disjoint hypergraphs and let H denote the hypergraph obtained from H_1 and H_2 by joining each vertex H_1 to each vertex of H_2 by an ordinary edge. Thus $\chi(H) = \chi(H_1) + \chi(H_2)$, and H is edge critical if and only if H_1 and H_2 are edge critical.

We shall now turn our attention to the constructions and properties of vertex critical hypergraphs. Several results describing the existence of vertex critical hypergraphs which are not edge critical have been shown in [22] by Brown and Corneil.

Theorem 3.4[22] For all $k, r \geq 3$, there is a vertex k-critical r-uniform hypergraph of order $\frac{1}{2}k(k-1)(r-2)+k$ that is not edge k-critical.

Theorem 3.5[22] For all $n \geq 3$, there is a vertex 3-critical 3-uniform hypergraph of order $2n+1$ that is not edge 3-critical.

Theorem 3.6[22] For all $n \geq 13$, there is a vertex 4-critical 3-uniform hypergraph of order n that is not edge 4-critical.

Theorem 3.7[22] For all $k \geq 4$ and all $n \geq 7.2^{k-3}-1$, there is a vertex k-critical 3-uniform hypergraph of order n that is not edge k-critical.

We consider an old conjecture, due to Dirac, which asks whether every vertex k-critical graph ($k \geq 2$) has a critical edge (i.e. an edge whose removal decrease the chromatic number). The following result is to show the corresponding question for hypergraphs has a negative answer.

Theorem 3.8[22] For all $n \geq 3$, there is a vertex 3-critical 3-uniform hypergraph of order $2n+1$ without critical edge.

If $H_1, H_2, \cdots, H_{r-1}$ are pairwise disjoint hypergraphs and v is a new vertex, then let $H(v, H_1, H_2, \cdots, H_{r-1})$ denote the hypergraph on $V(H_1) \cup V(H_2) \cup \cdots \cup V(H_{r-1}) \cup \{v\}$ formed by adding in all edges of the form $\{v, v_1, v_2, \cdots, v_{r-1}\}$, where $v_i \in V(H_i)$. In [22], the authors discussed the criticality of $H(v, H_1, H_2, \cdots, H_{r-1})$.

Theorem 3.9[22] Let H_i $(i = 1, 2 \cdots, r-1)$ be vertex k-critical hypergraphs. Then $H(v, H_1, H_2, \cdots, H_{r-1})$ is vertex $(k+1)$-critical; and $H(v, H_1, H_2, \cdots, H_{r-1})$ is edge $(k+1)$-critical if and only if $H_1, H_2, \cdots, H_{r-1}$ are edge k-critical.

What properties does a vertex critical hypergraph have? Berge[20] answered partly this question. His first result is to present a algebriac illustration for a vertex critical hypergraph.

Theorem 3.10[20] Let H be a vertex critical hypergraph with n vertices and m edges. Then $m \geq n$, and at least one of the $n \times n$ subdeterminants of the incident matrix A is not equal to 0.

The following two results expose the structure of cycles in a vertex critical hypergraph.

Theorem 3.11[20] Let H be a vertex critical hypergraph, and let $v_0 \in V(H)$. Then there exists an odd cycle $(v_1, E_1, v_2, \cdots, E_k, v_1)$ such that (1) $v_2 = v_0$; (2) $E_i \cap E_j = \emptyset$ if E_i and E_j are two non-consecutive edges; and (3) $E_1 \cap E_2 = \{v_0\}$.

Theorem 3.12[20] Let H be a vertex critical hypergraph. Then there exists an odd cycle $(v_1, E_1, v_2, \cdots, E_k, v_1)$ such that

(1) $|E_p \cap E_q \cap E_r| = 0$ for $p < q < r \leq k$;

(2) $|E_i \cap E_{i+1}| = 1$, $i = 1, 2, \cdots, k-1$;

(3) $|E_1 \cap E_k| \geq 1$.

4 Bicoloring of Hypergraphs

A 2-coloring of a hypergraph H is said to be a bicoloring of H. Since such coloring plays an important role in coloring theory of hypergraphs, it is necessary to study it in a whole section. What properties of bipartite graphs do bicolorable hypergraphs share? Fournier and Las Vergnas[37,38] gave some precise information on the obstructions to bicolorability.

Theorem 4.1[37,38] Let H be a hypergraph without odd cycle $(v_1, E_1, v_2, \cdots, E_k, v_1)$ satisfying three properties in Theorem 3.12. Then H is bicolorable.

Given a hypergraph H, we call a positional game on H the situation where two players, say A and B, play in turn at coloring a vertex of H, with the color red for A and the blue for B. A vertex already colored can not be recolored. The winner is the one who first colors an edge of H completely with his color. If neither of the players obtains a monochromatic edge then the game is a draw. It is easily seen that in a positional game on a hypergraph H which admits no uniform bicoloring, the first player A has a strategy which assures him a win. Moreover we have the following interesting result (see [18]).

Theorem 4.2 Let H be a hypergraph such that

$$\sum_{E^0 \in E(H)} 2^{-|E^0|} + \max_{v} \sum_{E' \in H(v)} 2^{-|E'|} < 1.$$

Then H is uniform bicolorable. Furthermore in the positional game on H the second player B has a strategy ensuring a draw.

Corollary 4.2.1[35] Let H be a hypergraph without loops such that the number of edges m and the maximum degree Δ satisfy $m+\Delta < 2^s$, where $s = \min_{E' \in E(H)} |E'|$. Then H is uniform bicolorable. Furthermore in the positional game on H the second player B has a strategy for forcing a draw.

Theorem 4.3 Let H be a hypergraph without loops, of order n such that

$$\sum_{E'\in E(H)} \binom{n-|E'|}{\lfloor\frac{n}{2}\rfloor} < \binom{n-1}{\lfloor\frac{n}{2}\rfloor}.$$

Then H is uniform bicolorable.

A proof for Theorem 4.3 is given in [18]. Hansen and Loréa (see [45]) extended this theorem to the general situation: Let H be a hypergraph of order $n \geq k$, and let $p = \lfloor\frac{n}{k}\rfloor$, $q = n - pk$. If

$$k\sum_{E'\in E(H)} \binom{n-|E'|}{n-p} + q\sum_{E'\in E(H)} \frac{|E'|}{p+1-|E'|} < \binom{n}{p},$$

then H admits a uniform k-coloring.

A cycle $(x_1, E_1, x_2, E_2, \cdots, E_k, x_1)$ of a hypergraph H is said to be a B-cycle if (1) k is odd; (2) $H' = (E_1, E_2, , \cdots, E_k)$ has maximum degree $\Delta(H') = 2$; (3) $|E_i \cap E_{i+1}| = 1 (i = 1, 2, \cdots, k-1)$; and (4) $|E_k \cap E_1| \geq 1$. It follows from the definition that a B-cycle of a hypergraph must be an odd cycle but not vice versa. Note that the projective plane P_7 and the complete hypergraph K^r_{2r-1}, which are not 2-colorable, contain B-cycles of length 3. Fournier and Las Vergnas[37] proved a general result below.

Theorem 4.4[37] Every non-bicolorable hypergraph contains a B-cycle.

Corollary 4.4.1[37] In a non-bicolorable hypergraph of rank ≤ 3, there exists a B-cycle such that every pair of two non-consecutive edges are disjoint.

Corollary 4.4.2[37] In a non-bicolorable hypergraph, there exists an odd cycle of maximum degree 2 such that every pair of two non-consecutive edges are disjoint.

Combining the previous results we can obtain the following characterization of the hypergraphs which contain no odd cycles. A short proof is given in [18].

Theorem 4.5 A hypergraph $H = (E_1, E_2, \cdots, E_m)$ has no odd cycles if and only if every hypergraph $H' = (E'_1, E'_2, \cdots, E'_m)$ with $E'_i \subset E_i$ for each i is bicolorable.

The class of hypergraphs without odd cycles has been studied from the view point of matrices by Commoner[27]; Yannakakis[73] has given a polynomial algorithm to test whether a given hypergraph in this class. In particular we obtain

Theorem 4.6[18] A hypergraph $H = (E_1, E_2, \cdots, E_m)$ is cycle-free if and only if for every nonempty subset J of $\{1, 2, \cdots, m\}$, we have

$$|\cup_{j\in J} E_j| > \sum_{j\in J}(|E_j| - 1).$$

A matrix $A = (a_{ij})$ is said to be totally unimodular if every square submatrix of A has determinant equal to 0, +1 or −1. A hypergraph is said to be unimodular if its incidence matrix is totally unimodular. A combinatorial property of unimodular hypergraph is related to the concept of an equitable coloring.

Theorem 4.7[27] A hypergraph H is unimodular if and only if for every $S \subset V(H)$ the subhypergraph H_S has an equitable bicoloring: That is to say a bipartition (S_1, S_2) of S such that each edge E' of H_S satisfies

$$\left\lfloor \frac{|E'|}{2} \right\rfloor \leq |E' \cap S_i| \leq \left\lceil \frac{|E'|}{2} \right\rceil \quad (i = 1, 2).$$

A direct consequence of Theorem 4.7 is that every hypergraph with odd cycles is unimodular. Furthermore de Werra[29] generalized the necessity of the theorem.

Theorem 4.8[29] A unimodular hypergraph H has an equitable k-coloring for every $k \geq 2$.

A hypergraph is said to be balanced if its each odd cycle has an edge containing three vertices of the cycle. It is easily checked that every unimodular hypergraph is balanced, but the converse is not true. However we know that a hypergraph of rank ≤ 3 is unimodular if and only if it is balanced. This is a direct corollary of the following result (see [18]).

Theorem 4.9 A hypergraph is balanced if and only if its induced subhypergraphs are bicolorable.

A hypergraph H is said to be normal if every partial hypergraph H' has the edge-coloring property, that is to say $\gamma(H') = \Delta(H')$ for every $H' \subseteq H$, where $\gamma(H')$ denote the chromatic index of H. (see Section 6).

Theorem 4.10[37] Every normal hypergraph is bicolorable.

In 1994, Berge further studied some properties of bicolorable hypergraphs. In order to introduce his work, we need a concept. Let x and y be two vertices of the hypergraph $H = (E_1, E_2, \cdots, E_m)$. We say that x is dependent on y if every edge containing x contains also y. A vertex of degree 0 or 1 is always a dependent vertex.

Theorem 4.11[20] Let H be a hypergraph and let A be the set of dependent vertices. If $H/(V(H) - A)$ is bicolorable, then every bicoloring of $H/(V(H) - A)$ can be extended to a bicoloring of H.

Corollary 4.11.1[20] A vertex critical hypergraph H contains no dependent vertices.

Corollary 4.11.2[20] The hypergraph H of the maximal cliques in a triangulated graph G is bicolorable.

Let H be a hypergraph such that one of the (induced) subhypergraphs obtained from H by removing successively a remaining dependent vertex is bicolorable. Obviously, H is bicolorable, and we call H a deeply bicolorable hypergraph. In [20], Berge also gave the following two results.

Theorem 4.12 A hypergraph H and all its partial hypergraphs are deeply bicolorable if and only if every odd cycle of H has three edges with a non-empty intersection.

Theorem 4.13 Let H be a hypergraph having no two intersecting edges of size ≥ 4 and no B-cycle, then H is bicolorable.

Alon and Bregman[8] investigated the bicolorability of r-uniform r-regular hypergraphs. They proved the following result.

Theorem 4.14[8] For every $r \geq 8$, every r-uniform r-regular hypergraph is bicolorable.

In fact, for $r \geq 9$, the proof of the above theorem can be completed by applying the Lovász Local Lemma to show that a random vertex coloring of the given hypergraph with 2 colors contains no monochromatic edges with positive probability[61]. However, for $r = 8$, Alon and

Bregman[8] presented a completely different proof from the probabilistic one. Obviously, if $r=2$, a 2-uniform 2-regular connected hypergraph is a graph that is a cycle. When this cycle is odd, it is not bicolorable. If $r=3$, we consider H to be a projective plane P_7. It is easy to see that P_7 is a 3-uniform 3-regular hypergraph, but it is not bicolorable. For $4\le r\le 7$, it is still a open problem. However, it seems plausible that in fact every 4-uniform 4-regular hypergraph is bicolorable. Thus we pose the following problem:

Problem 4.15 For $4\le r\le 7$, prove or disprove that every r-uniform r-regular hypergraph is bicolorable.

There are many applications of the bicoloring of hypergraphs. An interesting result related to the famous Four-Color Theorem is contributed by Berge[20]. For a simple graph G, let $H(G)$ denote the hypergraph on $V(G)$ whose edges are the minimal odd cycles of G. Then $H(G)$ is simple.

Theorem 4.16[20] A graph G is 4-colorable if and only if the hypergraph $H(G)$ is bicolorable.

Corollary 4.16.1[20] A graph G which is not 4-colorable contains a subgraph G_A such that the hypergraph $H(G_A)$ is vertex critical.

In addition, the question of deciding satisfiability of Boolean forms in conjunctive normal form (CNF) can be reduced to the bicolorability problem of a kind of special hypergraphs. The detailed description about it can be seen in [6].

A cycle $(v_1,E_1,v_2,E_2,\cdots,E_k,v_1)$ of a hypergraph H is said to be a hypercycle if the sets $E_{i+1}\setminus E_i$ (with $k+1\equiv 1$) form a partition of $E_1\cup E_2\cup\cdots\cup E_k$. We conclude this section with the following conjecture.

Conjecture 4.17 A non-bicolorable hypergraph contains an odd hypercycle such that $|E_i\cap E_{i+1}|=1$ for $i=1,2,\cdots,2k$.

5 Extremal Problems

As for graphs, there are a variety of extremal problems related to the chromatic number of hypergraphs. This field is of great importance in both theory and applications. Therefore we shall give a short survey for several main directions.

5.1 Extremal Problems Related to Number of Edges

We first introduce a few notations. Let

$$M_k(n,r)=\max_{\chi(H)\le k\ n(H)\le n} m(H)$$

denote the largest number of edges in an r-uniform hypergraph of order $\le n$ which is k-colorable. Similarly let

$$m_k(n,r)=\min_{\chi(H)>k\ n(H)\le n} m(H)$$

denote the smallest number of edges in an r-uniform hypergraph of order $\le n$ which is not k-colorable. Moreover, we denote by $M_k^0(n,r)$ the largest value of m for which there is an r-uniform hypergraph H with $n(H) \le n$, $m(H) = m$, and such that by adding a set of $n - n(H)$ isolated points we can find a uniform k-coloring; and by $m_k^0(n,r)$ the smallest number of edges in an r-uniform hypergraph of order $\le n$ which has no uniform k-coloring. By the definitions, we have immediately

$$1 \le m_k(n,r) \le M_k(n,r) \le C_n^r,$$

$$1 \le m_k^0(n,r) \le M_k^0(n,r) \le C_n^r,$$

$$m_k^0(n,r) \le m_k(n,r),$$

$$M_k^0(n,r) \le M_k(n,r).$$

It is easy to calculate $M_k(n,r)$ and $M_k^0(n,r)$, in fact, which are given by the following result.

Theorem 5.1[63] Let $H_{n,k}^r$ be an r-uniform hypergraph of order n on V defined by a uniform k-partion $(Y_1, Y_2, \cdots, Y_k)$ of V and by

$$H_{n,k}^r = (E' | E' \subset V; |E'| = r, E' \not\subset Y_i, i = 1, 2, \cdots, k).$$

Then we have $M_k(n,r) = M_k^0(n,r) = m(H_{n,r}^r)$. Morover, every r-uniform k-colorable hypergraph of order n with $M_k(n,r)$ edges is isomorphic to $H_{n,k}^r$.

In general, it is difficult to calculate $m_k(n,r)$. We however have trivially $m_2(n,2) = 3$ for $n \ge 3$; $m_2(5,3) \le 10$ and $m_2(n,3) = 7$ for $n \ge 7$. In the case of graphs we readily find that $m_k(n,2) = C_{k+1}^2$ for $n \ge k+1$ and the only extremal graph is K_{k+1}. Erdös[33] presented a lower bound of $m_k(n,r)$ for the quite general case.

Theorem 5.2[33] For $r \ge 2, k \ge 2, n \ge kr$, we have $m_k(n,r) \ge k^{r-1}$.

For $k = 2$, the best lower bound for $m_k(n,r)$ has been obtained by Beck[12]: For every $\delta > 0$ and every $n \ge n(\delta)$ we have $m_2(n,r) \ge 2^r r^{\frac{1}{3}-\delta}$. Further it was proved that $m_2(n,r) \le 2^r r^2$; $m_2(n,4) \le 23$ and $m_2(n,5) \le 51$. Applying a similar method to that of Theorem 4.3, we can prove the following theorem.

Theorem 5.3[45] Let H be a hypergraph of order n such that

$$\sum_{E \in H} k^{-|E|} \frac{k^2 - k + 1}{|E|} \cdot \sum_{E \in H} k^{|E|} < 1.$$

Then $\chi(H) \le k$.

Corollary 5.3.1[48] For $r \ge 2, k \ge 2, n \ge kr$, we have

$$m_k(n,r) \ge \frac{rk^r}{r + k(k-1)}.$$

Corollary 5.3.2[46] For $r \ge 2, n \ge 2r$, we have

$$m_2(n,r) \ge \frac{rk^r}{r+2}.$$

For $k=2$ we have upper bounds of $m_2(n,k)$ due to Erdös[33], Chavátal[25] and Beck[11]. In particular, Herzog and Schöheim[46] have made a good estimate of upper bounds of $m_k(n,r)$ for the general case.

Theorem 5.4[46]

$$m_k(n,r) \le \binom{kr-r+1}{r}.$$

Some bounds with the maximum degree are given by Erdös and Lovász[36]. We now propose to find some bounds for $m_k^0(n,r)$. Two nice upper bounds listed below can be referred to [18].

Theorem 5.5 If $p=\frac{n}{k}$ is an integer ($\ge r$), then

$$m_k^0(n,r) \ge \binom{n-1}{r-1} \cdot \binom{p-1}{r-1}^{-1}.$$

Theorem 5.6 Let $r\ge 2$, $k\ge 2$, $n\ge rk$. In an r-uniform k-partition of X with $|X|=n$, let q_1 be the number of classes of size $\lfloor \frac{n}{k} \rfloor$, and let q_2 be the number of size $\lceil \frac{n}{k} \rceil$, we have

$$m_k^0(n,k) \ge \binom{n}{r} \left\lfloor q_1 \binom{\lfloor \frac{n}{k} \rfloor}{r} + q_2 \binom{\lceil \frac{n}{k} \rceil}{r} \right\rfloor^{-1}.$$

Abbott and Hare[3] considered the special case in which H is a critical linear hypergraph. A k-critical r-uniform hypergraph H of order n is called a (n,r,k)-hypergraph. Let

$$m_k'(n,r) = \min_{\chi(H)>k \ n(H)\le n} \{m(H)|H \text{ is a linear hypergraph}\}.$$

$$\alpha'(r,k) = \lim_{n\to\infty} \frac{m_k'(n,r)}{n}, \quad \alpha(r,k) = \lim_{n\to\infty} \frac{m_k(n,r)}{n}.$$

If $r=2$, Dirac[30] proved that

$$m_k'(n,2) \ge \frac{1}{2}n(k-1) + \frac{1}{2}(k-3);$$

and Gallai[41] showed that

$$m_k'(n,2) \ge \frac{1}{2}n(k-1) + \frac{1}{2}n(k-3)(k^2-3)^{-1}.$$

In the other direction Hajós[44] gave a constrution which shows that

$$m_k'(n_1+n_2-1,2) \le m_k'(n_1,2) + m_k'(n_2,2) - 1$$

and from this it may be deduced that $\alpha'(2,k) = \lim_{n\to\infty} \frac{m_k(n,2)}{n}$ exists and satisfies

$$\frac{1}{2}(k-1) + \frac{k-3}{2(k^2-3)} \le \alpha'(2,k) \le \frac{k}{2} - \frac{1}{k-1}.$$

No value of $\alpha(2,k)$ has been determined, but it has been conjectured that the above equality holds on the right for all $k\ge 4$. In fact, for all $r\ge 3$, the result that $\alpha(r,3)=1$ is also shown in [3]. Furthermore we have

Theorem 5.7[3] For all $r, k \geq 3$, we have $\alpha(r, k+1) \leq \alpha(r, k) + 1$.

As an immediate consequence of Theorem 5.7, we can deduce that $\alpha(r, k) \leq k - 2$ for all $k \geq 4$.

Theorem 5.8[3] Let $r \geq 3, k \geq 3$. Let p and l be such that $(p, r-1, k+1)$ and (l, r, k)-hypergraphs exist. Let $q = m_{k+1}(p, r-1)$, and $t = m_k(l, r)$. Then

$$m_{k+1}(ql + p, r, k+1) \leq q(t + l).$$

From Theorem 5.8 we shall deduce the following results:

(a) $\alpha(r, k+1) \leq \alpha(r, k) + 1$;

(b) $\alpha(r, k) < k - 2$ for $r \geq 3$ and $k \geq 4$; and

(c) Let $t = t(r)$ be the least integer for which there exists a $(t, r, 3)$-hypergraph of size t, then for $r \geq 3$ we have

$$\alpha(r, 4) \leq 2 - \frac{2}{1 + \alpha(r-1, 4)t}.$$

Theorem 5.9[3] $\alpha(r, r+1) \geq 1 + \frac{r-2}{r^2}$.

Combining this theorem and the previous results we can obtain $\frac{10}{9} \leq \alpha(3, 4) \leq \frac{35}{19}$. It should be pointed out that we have not known if $\alpha'(r, k) < k - 2$ for $k \geq 4$, or $\alpha(r, k) < \alpha'(r, k)$. We remain this open problem to readers.

5.2 Extremal Problem Related to Number of Vertices

For $r \geq 3$, $k \geq 3$, let $M^*(r, k)$ denote the least integer such that for all $n \geq M^*(r, k)$ there exists a k-critical r-uniform linear hypergraph of order n. In [1] it is shown that for each pair of integers r and k, $r, k \geq 3$, there exists a k-critical r-uniform linear hypergraph with n vertices. It is also verified that linear $(n, 3, 3)$-hypergraph exist only when $n = 7$ or $n \geq 9$, so that $M^*(3, 3) = 9$. This is the only value of $M^*(r, k)$ that has been determined. However, Abbott and Hare obtained the bounds $M^*(4, 3) \leq 51$ and $M^*(3, 4) \leq 94$ (see [4]). A improved result was given in [5].

Theorem 5.10[5] $M^*(3, 4) \leq 56$.

Other results on this subject can be referred to [2, 57]. Moreover there exist many unsolved problems in the field, two of which are listed as follows.

Problem 5.11 What is the exact value of $M^*(3, 4)$?

Problem 5.12 For $r = 3, k \geq 5$ or $r \geq 4, k \geq 3$, what is the lower bound and the upper bound of $M^*(r, k)$?

5.3 Minimal Uniquely Colorable Hypergraphs

A hypergraph H is said to be uniquely strong k-colorable if there exists a strong k-coloring of H such that the partition of vertex set is uniquely determined. Let n, r, k be integers with $n \geq 2$, $k \geq r \geq 2$, and let $f_k^r(n)$ denote the minimum number of edges of an r-uniform hypergraph of order n which is uniquely strong k-colorable. Moreover we write $f_k(n) = f_k^k(n)$ for all integers $n \geq 2$, $k \geq 2$. Burosch et al.[23] gave the following result.

Theorem 5.13 Let n be an integer, then

(1) $f_2(n) = n - 1$ for $n \geq 2$;

(2) $f_3(n) = \lceil \frac{2n}{3} \rceil - 1$ for $n \geq 3$;

(3) $f_4(n) = \lfloor \frac{n}{2} \rfloor$ for $n \geq 4, n \not\equiv 0 \pmod 4$; and $2s - 1 \leq f_4(4s) \leq 2s$ for every integer $s \geq 1$;

(4) $f_k(n) \geq \frac{2n}{k} - 1$ for $n \geq k > 2$;

(5) $f_k(n) \leq \frac{2n}{k} + \frac{k}{2} - 1$ for all k;

(6) $\lim_{n\to\infty} \frac{f_k(n)}{n} = \frac{2}{k}$; and $f_k(k+1) = \lceil log_2 q \rceil + a_q$ for $k \geq 2$; $1 \leq q \leq k$, where $a_q = 2$ if $q = 2^r$ for some $r \in N$ and $a_q = 1$ otherwise.

6 Edge-coloring of Hypergraphs

What we have investigated in previous sections are aimed at the vertex coloring of hypergraphs. In fact, great advances have been made about the edge coloring of hypergraphs in recent thirty years. So the present section shall be devoted to this subject.

Let $k \geq 2$ be an integer. A weak k-coloring of edges of a hypergraph H is a coloring defined by a weak k-coloring of vertices of the dual hypergraph H^*. Thus it is a partition $H = H_1 + H_2 + \cdots + H_k$ (edge-disjoint sum) such that for every vertex v with $d_H(v) > 1$, the star $H(v)$ has at least two edges of different colors. A good k-coloring of the edges of H is a weak k-coloring of the edges of H such that if $d_H(v) \geq k$, the star $H(v)$ contains at least one edge of each of the colors, and if $d_H(v) \leq k$, the edges of $H(v)$ all have different colors. A strong k-coloring of the edges of H is a partition $H = H_1 + H_2 + \cdots + H_k$ such that edges of the star $H(v)$ all have different colors. The chromatic index $\gamma(H)$ of H is the smallest value of k for which a strong k-coloring of edges exists.

By the definition, $\gamma(H) \geq \Delta(H)$ is trivial. Further, if $\gamma(H) = \Delta(H)$, we say that H has the edge-coloring property. In general, it is very difficult to determine the exact value of the chromatic index of a given hypergraph. We do not know what hypergraphs possess the edge-coloring property. However, the following theorem summarizes the minimax types properties of balanced hypergraphs (see [18]).

Theorem 6.1 For a hypergraph H, the following properties are equivalent:

(1) H is balanced;

(2) H^* is balanced;

(3) Every subhypergraph of H is bicolorable;

(4) Every subhypergraph of H has the König property (i.e.the transversal number is equal to the matching number).

(5) Every subhypergraph of H has the edge-coloring property;

(6)The blocker of any subhypergraph of H has the König property.

The edge-coloring property on the complete multi-partite hypergraphs and the complete uniform hypergraphs has been settled completely.

Theorem 6.2[51] For every $k \geq 2$, $K^r_{n_1,n_2,\cdots,n_k}$ admits a good k-coloring of the edges.

Baranyai[10] extended the above theorem to the stronger result that for every $k \geq 2$, the edges of $K^r_{n_1,n_2,\cdots,n_r}$ admits an equitable k-coloring which is uniform.

Corollary 6.2.1[14,51] Let $k \geq 2$ and $n_1, n_2, \cdots, n_r \geq 1$ be integers with $n_1 \leq n_2 \leq \cdots \leq n_r$. Then $\gamma(K^r_{n_1,n_2,\cdots,n_r}) = n_2 n_3 \cdots n_r = \Delta(K^r_{n_1,n_2,\cdots,n_r})$.

The existence of good k-colorings of edges of K^r_n has been already proved by Baranyai.

Theorem 6.3[9] Let n, r be integers, $n \geq r \geq 2$, and let $m_1, m_2, \cdots, m_t \geq 0$ be integers with $m_1 + m_2 + \cdots + m_t = C^r_n$. Then K^r_n is the edge-disjoint sum of t hypergraphs H_j, each satisfying

(1) $m(H_j) = m_j$;

(2) $\lfloor \frac{rm_j}{n} \rfloor \leq d_{H_j}(x) \leq \lceil \frac{rm_j}{n} \rceil$ for each $x \in V(H)$.

From the above theorem we can obtain the following four corollaries (see [18]).

Corollary 6.3.1 K^r_n is the edge-disjoint sum of partial h-regular hypergraphs H_j if and only if $r|hn$ and $\frac{hn}{r}$ divides C^r_n. In this case, H_j's make up a uniform coloring of the edges of K^r_n.

As a special case of Corollary 6.3.1, we clearly have that the complete graph K_n is the sum of h-regular graphs if and only if hn is even and $\frac{hn}{2}$ divide C^2_n.

Corollary 6.3.2 $\gamma(K^r_n) = \lceil C^r_n \lfloor \frac{n}{r} \rfloor^{-1} \rceil$.

Corollary 6.3.3 K^r_n has the edge-coloring property if and only if $r|n$. In this case, there exists an optimal coloring of edges of K^r_n which is uniform.

Corollary 6.3.4 There exists a good k-coloring of the edges of K^r_n if and only if either $k \leq \lfloor C^r_n \lceil \frac{n}{r} \rceil^{-1} \rfloor$ or $k \geq \lceil C^r_n \lfloor \frac{n}{r} \rfloor^{-1} \rceil$.

In 1989, Pippenger and Spencer[56] studied the asymptotic behavior of the chromatic index for hypergraphs. A remarkable result is shown as follows.

Theorem 6.4[56] Let H be an r-uniform hypergraph of order n such that any vertex belongs to $d(1+o(1))$ edges and any two different vertices are in $o(d)$ edges (r is fixed, $d = d(n) \to \infty$). Then

(1) $\gamma(H) = d(1+o(1))$, and

(2) Edges of H can be partitioned into $d(1+o(1))$ edges covers.

If a hypergraph H is endowed with a partition $V_1, V_2, \cdots, V_p$ of its vertex set, we say H is p-partitioned. Given a mapping $M : N^p \to N$ and integers $n_i = |V_i|$, the complete p-partitioned hypergraph $K^M_{n_1,n_2,\cdots,n_p}$ is the multiset where every subset S of V occurs with multiplicity $M(|S \cap V_1|, |S \cap V_2|, \cdots, |S \cap V_p|)$. If M takes value 1 for a single p-tuple $(r_1, r_2, \cdots, r_p)$ and is 0 otherwise, the standard notation is $K^{r_1,r_2,\cdots,r_p}_{n_1,n_2,\cdots,n_p}$, abridged to $K^p_{n_1,n_2,\cdots,n_p}$ if $r_1 = r_2 = \cdots = r_p = 1$ (complete p-partite hypergraph) and to K^r_n if $p = 1$ (complete r-uniform hypergraph of order n). To any (possibly improper) coloring of edges of a complete p-partitioned hypergraph H corresponds a color-chart Λ which matches to every color c the multiset $\Lambda(c)$ formed with the the tinctures of the edges colored c; for $1 \leq k \leq p$, the sum of the kth components of $\Lambda(c)$ is denoted by $\Lambda_k(c)$.

Theorem 6.5[43] Let $H = (V, (E_i)_{i \in I})$ be a complete p-partitioned hypergraph with partition $V = V_1 \cup V_2 \cup \cdots \cup V_p$. Let $V' \subset V$ and $V'_k = V' \cap V_k$. Let λ' denote a coloring of edges, with color chart Λ', of the subhypergraph $H' = (V', (E_i \cap V')_{i \in I})$ and assume that λ' extends to some (possibly improper) edge-coloring of H, with color chart Λ. Then, a necessary and sufficient condition for λ' to be extendable to a coloring of edges of H is that, for every color c, we have $\Lambda_k(c) - \Lambda'_k(c) \leq |V_k| - |V'_k|$ for $1 \leq k \leq p$.

This theorem has many interesting consequences. In fact, Theorem 6.3 is one of its corollaries. Furthermore we have

Corollary 6.5.1 Let $H = K^p_{q,q,\cdots,q}$ be the complete p-partite p-uniform hypergraph on pq vertices. Let $A \subset V(H)$ and $H_A = (E \cap A)_{A \in H}$. Then every edge-coloring of H_A where each color occurs q times can be extended to all of H.

It follows from Theorem 6.5 that complete p-partite hypergraphs and also their hereditary closure have the edge-coloring property. The hereditary closure $\hat{H}$ of a simple hypergraph $H = (E_1, E_2, \cdots, E_m)$ on a set V is a hypergraph on V whose edge set is the set of all nonempty subsets $F \subset V$ such that $F \subset E_i$ for at least one index i.

Conjecture 6.6 [16] If H is a linear hypergraph, then $\gamma(\hat{H}) = \Delta(\hat{H})$, where $\hat{H}$ denotes the hereditary closure of a hypergraph H.

Conjecture 6.6 is true when H is a star (Berge, see [16]) and, as Berge observed, is equivalent to the famous Vizing theorem when H is a graph. In order to give the following conjecture we need a useful term. For two positive integers k and r with $k \leq r$, the k-section of a simple hypergraph H on V of rank r is a hypergraph $H_{[k]}$ whose edges are the sets $F \subset V$ such that either $|F| = k$, and $F \subseteq E'$ for some $E' \in E(H)$; or $|F| < k$ and $F = E'$ for some $E' \in E(H)$. It is easily seen that the 2-section $H_{[2]}$ of a hypergraph H is a simple graph which is obtained by joining two vertices of V if they belong to the same edge of H.

Conjecture 6.7 [40] If H is a simple linear hypergraph, then $\gamma(\hat{H}) \leq \Delta(H_{[2]}) + 1$.

Berge and Hilton[19] proved Conjecture 6.7 if the edges of H with more than 2 elements are assumed pairwise disjoint.

Conjecture 6.8 Let H be a linear hypergraph of order n, then $\gamma(H) \leq n$.

Conjecture 6.8 was proved up to $n = 10$ (Hindman [47]), also in case H is a cyclic Steiner system (Colbourn and Colbourn [26]) and in case H is intersecting (Füredi [40]).

7 Other Colorings of Hypergraphs

Just as for the colorings of graphs, there exist many other kind of colorings for hypergraphs which are, in general, more complex than the vertex coloring and edge coloring. The several directions selected in the following reflect to some extent the development of coloring theory for hypergraphs.

7.1 Total Chromatic Number

The total chromatic number of hypergraphs is a natural generalization of the total chromatic number of graphs. A weak (strong, resp.) total coloring of a hypergraph H is a mapping $\phi : V(H) \cup E(H) \to \{1, 2, \cdots, k\}$ which induces a weak (strong, resp.) vertex coloring and a strong edge coloring of H and if $v \in V(H)$ is in some edge $E_i \in E(H)$ then $\phi(v) \neq \phi(E_i)$. The least m for which such a mapping exists is the weak (strong, resp.) total chromatic number $\chi^w_T(H)$ ($\chi^s_T(H)$, resp.).

According to the definition, the following two results are trivial.

Proposition 7.1 For each hypergraph H, $\max\{\chi(H), \gamma(H), \Delta(H)+1\} \le \chi_T^w(H) \le \chi(H)+\gamma(H)$.

Proposition 7.2 For each hypergraph H, $\max\{\chi_s(H), \gamma(H), \Delta(H)+1\} \le \chi_T^s(H) \le \chi_s(H)+\gamma(H)$.

So far there have existed very few known results about the total colorings of hypergraphs, except for the colmplete r-uniform hypergraphs and complete h-partite hypergraphs.

Theorem 7.3[52] Let n, r be integers with $2 \le r \le n$, then

$$\chi_T^w(K_n^r) = \begin{cases} \binom{n-1}{r-1} + 2 & \text{if } n \equiv 0 (\bmod\ r), \\ \left\lceil \binom{n}{r} \left\lfloor \frac{n}{r} \right\rfloor^{-1} \right\rceil & \text{if } n \not\equiv 0 (\bmod\ r). \end{cases}$$

Theorem 7.4 [28] Let n, r be integers with $2 \le r \le n$, then

$$\chi_T^s(K_n^r) = \begin{cases} \binom{n-1}{r-1} + r & \text{if } n \equiv 0 (\bmod\ r), \\ \left\lceil \binom{n}{r} \left\lfloor \frac{n}{r} \right\rfloor^{-1} \right\rceil & \text{if } n \not\equiv 0 (\bmod\ r). \end{cases}$$

Theorem 7.5[52] Let $r \ge 2$ and $n_1, n_2, \cdots, n_r \ge 1$ be integers with $n_1 \le n_2 \le \cdots \le n_r$. Then $\chi_T^w(K_{n_1,n_2,\cdots,n_r}^r) = n_2 n_3 \cdots n_r + 1$ if $n_1 < n_r$ and $\chi_T^w(K_{n_1,n_2,\cdots,n_r}^r) = n_2 n_3 \cdots n_r + 2$ if $n_1 = n_r$.

Theorem 7.6[28] Let $r \ge 2$ and $n_1, n_2, \cdots, n_r \ge 1$ be integers with $n_1 \le n_2 \le \cdots \le n_r$ and let $k = \max\{i | n_i = n_1\}$. Then $\chi_T^s(K_{n_1,n_2,\cdots,n_r}^r) = n_2 n_3 \cdots n_r + k$.

We would like to see that the following problems can be settled in the future.

Problem 7.7 What are the exact upper bounds of $\chi_T^s(H)$ and $\chi_T^w(H)$ for a hypergraph H?

Problem 7.8 Determine or estimate the strong and weak total chromatic number of 3-uniform hypergraphs.

If G is a graph, then the weak total coloring and the strong total coloring of G both correspond to the classical notion of total coloring. Behzad[13] and Vizing[70] raised independently that

Total Coloring Conjecture: For each graph G, $\Delta(G)+1 \le \chi_T(G) \le \Delta(G)+2$.
This conjecture is still open. The detailed description on the progress of total coloring of graphs can be referred to the monograph[74] by Yap.

7.2 Star Chromatic Number

The concept of star chromatic number of a graph, introduced by Vince[69], is a natural generalization of the chromatic number of a graph. This concept was studied from a pure combinatorical point of view by Bondy and Hell[21]. Recently, Haddad and Zhou[42] have further extended the star chromatic number to hypergraphs.

Let $r \geq 2$ and H be an r-uniform hypergraph, k and d be positive integers such that $k \geq 2d$. A mapping ϕ: $V(H) \to K = \{0, 1, \cdots, k-1\}$ is a strong (k,d)-coloring of H if, for every pair $u, v \in V(H)$, we have $|\phi(u) - \phi(v)|_k \geq d$, where $|x|_k = \min\{|x|, k - |x|\}$, whenever the $\{u, v\}$ appear in some edge of H. The mapping ϕ is called a weak (k,d)-coloring of H if the following conditions are satisfied: (1) no edge of H is monochromatic and (2) if a pair $\{u,v\} \subseteq V(H)$ appear in some edge of H, then $\phi(u) \neq \phi(v)$ implies $|\phi(u) - \phi(v)|_k \geq d$.

Note that a strong (or weak) $(k,1)$-coloring of H is just a strong (or weak) k-coloring of H. The strong star chromatic number of H is defined by

$$\chi_s^*(H) = \inf \left\{ \frac{k}{d} \middle| H \text{ has a strong } (k,d)\text{-coloring} \right\}.$$

Similarly, we can define the weak star chromatic number $\chi^*(H)$ of H. Obviously, when $d = 1$, $\chi_s^*(H) = \chi_s(H)$ and $\chi^*(H) = \chi(H)$.

Haddad and Zhou[42] studied the basic properties of strong star chromatic number. We shall display the main part of their work (from Theorem 7.9 to Theorem 7.14).

Theorem 7.9 If an r-uniform hypergraph H has a strong (k,d)-coloring, then it has a strong (k',d')-coloring for all positive k', d' with $\frac{k'}{d'} \geq \frac{k}{d}$.

Corollary 7.9.1 If an r-uniform hypergraph H has a strong (k,d)-coloring, then it has a strong (k',d')-coloring with $\frac{k'}{d'} = \frac{k}{d}$ and $gcd(k',d') = 1$.

Theorem 7.10 If an r-uniform hypergraph H has a strong (k,d)-coloring with $\gcd(k,d) = 1$ and $k > |V(H)|$, then it has a strong (k',d')-coloring with $k' < k$ and $\frac{k'}{d'} < \frac{k}{d}$.

Combining Corollary 7.9.1 and Theorem 7.10, we can obtain the following simplified expression of the strong star chromatic number, which will play an important role in calculating or estimating the value of $\chi_s^*(H)$.

Theorem 7.11 Let H be an r-uniform hypergraph, then

$$\chi_s^*(H) = \min \left\{ \frac{k}{d} \middle| H \text{ has a strong } (k,d)\text{-coloring and } k \leq |V(H)| \right\}.$$

Theorem 7.12 Let H be an r-uniform hypergraph, then $\chi_s(H) - 1 < \chi_s^*(H) \leq \chi_s(H)$.

It should be pointed out that Theorems 7.9 and 7.10 also hold when replacing $\chi_s(H)$ and $\chi_s^*(H)$ by $\chi(H)$ and $\chi^*(H)$ respectively. Note that if $k \geq rd$ and H is an r-uniform hypergraph, then a strong (k,d)-coloring of H is just a hypergraph homomorphism $H \to H_r(k,d)$, i.e. a map $\phi : V(H) \to V(H_r(k,d))$ so that $\{\phi(u_1), \phi(u_2), \cdots, \phi(u_r)\}$ is an edge of $H_r(k,d)$ whenever $\{u_1, u_2, \cdots, u_r\}$ is an edge of H.

Theorem 7.13 $\chi_s^*(H_r(k,d)) = \frac{k}{d}$ for all r, k, d with $k \geq rd$.

Corollary 7.13.1 For each $r \geq 2$ there exists an infinite family $(H_n)_{n\geq 1}$ of r-uniform hypergraphs with $\chi_s^*(H_n) < \chi_s(H_n)$ for all $n \geq 1$.

Theorem 7.14 Deciding whether a simple r-uniform hypergraph has a strong (k,d)-coloring is NP-complete for all fixed k, r, d such that either $r > 2$ and $k \geq rd$ or $r = 2$ and $k > 2d$.

If we restrict ourselves to the case of graphs, the following two results are very interesting.

Theorem 7.15[76] For any rational number $r \geq 2$ and any integer $g \geq 3$, there is a graph G of girth at least g and $\chi^*(G) = r$.

Theorem 7.16[54,77] If r is a rational number between 2 and 4, then there exists a plane graph with star chromatic number r.

A very nice survey on the star chromatic number of hypergraphs and graphs is given recently by Zhu[78]. In the survey 30 open problems are presented, two of which are as follows:

Problem 7.17 Which graphs G has the property $\chi^*(G) = \chi(G)$?

Problem 7.18 What is the least integer $g(n)$ such that any n-critical graph G with girth at least $g(n)$ has $\chi^*(G) < \chi(G)$?

7.3 Upper Chromatic Number

In order to solve a scheduling problem in reality, Voloshin[71] introduced the notions of the mixed hypergraphs and upper chromatic number. A mixed hypergraph $H = (V, S)$ is a hypergraph with $S = A \cup E$, where A and E are the families of subsets in V and $A \cap E = \emptyset$. We call each subset E_j of E an edge of H and each subset A_i of A a co-edge of H. If $A = \emptyset$, $H = H_E$ is the ordinary hypergraph, and if $E = \emptyset$, $H = H_A$ is called a co-hypergraph. A free k-coloring of a mixed hypergraph $H = (V, A \cup E)$ with k colors is a map $\phi : V \to \{1, 2, \cdots, k\}$ such that every edge in E is not monochromatic and every edge in A has at least two vertices of the same color. A free coloring of a hypergraph H with $i(\geq 0)$ colors is said to be a strict coloring if exactly i colors are used. The maximum i for which there exists a strict coloring of a mixed hypergraph H with i colors is called the upper chromatic number of H and is denoted by $\overline{\chi}(H)$.

Let $p(H, \lambda)$, $\lambda \geq 0$, be the chromatic polynomial of a mixed hypergraph H, which is the number of different free colorings of H with λ colors. Let $\gamma_i(H)$ be the number of strict colorings of H with $(i \geq 1)$ colors.

Theorem 7.19[72] For any mixed hypergraph H,

$$p(H, \lambda) = \sum_{i=\chi(H)}^{\overline{\chi}(H)} \gamma_i(H) \lambda^{(i)},$$

where $\lambda^{(i)} = \lambda(\lambda - 1) \cdots (\lambda - i + 1)$.

Theorem 7.20[72] $\chi(H_E) \leq \chi(H) \leq \overline{\chi}(H) \leq \overline{\chi}(H_A)$.

The cardinality of maximum stable set of an all-vertex partial hypergraph generated by co-edges is called the co-stability number $\alpha_A(H)$. A hypergraph H is called co-perfect if $\overline{\chi}(H') = \alpha_A(H')$ for all its wholly-edge subhypergraphs H'.

Problem 7.21 Is there any relationship between perfect graphs and uniform co-perfect co-hypergraphs?

Problem 7.22 What is the upper chromatic number of co-and maxied hypergraphs without cycles, and of unimodular, balanced, normal co-and mixed hypergraphs?

Problem 7.23 Find the meaning of the chromatic polynomail's coefficients for co-and mixed hypergraphs.

Problem 7.24 Let H ba a mixed hypergraph such that its dual hypergraph H^* represents a multigraph. In this case $\chi(H)$ and $\overline{\chi}(H)$ can be called the lower and upper chromatic indexes of a graph respectively. What are they equal to ?

Note that Vizing Theorem on the edge coloring of graphs resolves only the special case of final problem. The other results and problems of the mixed hypergraphs and their upper chromatic numbers can be referred to [71, 72].

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超图中的着色问题

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摘要 本文是近三十年来有关超图中涉及的着色问题的综述. 它包含了有关超图着色中的基本结果，临界可着色性， 2- 可着色性，非 2- 可着色性以及在超图中与顶点着色、边着色和其它着色相关的极值问题.

关键词 超图；着色