A LOCAL INTERSECTION CONDITION FOR  
\textit{n-EXTENDABLE GRAPHS}

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Abstract Let $G$ be a connected graph with even order. Let $a_k^v$ denote independence number of the subgraph induced by $N^k(v)$ in $G$, where $N^k(v)=\{v'|v\in V(G)\text{ and } d(u,v)=k\}$. For any $uv \in E(G)$, we define $\lambda_{uv}=|N(u) \cap N(v)|$, $T^1_v(v)=N^2(u) \setminus N(v)$, $t^1_{uv}=\min\{|T^1_v(v)|, |T^1_u(u)|\}$ and $a^1_{uv} = \min\{a^2_u, a^2_v\}$. It is proved that if $\lambda_{uv} \geq \min\{a^1_{uv}, t^1_{uv}\} + 2n$ for any two vertices $u$ and $v$ with $d(u,v)=2$, then $G$ is $n$-extendable.

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1 Introduction

Graphs considered in this paper are finite, undirected and simple.

Let $G$ be a graph, for $S \subseteq V(G)$, set $E(S)=\{uv \in E(G) | u,v \in S\}$. For $M \subseteq E(G)$, set $V(M)=\{v \in V(G) | \text{ there is } x \in V(G) \text{ such that } vx \in M\}$. Edges $e,f \in E(G)$ are called independent if $V(e) \cap V(f) = \emptyset$. $M \subseteq E(G)$ is a matching of $G$ if $V(e) \cap V(f) = \emptyset$ for every pair of edges $e,f \in M$. A matching $M$ of $G$ is perfect if $V(M)=V(G)$.

Plummer$^{[4]}$ introduced the concept of $n$-extendable graphs.

Let $G$ be a graph of order $p$ with a perfect matching and let $n$ be a positive integer such
that \(n \leq (p - 2)/2\), \(G\) is said to be \(n\)-extendable if \(G\) has \(n\) independent edges and any \(n\) independent edges of \(G\) are contained in a perfect matching of \(G\).

Let \(G\) be a connected graph, \(v\) be a vertex in \(G\). We define \(N(v) = \{u | u \in V(G), uv \in E(G)\}\), \(N^k(v) = \{u | u \in V(G)\) and \(d(u, v) = k\}\) and \(M^k(v) = \{u | u \in V(G)\) and \(d(u, v) \leq k\}\). Let \(a_v\) denote the number of vertices in the largest independent set containing the vertex \(v\) in \(G\) and \(\alpha^2_v\) denote independence number of the subgraph induced by \(N^2(v)\) in \(G\). For any \(uv \in E(G)\), we defined \(T_{uv} = V \setminus N(u) \cup N(v) \cup \{u, v\}\), \(\lambda_{uv} = |N(u) \cap N(v)|\), \(\alpha_{uv} = \min(\alpha_u, \alpha_v)\), \(T_{uv}^2 = N^2(u) \setminus N(v)\), \(t_{uv} = |T_{uv}^2|\), \(\alpha_{uv}^2 = \min(|T_{uv}^2(v)|, |T_{uv}^2(u)|)\) and \(\alpha_{uv}^2 = \min(\alpha_{uv}^2, \alpha_{uv}^2)\). It is clear that \(\alpha_{uv}^2 \leq \alpha_{uv} - 1 \leq \alpha - 1\).

We use \(\omega(G)\) to denote the number of components of \(G\) and use \(\sigma(G)\) to denote the number of odd components of \(G\).

For any terminology and notation used, but not defined here, the reader is referred to [1].

In [3] Lam and Song introduced a local condition for Hamiltonian graphs.

**Theorem 1.1** Let \(G\) be a 2-connected simple graph of order \(p (\geq 3)\). If \(\lambda_{uv} \geq \min(\alpha_{uv}^2 + 1, \alpha_{uv}^2 + 1)\) for any two vertices \(u\) and \(v\) with \(d(u, v) = 2\), then \(G\) is Hamiltonian.

However, \(n\)-extendable graphs and Hamiltonian graphs have many similar properties. In this paper, we give a sufficient condition for \(n\)-extendable graphs which is a Lam and Song's type condition. This condition implies a result of Plummer that for a connected graph \(G\) with even order \(p\), if \(\sigma(G) \geq p/2 + n\), then \(G\) is \(n\)-extendable[4].

## 2 Main Results

**Theorem 2.1** Let \(G\) be a connected graph and let \(k \geq 0\). If \(\lambda_{uv} \geq \min(\alpha_{uv}^2 + 1, \alpha_{uv}^2 + 1)\) for any two vertices \(u\) and \(v\) of \(G\) with \(d(u, v) = 2\), then \(\omega(G - S) \leq |S| - k\) for all cutset \(S \subseteq V(G)\).

**Lemma 2.2** Let \(G\) be a connected graph and let \(k \geq 1\). If \(\lambda_{uv} \geq \min(\alpha_{uv}^2, \alpha_{uv}^2) + k\) for any two vertices \(u\) and \(v\) of \(G\) with \(d(u, v) = 2\), then \(G\) is \((k + 1)\)-connected.

**Proof** Suppose that \(G\) is not \((k + 1)\)-connected. Let \(S \subseteq V(G)\) be a minimum vertex cutset such that \(|S| \leq k\). Assume \(C_1\) and \(C_2\) are two components of \(G - S\). Let \(uv \in S\). By the minimality of \(|S|\), \(uv\) is adjacent to a vertex \(u\) in \(C_1\) and a vertex \(v\) in \(C_2\). Then \(d(u, v) = 2\). So \(\lambda_{uv} \geq \min(\alpha_{uv}^2, \alpha_{uv}^2) + k\). Since \(\{u, v\} \subseteq N^2(u) \cup \{u\}\) is independent and \(\{u, v\} \subseteq N^2(v) \cup \{v\}\) is also independent, \(\alpha_{uv}^2 \geq 1\). Since \(v \in T_{uv}^2(v)\) and \(u \in T_{uv}^2(u)\), \(t_{uv} \geq 1\). But \(N(u) \cap N(v) \subseteq S\). Hence \(\alpha_{uv}^2 \geq \min(\alpha_{uv}^2, \alpha_{uv}^2) + k \geq k + 1\), a contradiction. The lemma follows.

**Proof of Theorem 2.1** Let \(S \subseteq V(G)\) be a vertex cutset of \(G\). We choose \(S \subseteq V(G)\) such that \(|S| - \omega(G - S)\) is as small as possible. By Lemma 2.2, \(|S| \geq k + 2\). Let \(|S| = s\), \(\omega(G - S) = t\) and \(C_1, C_2, \ldots, C_s\) be the components of \(G - S\). Let \(S = \{v_1, v_2, \ldots, v_s\}\) and \(k\) be the number...
ber of components in $G-S$ which are adjacent to $v_i$. Without loss of generality, assume that $k_j \leq k_2 \leq \cdots \leq k_s$. Let $k_{m_i} = \max \{ k_i | v_i \text{ is adjacent to } C_j \text{ and } 1 \leq i \leq s \} (j = 1, 2, \cdots, t)$. Without loss of generality, assume that $k_{m_1} \leq k_{m_2} \leq \cdots \leq k_{m_t}$.

Claim 1 \hspace{1em} $k \geq 2$ for $1 \leq i \leq s$.

Suppose there is $k_i (1 \leq i \leq s)$ such that $k_i \leq 1$. We replace $S$ by $S' = S \setminus \{ v_i \}$. Then $\omega(G-S') \geq \omega(G-S)$. But $|S'| = |S| - 1$. So $|S'| - \omega(G-S') < |S| - \omega(G-S)$, this contradicts the choice of $S$.

Assume $v_j$ is adjacent to a vertex $u$ in component $C$ and a vertex $v$ in component $C'$. It is easy to see that $\min \{ d^2_{m+1} + 1, t^2_{m+1} + 1 \} \geq k_j$. By the hypothesis of the theorem, $u$ and $v$ have at least $k_i + k$ common neighbours in $S$. So for each component adjacent to $v_i$, the component is adjacent to at least $k_i + k$ vertices in $S$. Considering all vertices in $S$ which are adjacent to component $C_j$, we know that $C_j$ is adjacent to at least $k_{m_j} + k$ vertices in $S$. Hence the components of $G-S$ have at least $(k_{m_1} + k) + (k_{m_2} + k) + \cdots + (k_{m_t} + k)$ edges to $S$. But the vertices in $S$ have $k_1 + k_2 + \cdots + k_t$ edges to the components of $G-S$. So we have

$$k_1 + k_2 + \cdots + k_t \geq (k_{m_1} + k) + (k_{m_2} + k) + \cdots + (k_{m_t} + k).$$

(1)

Claim 2 \hspace{1em} $\sum_{i=1}^{t} k_i \leq \sum_{i=1}^{t} k_{m_i}$.

We shall prove $k_{m_i} \geq k_i (i = 1, 2, \cdots, t)$ by induction on $i$, and the Claim 2 holds. By the definition of $k_{m_i}$, we have $k_{m_i} \geq k_i$. Assume $k_{m_i} \geq k_i$ for all $i < j$. Now assume $i = j$. If there is a component $C_j \subseteq \{ C_1, C_2, \cdots, C_t \}$ such that $C_j$ is adjacent to $v_q$ for some $q \geq j$, then $k_{m_j} \geq k_q \geq k_j$. Otherwise, $C_1, C_2, \cdots, C_j$ are only adjacent to $v_1, v_2, \cdots, v_{j-1}$. Then $k_1 + k_2 + \cdots + k_{j-1} \geq (k_{m_1} + k) + (k_{m_2} + k) + \cdots + (k_{m_j} + k)$. By the induction hypothesis, $k_{m_j} \geq k_i (i = 1, 2, \cdots, j-1)$ and $k_{m_j} \geq 2$. So $k_{m_1} + k_{m_2} + \cdots + k_{m_{j-1}} + k_{m_j} \geq k_1 + k_2 + \cdots + k_{j-1}$, a contradiction. Claim 2 follows.

By (1) and Claim 2, it is easy to see that $s \geq t$ and we have

$$\sum_{j=t+1}^{s} k_j \geq tk.$$ 

(2)

But at most $t$ components are adjacent to each of $v_1, v_2, \cdots, v_s$, hence

$$k_i \leq t \hspace{1em} (i = 1, 2, \cdots, s).$$

(3)

By (2) and (3), we have

$$(s - t)t \geq \sum_{j=t+1}^{s} k_j \geq tk.$$ 

So $s - t \geq k$, i.e. $t \leq s - k$. Therefore $\omega(G-S) \leq |S| - k$. This completes the proof of Theorem 2.1.

Theorem 2.3 \hspace{1em} Let $G$ be a 2-connected graph with even order $p$. If $\lambda_{uv} \geq \min \{ \alpha_{uv}^2 + 2n \}$ for any two vertices $u$ and $v$ with $d(u, v) = 2$, then $G$ is $n$-extendable.

Proof \hspace{1em} By Theorem 1.1, $G$ has Hamiltonian cycle. Since $G$ has even vertices, $G$ has a perfect matching containing $p/2$ independent edges. Suppose $G$ is not $n$-extendable. There are
$n$ independent edges $e_i = u_i v_i (i = 1, 2, \cdots, n)$ such that

$$G' = G - \{ u_i, v_i | i = 1, 2, \cdots, n \}$$

has no perfect matching. By Tutte's Theorem, there is a set $S' \subseteq V(G')$ such that

$$o(G' - S') \geq |S'| + 2.$$ 

By the parity, $o(G' - S') \geq |S'| + 2$. Let

$$T = S' \cup \{ u_i, v_i | i = 1, 2, \cdots, n \}.$$ 

Then

$$w(G - T) = w(G' - S') \geq o(G' - S') \geq |S'| + 2 = |T| - 2n + 2.$$ 

But by Theorem 2.1, $w(G - T) \leq |T| - (2n - 1) = |T| - 2n + 1$, a contradiction.

The bound of $\lambda_w$ in Theorem 2.3 is sharp. Let $H = K_2$, and let $u$ and $v$ be two vertices not in $V(H)$. We construct $G$ by joining each of $u$ and $v$ to all vertices of $H$. Then we have only two vertices of distance 2 and $\lambda_w = 2n = \min \{ \lambda_w^2, \lambda_v^2 \} + 2n - 1$. However, $G$ is not $n$-extendable since there are $n$ independent edges in $H$ which are not contained in a perfect matching of $G$.

Let $G$ be a connected graph. For any vertices $u, v$ with

$$d(u, v) = 2, \quad d(u) + d(v) \geq \min \{ |M^1(u)|, |M^1(v)| \} \implies \lambda_w$$

$$\geq \min \{ \lambda_w^1 + 1, \lambda_v^1 + 1 \}.$$ 

So we have the following corollary which is corresponding to a local condition for Hamiltonian graphs of A. S. Hasratian and N. K. Khachatryan in [2].

**Corollary 2.4** Let $G$ be a connected graph with even order. If

$$d(u) + d(v) \geq \min \{ |M^1(u)|, |M^1(v)| \} + 2n - 1$$

for any two vertices $u$ and $v$ with $d(u, v) = 2$, then $G$ is $n$-extendable.

**Corollary 2.5** Let $G$ be a connected graph with even order $p$. If

$$d(u) + d(v) \geq p + 2n - 1$$

for any two vertices $u$ and $v$ with $d(u, v) = 2$, then $G$ is $n$-extendable.

Corollary 2.5 implies a result of Plummer. The following corollary is due to Plummer [4].

**Corollary 2.6** Let $G$ be a connected graph of even order $p$. If

$$\delta(G) \geq \frac{p}{2} + n,$$

then $G$ is $n$-extendable.

References

一个 $n$-可扩图的局部交条件

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摘要 设 $G$ 是一个有偶数个顶点的连通图，$\alpha$ 表示由 $N^2(u)$ 导出的子图的独立数，其中

$$N^4(u) = \{ u \mid u \in V(G) 并且 d(u, v) = k \},$$

对任意 $uv \in E(G)$，我们定义 $\lambda_v = |N(u) \cap N(u)|$, $T^0_v(v) = N^2(u) \setminus N(v)$, $t^2_w = \min \{|T^0_v(v)|, |T^0_u(u)|\}$ 和 $\alpha_v = \min \{ \alpha_v^2, \alpha_v^2 \}$. 本文证明如果对任意 $d(u, v) = 2$ 的 $u, v$, 有 $\lambda_v \geq \alpha_v^2 + t^2_w + 2n$, 则 $G$ 是 $n$-可扩的.

关键词 局部条件，$n$-可扩图

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