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LOCALIZATION THEOREM ON HAMILTONIAN GRAPHS [1]

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Abstract Let G be a 2-connected graph of order $n(\geq 3)$. If $I(u,v) \geq S(u,v)$ or $\max\{d(u),d(v)\} \geq n/2$ for any two vertices u, v at distance two in an induced subgraph $K_{1,3}$ or P_3 of G , then G is hamiltonian. Here $I(u,v) = |N(u) \cap N(v)|$, $S(u,v)$ denotes the number of edges of maximum star containing u, v as an induced subgraph in G.

Key words Local condition, Hamilton cycle

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1 Introduction

In this paper, We use [1] for terminology and notation not defined here and consider finite simple graphs only.

The distance between vertices u and v is denoted by $d(u,v)$. For each vertex $u \in V(G)$, we denote by $N(u)$ the set of all vertices of G adjacent to u. The subgraph of G induced by $N(u) \cup \{u\}$ is denoted by $G(u)$. If $uv \notin E(G)$, we denote by $S(u,v)$ the number of edges of maximum star including u, v as an induced subgraph in G. Let x and y be two vertices in G with $d(x,y) = 2$, we define $I(x,y) = |N(x) \cap N(y)|$. Let C be a cycle of G with a fixed cyclic orientation. For $u \in V(C)$, let u^+ be the successor and u^- be the predecessor of u in the chosen direction on C.

Theorem A[2] Let G be a 2-connected graph of order $n(\geq 3)$. If $I(u,v) \geq S(u,v)$ whenever $d(u,v) = 2$ and $\max\{d(u),d(v)\} < n/2$, then G is hamiltonian.

In this paper, we obtain the following result.

Theorem Let G be a 2-connected graph of order $n(\geq 3)$. If $I(u,v) \geq S(u,v)$ or $\max\{d(u),d(v)\} \geq n/2$ for any two vertices u, v at distance two in an induced subgraph $K_{1,3}$ or P_3 of G, then G is hamiltonian, where P_3 is a path with length 3.

Consider the graph G_1 obtained from K_{n-3} and $\{x,y,z\}$ by adding an edge set $\{xy, yz, yu, yv, xu, zv\}$, where $\{u,v\} \subseteq V(K_{n-3})$. Obviously, G_1 satisfies the condition of Theorem, but does not satisfy the condition of Theorem A. Hence Theorem generalizes Theorem A.

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2 Proof of Theorem

In the proof, we need the following Lemma.

Lemma[3] Let G be a 2-connected graph of order n, then G contains a cycle passing through all vertices of degree at least $n/2$.

Proof of Theorem By contradiction, let G be a nonhamiltonian graph with maximum number of edges satisfying the condition of Theorem. Let $A = \{u \in V(G) | d(u) \geq n/2\}$. If $A \neq \emptyset$, then by the choice of G, the induced subgraph $G[A]$ is complete.

By Lemma G contains a cycle passing through all vertices of degree at least $\frac{n}{2}$. Among such cycles, take C a longest cycle with a fixed cyclic orientation. Set $R = V(G) \backslash V(C)$, then $R \neq \emptyset$. Let B be a connected component in $G[R]$ and let $v_1, v_2, \cdots, v_m$ be the elements of $N_C(B)$ occuring on $\vec{C}$ in consecutive order. Since G is 2-connected, we have $m \geq 2$. Let x_j be a vertex of B which is adjacent to v_j (for $i \neq j$, possibly $x_i = x_j$). It is easy to show that for any $1 \leq i < j \leq m$, $v_i^+ v_j^+ \notin E(G)$, $v_i^- v_j^- \notin E(G)$. If $v_i^+ v_i^- \in E(G)$, then $v_{i+1}^- v_i \notin E(G)$. Choose $a_i \in \{v_i^+, v_i^{++}, \cdots, v_{i+1}^-\}$ such that for any $v \in \{v_i^+, \cdots, a_i\}$, $vv_i \in E(G)$ but $a_i^+ v_i \notin E(G)$. If $v_i^+ v_i^- \notin E(G)$, set $a_i = v_i^+$. By the choice of C, it is easy to check that for any $1 \leq i \leq m$, $\{a_i, x_i\}$ is in an induced subgraph $K_{1,3}$ or P_3 of G with $d(a_i, x_i) = 2$. For any $1 \leq i < j \leq m$, we denote the vertices of $a_i^+ \vec{C} a_j$ by S_1 and the vertices of $a_j^+ \vec{C} a_i$ by S_2. For any $x \in V(B)$, by the choice of C the sets $N_{S_1}(a_i)$, $N_{S_1}^+(a_j)$, $N_{S_2}^+(a_i)$, $N_{S_2}(a_j)$, $N_R(a_i)$, $N_R(a_j)$, $\{x\}$ are pairwise disjoint. So we have $n \geq |N_{S_1}(a_i)| + |N_{S_1}^+(a_j)| + |N_{S_2}^+(a_i)| + |N_{S_2}(a_j)| + |N_R(a_i)| + |N_R(a_j)| + |\{x\}| = |N_C(a_i)| + |N_C(a_j)| + |N_R(a_i)| + |N_R(a_j)| + |\{x\}| = d(a_i) + d(a_j) + 1$. So $\min\{d(a_i), d(a_j)\} < n/2$, say $d(a_i) < n/2$. By the condition of Theorem we have $I(a_i, x_i) \geq S(a_i, x_i) \geq 2$. Since $N_R(a_i) \cap N_R(x_i) = \emptyset$ by the choice of C, $|N_C(x_i)| \geq 2$. Hence there is a v such that $N(v) \cap V(C) = W = \{w_1, w_2, \cdots, w_p\} \neq \emptyset$ and $p \geq 2$.

It is easy to show that for any $1 \leq i < j \leq p$, $w_i^+ w_j^+ \notin E(G)$ $w_i^- w_j^- \notin E(G)$. If $w_i^+ w_i^- \in E(G)$, then $w_{i+1}^- w_i \notin E(G)$. Choose $u_i \in \{w_i^+, w_i^{++}, \cdots, w_{i+1}^-\}$ such that for any $u \in \{w_i^+, \cdots, u_i\}$, $uw_i \in E(G)$ but $u_i^+ w_i \notin E(G)$. If $w_i^+ w_i^- \notin E(G)$, set $u_i = w_i^+$. Let $U = \{u_1, u_2, \cdots, u_p\}$. By the choice of C, it is easy to check that for any $1 \leq i \leq p$, $\{u_i, v\}$ is in an induced subgraph $K_{1,3}$ or P_3 of G with $d(v, u_i) = 2$ and $U \cup \{v\}$ is an independent set of G. Since the induced subgraph $G[A]$ is complete, there is at most one vertex of degree at least $n/2$ in the set $U \cup \{v\}$. If such vertex exists, we assume $d(u_s) \geq n/2$, $1 \leq s \leq p$.

Now we consider the following two cases.

Case 1 For any i, $1 \leq i \leq p$, $d(u_i) < n/2$.

Consider the following iterated definition. Let $A_j^1 = \{v, u_j\}$, $B_j^1 = \{w_j\}$, $j = 1, 2, \cdots, p$. Clearly $A_j^1 \subseteq N(w_j) \cap (U \cup \{v\})$, $B_j^1 \subseteq N(u_j) \cap W$, and $|A_j^1| > |B_j^1|$.

Assume sets A_j^k, B_j^k, with $A_j^k \subseteq N(w_j) \cap (U \cup \{v\})$, $B_j^k \subseteq N(u_j) \cap W$ and $|A_j^k| \geq |B_j^k|$, $j = 1, 2, \cdots, p$, $k \geq 1$ are well defined. If there exists t $(1 \leq t \leq p)$ such that $|A_t^k| > |B_t^k|$, then by $d(v, u_t) = 2$ and $S(v, u_t) \geq |A_t^k|$ we have $I(v, u_t) \geq S(v, u_t) \geq |A_t^k| \geq |B_t^k| + 1$. So $|(N(v) \cap N(u_t)) \backslash B_t^k| \geq 1$. Thus there exists r $(1 \leq r \leq p)$ such that $w_r \in (N(v) \cap N(u_t)) \backslash B_t^k$. Hence we can define $A_j^{k+1} = A_j^k, B_j^{k+1} = B_j^k$, when $j \neq r, t$ and $1 \leq j \leq p$; $A_t^{k+1} = A_t^k, B_t^{k+1} = B_t^k \cup \{w_r\}$ and $A_r^{k+1} = A_r^k \cup \{u_t\}, B_r^{k+1} = B_r^k$. Clearly $A_j^{k+1} \subseteq N(w_j) \cap (U \cup \{v\})$, $B_j^{k+1} \subseteq N(u_j) \cap W$ and $|A_j^{k+1}| \geq |B_j^{k+1}|$, $j = 1, 2, \cdots, p$. Particularly $|A_r^{k+1}| > |B_r^{k+1}|$ and $|B_t^{k+1}| = |B_t^k| + 1$.

So the above iterative process can be done infinitely.

Set $b_k = \sum\limits_{j=1}^{p} |B_j^k|$, $k = 1, 2, \cdots$, then $0 < b_1 < b_2 < \cdots < b_k < \cdots$. On the other hand, $b_k = \sum\limits_{j=1}^{p} |B_j^k| \le p^2$, $k = 1, 2, \cdots$, since $B_j^k \subseteq W$, a contradiction.

Case 2 $d(u_s) \ge \frac{n}{2}$.

Let $I = \{1, 2, \cdots, p\}$ and $J = \{j | j \in I \text{ and } w_s u_j \in E(G)\}$. In this case we define $A_j^1 = \{v, u_j\}$, $B_j^1 = \{w_j\}$ if $j \in \{1, 2, \cdots, p\} \backslash (J \cup \{s\})$; $A_j^1 = \{v, u_j, u_s\}$, $B_j^1 = \{w_j, w_s\}$ if $j \in J \backslash \{s\}$. A_j^k, B_j^k, $k > 1$, $j = 1, 2, \cdots, s-1, s+1, \cdots, p$ can be defined as case 1. We will also deduce a similar contradiction as case 1. This completes the proof of Theorem.

Corollary 1[4] Let G be a 2-connected graph of order n. If $\max\{d(u), d(v)\} \ge \frac{n}{2}$ for any two vertices u, v with $d(u, v) = 2$, then G is hamiltonian.

Corollary 2 Let G be a 2-connected graph of order n. If $\max\{d(u), d(v)\} \ge \frac{n}{2}$ for any two vertices u, v at distance two in an induced subgraph $K_{1,3}$ or P_3 of G, then G is hamiltonian.

Corollary 3 Let G be a 2-connected graph. If G has neither $K_{1,3}$ nor P_3 as induced subgraph, then G is hamiltonian.

Corollary 4[5] Let G be a connected graph of order n. If $d(u) + d(v) \ge n$ for each pair u, v of nonadjacent vertices, then G is hamiltonian.

Corollary 5[2] Let G be a connected graph of order n. If $d_{G(u)}(x) + d_{G(u)}(y) \ge d(u) + 1$ for any $u \in V(G)$, $\{x, y\} \subseteq V(G(u))$, $xy \notin E(G)$, then G is hamiltonian.

Proof It is sufficient to prove that G satisfies the condition of Theorem. In fact we can prove that $I(u, v) \ge S(u, v)$ for any two vertices u, v with $d(u, v) = 2$. Let $u, v \in V(G)$, $d(u, v) = 2$. By the definition of $S(u, v)$, there exists $w \in V(G)$ such that $\{u, v, x_1, \cdots, x_{s-2}\} \subseteq N(w)$ is an independent set, where $s = S(u, v) \ge 2$. By the condition of Corollary 5, we have

$$d_{G(w)}(u) + d_{G(w)}(v) \ge d(w) + 1 = |V(G(w))|,$$

$$V(G(w)) = N(w) \cup \{w\} \supseteq (N_{G(w)}(u) \cup N_{G(w)}(v)) \cup \{u, v, x_1, \cdots, x_{s-2}\},$$

$$(N_{G(w)}(u) \cup N_{G(w)}(v)) \cap \{u, v, x_1, \cdots, x_{s-2}\} = \emptyset.$$

So

$$\begin{aligned} |V(G(w))| &\ge |N_{G(w)}(u) \cup N_{G(w)}(v)| + s \\ &= |N_{G(w)}(u)| + |N_{G(w)}(v)| - |N_{G(w)}(u) \cap N_{G(w)}(v)| + s. \end{aligned}$$

Hence

$$\begin{aligned} I(u, v) = |N(u) \cap N(v)| &\ge |N_{G(w)}(u) \wr ap N_{G(w)}(v)| \\ &\ge |N_{G(w)}(u)| + |N_{G(w)}(v)| - |V(G(w))| + s \\ &\ge d_{G(w)}(u) + d_{G(w)}(v) - (d(w) + 1) + s \ge s = S(u, v). \end{aligned}$$

On the other hand, $|N(u) \cap N(v)| \ge S(u, v) \ge 2$ implies that G is 2-connected. Therefore Corollary 5 follows from Theorem.

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