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# A LOCAL INTERSECTION CONDITION FOR *n*-EXTENDABLE GRAPHS*

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**Abstract** Let $G$ be a connected graph with even order. Let $\alpha_v^2$ denote independence number of the subgraph induced by $N^2(v)$ in $G$, where $N^k(v)=\{v|v\in V(G)$ and $d(u,v)=k\}$. For any $uv\notin E(G)$, we define $\lambda_{uv}=|N(u)\cap N(v)|$, $T_u^2(v)=N^2(u)\setminus N(v)$, $t_{uv}^2=\min\{|T_u^2(v)|,|T_v^2(u)|\}$ and $\alpha_{uv}^2=\min\{\alpha_u^2,\alpha_v^2\}$. It is proved that if $\lambda_{uv}\geqslant\min\{\alpha_{uv}^2,t_{uv}^2\}+2n$ for any two vertices $u$ and $v$ with $d(u,v)=2$, then $G$ is $n$-extendable.

**Keywords** Local condition, $n$-extendable graphs

**AMS(1991) subject classifications** 05C70, 05C38

## 1 Introduction

Graphs considered in this paper are finite, undirected and simple.

Let $G$ be a graph, for $S\subseteq V(G)$, set $E(S)=\{uv\in E(G)|u,v\in S\}$. For $M\subseteq E(G)$, set $V(M)=\{v\in V(G)|$ there is $x\in V(G)$ such that $vx\in M\}$. Edges $e,f\in E(G)$ are called independent if $V(e)\cap V(f)=\varnothing$. $M\subseteq E(G)$ is a matching of $G$ if $V(e)\cap V(f)=\varnothing$ for every pair of edges $e,f\in M$. A matching $M$ of $G$ is perfect if $V(M)=V(G)$.

Plummer[4] introduced the concept of $n$-extendable graphs.

Let $G$ be a graph of order $p$ with a perfect matching and let $n$ be a positive integer such

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that $n \leqslant (p-2)/2$. $G$ is said to be $n$-extendable if $G$ has $n$ independent edges and any $n$ independent edges of $G$ are contained in a perfect matching of $G$.

Let $G$ be a connected graph, $v$ be a vertex in $G$. We define $N(v)=\{u \mid u \in V(G), uv \in E(G)\}$, $N^k(v)=\{u \mid u \in V(G)$ and $d(u,v)=k\}$ and $M^k(v)=\{u \mid u \in V(G)$ and $d(u,v) \leqslant k\}$. Let $\alpha_v$ denote the number of vertices in the largest independent set containing the vertex $v$ in $G$ and $\alpha_v^2$ denote independence number of the subgraph induced by $N^2(v)$ in $G$. For any $uv \notin E(G)$, we defined $T_{uv}=V \setminus N(u) \cup N(v) \cup \{u,v\})$, $\lambda_{uv}=|N(u) \cap N(v)|$, $\alpha_{uv}=\min\{\alpha_u, \alpha_v\}$, $T_u^2(v)=N^2(u) \setminus N(v)$, $t_{uv}=|T_{uv}|$, $t_{uv}^2=\min\{|T_u^2(v)|, |T_v^2(u)|\}$ and $\alpha_{uv}^2=\min\{\alpha_u^2, \alpha_v^2\}$. It is clear that $\alpha_u^2 \leqslant \alpha_v - 1 \leqslant \alpha - 1$, $\alpha_{uv}^2 \leqslant \alpha_{uv} - 1 \leqslant \alpha_v - 1 \leqslant \alpha - 1$.

We use $\omega(G)$ to denote the number of components of $G$ and use $o(G)$ to denote the number of odd components of $G$.

For any terminology and notation used, but not defined here, the reader is refered to [1].

In [3] Lam and Song introduced a local condition for Hamiltonian graphs.

**Theorem 1.1**[3] Let $G$ be a 2-connected simple graph of order $p(\geqslant 3)$. If $\lambda_{uv} \geqslant \min\{\alpha_{uv}^2+1, t_{uv}^2+1\}$ for any two vertices $u$ and $v$ with $d(u,v)=2$, then $G$ is Hamiltonian.

However, $n$-extendable graphs and Hamiltonian graphs have many similar properties. In this paper, we give a sufficient condition for $n$-extendable graphs which is a Lam and Song's type condition. This condition implies a result of Plummer that for a connected graph $G$ with even order $p$, if $\delta(G) \geqslant p/2+n$, then $G$ is $n$-extendable[4].

## 2 Main Results

**Theroem 2.1** Let $G$ be a connected graph and let $k \geqslant 0$. If $\lambda_{uv} \geqslant \min\{\alpha_{uv}^2, t_{uv}^2\}+k+1$ for any two vertices $u$ and $v$ of $G$ with $d(u,v)=2$, then $\omega(G-S) \leqslant |S|-k$ for all cutset $S \subseteq V(G)$.

**Lemma 2.2** Let $G$ be a connected graph and let $k \geqslant 1$. If $\lambda \geqslant \min\{\alpha^2, t^2\}+k$ for any two vertices $u$ and $v$ of $G$ with $d(u,v)=2$, then $G$ is $(k+1)$-connected.

**Proof** Suppose that $G$ is not $(k+1)$-connected. Let $S \subseteq V(G)$ be a minimum vertex cutset such that $|S| \leqslant k$. Assume $C_1$ and $C_2$ are two components of $G-S$. Let $w \in S$. By the minimality of $|S|$, $w$ is adjacent to a vertex $u$ in $C_1$ and a vertex $v$ in $C_2$. Then $d(u,v)=2$. So $\lambda_{uv} \geqslant \min\{\alpha_{uv}^2, t_{uv}^2\}+k$. Since $\{u,v\} \subseteq N^2(u) \cup \{u\}$ is independent and $\{u,v\} \subseteq N^2(v) \cup \{v\}$ is also independent, $\alpha_{uv}^2 \geqslant 1$. Since $v \in T_u^2(v)$ and $u \in T_v^2(u)$, $t_{uv}^2 \geqslant 1$. But $N(u) \cap N(v) \subseteq S$. Hence $|S| \geqslant \min\{\alpha_{u,v}^2, t_{u,v}^2\}+k \geqslant k+1$, a contradiction. The lemma follows.

**Proof of Theorem 2.1** Let $S \subseteq V(G)$ be a vertex cutset of $G$. We choose $S \subseteq V(G)$ such that $|S|-\omega(G-S)$ is as small as possible. By Lemma 2.2, $|S| \geqslant k+2$. Let $|S|=s$, $\omega(G-S)=t$ and $C_1, C_2, \cdots, C_t$ be the components of $G-S$. Let $S=\{v_1, v_2, \cdots, v_s\}$ and $k_i$ be the number of components in $G-S$ which are adjacent to $v_i$. Without loss of generality, assume that $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s$. Let $k_{m_j}=\max\{k_i \mid v_i$ is adjacent to $C_j$ and $1\leqslant i\leqslant s\}$ $(j=1,2,\cdots,t)$. Without loss of generality, assume that $k_{m_1}\leqslant k_{m_2}\leqslant\cdots\leqslant k_{m_t}$.

**Claim 1**　$k_i\geqslant 2$ for $1\leqslant i\leqslant s$.

Suppose there is $k_i(1\leqslant i\leqslant s)$ such that $k_i\leqslant 1$. We replace $S$ by $S'=S\setminus\{v_i\}$. Then $\omega(G-S')\geqslant\omega(G-S)$. But $|S'|=|S|-1$. So $|S'|-\omega(G-S')<|S|-\omega(G-S)$, this contradicts the choice of $S$.

Assume $v_j$ is adjacent to a vertex $u$ in component $C$ and a vertex $v$ in component $C'$. It is easy to see that min $\{\alpha_{uv}^2+1, t_{uv}^2+1\}\geqslant k_j$. By the hypothesis of the theorem, $u$ and $v$ have at least $k_j+k$ common neighbours in $S$. So for each component adjacent to $v_i$, the component is adjacent to at least $k_j+k$ vertices in $S$. Considering all vertices in $S$ which are adjacent to component $C_j$, we know that $C_j$ is adjacent to at least $k_{m_j}+k$ vertices in $S$. Hence the components of $G-S$ have at least $(k_{m_1}+k)+(k_{m_2}+k)+\cdots+(k_{m_t}+k)$ edges to $S$. But the vertices in $S$ have $k_1+k_2+\cdots+k_s$ edges to the components of $G-S$. So we have

$$k_1+k_2+\cdots+k_s\geqslant(k_{m_1}+k)+(k_{m_2}+k)+\cdots+(k_{m_t}+k). \tag{1}$$

**Claim 2**　$\sum_{i=1}^{t}k_i\leqslant\sum_{i=1}^{t}k_{m_i}$.

We shall prove $k_{m_i}\geqslant k_i(i=1,2,\cdots,t)$ by induction on $i$, and the Claim 2 holds. By the definition of $k_{m_i}$, we have $k_{m_1}\geqslant k_1$. Assume $k_{m_i}\geqslant k_i$ for all $i<j$. Now assume $i=j$. If there is a component $C_p\in\{C_1,C_2,\cdots,C_j\}$ such that $C_p$ is adjacent to $v_q$ for some $q\geqslant j$, then $k_{m_j}\geqslant k_{m_p}\geqslant k_q\geqslant k_j$. Otherwise, $C_1,C_2,\cdots,C_j$ are only adjacent to $v_1,v_2,\cdots,v_{j-1}$. Then $k_1+k_2+\cdots+k_{j-1}\geqslant(k_{m_1}+k)+(k_{m_2}+k)+\cdots+(k_{m_j}+k)$. By the induction hypothesis, $k_{m_i}\geqslant k_i(i=1,2,\cdots,j-1)$ and $k_{m_j}\geqslant 2$. So $k_{m_1}+k_{m_2}+\cdots+k_{m_{j-1}}+k_{m_j}>k_1+k_2+\cdots+k_{j-1}$, a contradiction. Claim 2 follows.

By (1) and Claim 2, it is easy to see that $s>t$ and we have

$$\sum_{j=t+1}^{s}k_j\geqslant tk. \tag{2}$$

But at most $t$ components are adjacent to each of $v_1,v_2,\cdots,v_s$, hence

$$k_i\leqslant t\qquad(i=1,2,\cdots,s). \tag{3}$$

By (2) and (3), we have

$$(s-t)t\geqslant\sum_{j=t+1}^{s}k_j\geqslant tk.$$

So $s-t\geqslant k$, i.e. $t\leqslant s-k$. Therefore $\omega(G-S)\leqslant|S|-k$. This completes the proof of Theorem 2.1.

**Theorem 2.3**　Let $G$ be a 2-connected graph with even order $p$. If $\lambda_{uv}\geqslant\min\{\alpha_{uv}^2, t_{uv}^2\}+2n$ for any two vertices $u$ and $v$ with $d(u,v)=2$, then $G$ is $n$-extendable.

**Proof**　By Theorem 1.1, $G$ has Hamiltonian cycle. Since $G$ has even vertices, $G$ has a perfect matching containing $p/2$ independent edges. Suppose $G$ is not $n$-extendable. There are

$n$ independent edges $e_i = u_i v_i (i=1,2,\cdots,n)$ such that

$$G' = G - \{u_i, v_i | i = 1,2,\cdots,n\}$$

has no perfect matching. By Tutte's Theorem, there is a set $S' \subseteq V(G')$ such that

$$o(G' - S') > |S'|.$$

By the parity, $o(G' - S') \geqslant |S'| + 2$. Let

$$T = S' \cup \{u_i, v_i | i = 1,2,\cdots,n\}.$$

Then

$$\omega(G - T) = \omega(G' - S') \geqslant o(G' - S') \geqslant |S'| + 2 = |T| - 2n + 2.$$

But by Theorem 2.1, $\omega(G-T) \leqslant |T| - (2n-1) = |T| - 2n + 1$, a contradiction.

The bound of $\lambda_{uv}$ in Theorem 2.3 is sharp. Let $H = K_{2n}$ and let $u$ and $v$ be two vertices not in $V(H)$. We construct $G$ by joining each of $u$ and $v$ to all vertices of $H$. Then we have only two vertices of distance 2 and $\lambda_{uv} = 2n = \min\{\alpha^2_{uv}, t^2_{uv}\} + 2n - 1$. However, $G$ is not $n$-extendable since there are $n$ independent edges in $H$ which are not contained in a perfect matching of $G$.

Let $G$ be a connected graph. For any vertices $u$, $v$ with

$$d(u,v) = 2,\ d(u) + d(v) \geqslant \min\{|M^3(u)|, |M^3(v)|\} \text{ implies } \lambda_{uv} \geqslant \min\{\alpha^2_{uv} + 1, t^2_{uv} + 1\}.$$

So we have the following corollary which is corresponding to a local condition for Hamiltonian graphs of A. S. Hasratian and N. K. Khachatrian in [2].

**Corollary 2.4** Let $G$ be a connected graph with even order. If

$$d(u) + d(v) \geqslant \min\{|M^3(u)|, |M^3(v)|\} + 2n - 1$$

for any two vertices $u$ and $v$ with $d(u,v) = 2$, then $G$ is $n$-extendable.

**Corollary 2.5** Let $G$ be a connected graph with even order $p$. If

$$d(u) + d(v) \geqslant p + 2n - 1$$

for any two vertices $u$ and $v$ with $d(u,v) = 2$, then $G$ is $n$-extendable.

Corollary 2.5 implies a result of Plummer. The following corollary is due to Plummer [4].

**Corollary 2.6** Let $G$ be a connected graph of even order $p$. If

$$\delta(G) \geqslant p/2 + n,$$

then $G$ is $n$-extendable.

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# 一个 $n$-可扩图的局部交条件

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**摘要**　设 $G$ 是一个有偶数个顶点的连通图，$\alpha_v^2$ 表示由 $N^2(v)$ 导出的子图的独立数，其中

$$N^k(v) = \{u \mid u \in V(G) \text{ 并且 } d(u,v) = k\},$$

对任意 $uv \notin E(G)$，我们定义 $\lambda_{uv} = |N(u) \cap N(u)|$，$T_v^2(v) = N^2(u) \setminus N(v)$，$t_{uv}^2 = \min\{|T_u^2(v)|, |T_v^2(u)|\}$ 和 $\alpha_{uv}^2 = \min\{\alpha_u^2, \alpha_v^2\}$. 本文证明如果对任意 $d(u,v)=2$ 的 $u,v$，有 $\lambda_{uv} \geq \{\alpha_{uv}^2, t_{uv}^2\} + 2n$，则 $G$ 是 $n$-可扩的.

**关键词**　局部条件，$n$-可扩图

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