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The Minimal Solutions of Boolean Matrix-equation $A^k = J$

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Abstract: Let A be a primitive Boolean matrix. $\gamma(A)$ is the least number k such that $A^k = J$. $\sigma(A)$ is the number of 1-entries in A. In this paper, the parameter $N'(k, n) = \min\{\sigma(A) \mid A^T = A, \operatorname{trace}(A) = 0, \gamma(A) = k\}$ is considered. Furthermore, we describe the set $EG(k, n) = \{G(A) \mid \sigma(A) = N'(k, n), A^T = A, \operatorname{trace}(A) = 0, \gamma(A) = k\}$ and obtain a characterization of the minimal solutions with zero trace of the Boolean matrix equation $A^k = J$.

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Let $B = \{0, 1\}$ be the usual binary Boolean algebra. The matrices over B are called Boolean matrices. An $n \times n$ Boolean matrix A is called a primitive matrix if there exists a positive integer k such that $A^k = J$ (where J is the universal matrix). The least such k is called the exponent of A, denoted by $\gamma(A)$.

In projective plane theory, although a lot of results are obtained on Boolean matrix equation, it is still a famous open problem to find the square roots of a Boolean matrix.

Let A be an $n \times n$ Boolean matrix. Define the norm of A, denoted by $\sigma(A)$, to be the number of 1-entries in A. Clearly, σ satisfies the norm axioms. As you know, a special Boolean matrix-equation $A^k = J$ has solutions. In general, it is very difficult to solve this equation. So, the parameter

$$N'(k, n) = \min\{\sigma(A) \mid A^T = A, \operatorname{trace}(A) = 0, \gamma(A) = k\}$$

must be considered. And we need the following concepts and propositions. The associated graph of an $n \times n$ symmetric Boolean matrix $A = (a_{ij})$, denoted by $G(A)$, is the graph

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with vertex set $V = \{1, 2, \cdots, n\}$ such that there is an edge arc between i and j in $D(A)$ if and only if $a_{ij} = a_{ji} = 1$. A graph G is primitive if there exists an integer $k > 0$ such that for all pairs of vertices $i, j \in V(G)$ (not necessary distinct), there is a walk from i to j with length k. The least such k is called the exponent of G, denoted by $\gamma(G)$. Clearly, a symmetric Boolean matrix A is primitive if and only if its associated graph $G(A)$ is primitive and $\gamma(A) = \gamma(G(A))$.

Let G be a primitive graph with order n. For any $i, j \in V(G)$, the local exponent from i to j, denoted by $\gamma(i, j)$, is the least integer k such that there exists a walk of length m from i to j for all $m \geq k$. It is obvious that $\gamma(G) = \max\limits_{i, j \in V(G)} \gamma(i, j)$.

Proposition[1] *The exponent set of symmetric primitive $(0,1)$-matrices with zero trace is $E_n = \{2, 3, \cdots, 2n-4\} - Y$, where Y is the set of all odd numbers in $\{n-2, n-1, \cdots, 2n-5\}$.*

In this paper, we describe the set

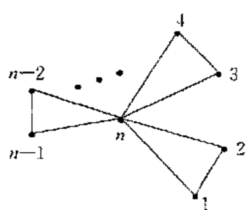
$$EG(k, n) = \{G(A) \mid \sigma(A) = N(k, n),\ A^T = A,\ \mathrm{trace}(A) = 0,\ \gamma(A) = k\}$$

and give a characterization of the minimal symmetric solutions with zero trace of $A^k = J$. Other terms and notations not defined here, we refer the reader to [2].

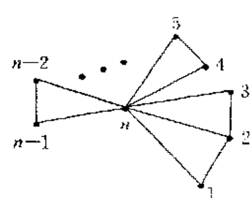
According to the definition of $N'(k, n)$ and Proposition 3, $k \in E_n$.

Theorem 1 $N'(2, n) = 2\left[\dfrac{3n-3}{2}\right]$ *for $n \geq 3$. Moreover, $G \in EG(2, n)$ is unique in the sense of permutation similarity.*

Proof Since $\gamma(G_1) = \gamma(G_2) = 2$ for $n \geq 3$ (see Figure 1), we have $N'(2, n) \leq 2\left[\dfrac{3n-3}{2}\right]$.



G_1 (n = odd)



G_2 (n = even)

Figure 1

If $A^T = A$, $A^2 = J$ and $\mathrm{trace}(A) = 0$, then there is a walk with length 2 from one vertex to another in $G(A)$. So any vertex of $G(A)$ is on some 3-cycle. Otherwise, there would exist a vertex $u \in V(G(A))$ such that u is a pendant vertex or u is on an s-cycle ($s \geq 4$). Let v be a neighbouring vertex of u. Then $\gamma(v, u) > 2$, a contradiction. Since $A^2 = J$, for any $u, v \in V(G(A))$ there exists 3-cycles C_1, C_2 such that $u \in C_1$, $v \in C_2$ and $C_1 \cap C_2 \neq \varnothing$. For any 3-cycle C, let

$$t = \left|\{u : \text{There exists a 3-cycle } C' \text{ containing } u \text{ such that } C' \cap C \text{ is exactly one vertex}\}\right|.$$

Thus we have

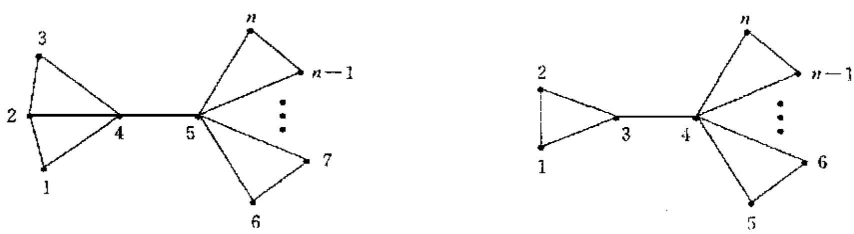
$$\varepsilon(G(A)) \geq 3 + \left\lfloor \frac{3t}{2} \right\rfloor + 2(n-3-t) = \left\lfloor \frac{2n-3-t}{2} \right\rfloor$$
$$\geq \left\lfloor 2n-3-\frac{n-3}{2} \right\rfloor = \left\lfloor \frac{3n-3}{2} \right\rfloor . \qquad (1)$$

So $\sigma(A) \geq 2\left\lfloor \frac{3n-3}{2} \right\rfloor$. Hence $N'(2, n) = 2\left\lfloor \frac{3t}{2} \right\rfloor$.

If $G \in EG(2, n)$, then (1) is an equality. So $G \cong G_1$ when n is odd and $G \cong G_2$ when n is even.

Theorem 2 $N'(3, n) = 2\left\lfloor \frac{3n-4}{2} \right\rfloor$ *for* $n \geq 6$. *Further,* $G \in EG(3, n)$ *is unique in the sense of permutation similarity.*

Proof Since $\gamma(G_3) = \gamma(G_4) = 3$ for $n \geq 6$ (see Figure 2), we have $N'(3, n) \leq 2\left\lfloor \frac{3n-4}{2} \right\rfloor$.



G_3 (n = odd) G_4 (n = even)

Figure 2

Let $A^T = A$, trace$(A) = 0$ and $\gamma(A) = 3$. Then for any $u \in V(G(A))$ there is a 3-cycle containing u. Otherwise, $\gamma(u, u) > 3$. By $\gamma(A) = 3$, the diameter of $G(A)$ is less then 4 and greater then 1.

Case 1 The diameter of $G(A)$ is equal to 3.

We claim that $G(A)$ has a subgraph H (see Figure 3).

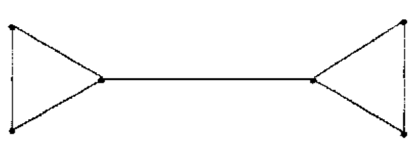


Figure 3 H

We take any two 3-cycles C_1, C_2, and let $d = d_G(C_1, C_2) \leq 3$. . Thus there exists a path with length d. For $d = 3$ (similary for $d = 2$), let $P = xuvy$, $x \in C_1$ and $y \in C_2$. If there is a 3-cycle C_3 containing v such that $C_2 \cap C_3 = \varnothing$, then the claim holds. Hence $C_2 \cap C_3 \neq \varnothing$. Now, we consider a 3-cycle C_4 with $u \in C_4$. Thus we have $C_1 \cap C_4 \neq \varnothing$ and $C_3 \cap C_4 \neq \varnothing$. Hence $d_G(C_1, C_2) \leq 2$, a contradiction. So $G(A)$ contains a subgraph H.

Case 2 The diameter of $G(A)$ is equal to 2.

If there exist two 3-cycles C_1, C_2 such that $u \in C_1$, $v \in C_2$ and $C_1 \cap C_2 \neq \varnothing$ for any u, $v \in V(G(A))$, then $\gamma(A) = 2$. This is a contradiction. If there exist two vertices u, v and two 3-cycles C_1, C_2 such that $u \in C_1$, $v \in C_2$ and $C_1 \cap C_2 = \varnothing$, then we can prove that

$G(A)$ has a subgraph H as in the proof of the case 1. Let

$$t = |\{u: \text{There exists a 3-cycle } C \text{ containing } u \text{ such that } C \cap H \text{ is exactly one vertex}\}|.$$

Thus we have

$$\varepsilon(G(A)) \geq 7 + \left[\frac{3t}{2}\right] + 2(n-6-t) = \left[\frac{2n-5-t}{2}\right] \geq \left[2n-5-\frac{n-6}{2}\right] = \left[\frac{3n-4}{2}\right]. \qquad (2)$$

So $\sigma(A) \geq 2\left[\frac{3n-4}{2}\right]$. Hence $N'(3, n) = 2\left[\frac{3n-4}{2}\right]$.

If $G \in EG(3, n)$, then (2) is an eqality. So $G \cong G_3$ when n is odd and $G \cong G_4$ when n is even.

Theorem 3 $N'(2q, n) = 2n$ *for* $2 \leq q \leq n-2$.

Proof Since $\gamma(G_5) = 2q$ for $2 \leq q \leq n-2$ (see Figure 4), we have $N'(2q, n) \leq 2n$ for $2 \leq q \leq n-2$.

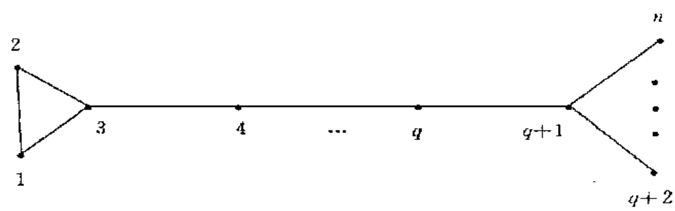


Figure 4

If A is a symmetric primitive matrix with zero trace, then $\gamma(A) \geq 2n$. Hence $N'(2q, n) = 2n$.

Lemma *Let G be a primitive undirected graph with exactly one cycle C. Then $\gamma(G)$ is even.*

Proof Let $G_1, G_2, \cdots, G_r$ be the components of $G - E(C)$, and c the length of C. Let

$$t_i = \max_{v_i \in V(C_i)} \min_{u \in V(C)} d(u, v_i), \qquad t = \max\{t_1, t_2, \cdots, t_r\}.$$

It is obvious that there exists a vertex u_0 such that the distance from u_0 to C is t. Hence

$$\gamma(u_0, u_0) = 2t + c - 1.$$

If $u, v \in V(G_i)$, then

$$\gamma(u, v) \leq 2t_i + c - 1 \leq 2t + c - 1.$$

If $u \in V(G_i)$, $v \in V(G_j)$ $(j \neq i)$, then

$$\gamma(u, v) \leq t_i + t_j + c - 1 \leq 2t + c - 1.$$

Therefore $\gamma(G) = 2t + c - 1$ is even.

Theorem 4 $N'(2q+1, n) = 2(n+1)$ *for* $2 \leq q \leq \left[\frac{n-4}{2}\right]$.

Proof If $\gamma(A)$ is odd, then $G(A)$ has at least two odd cycles by the above Lemma. So $N'(2q+1, n) \geq 2(n+1)$.

Since $\gamma(G_6) = 2q+1$ for $2 \leq q \leq \left[\frac{n-4}{2}\right]$ (see Figure 5), we have

$$N'(2q+1, n) = 2(n+1) \quad \text{for } 2 \le q \le \left[\frac{n-4}{2}\right].$$

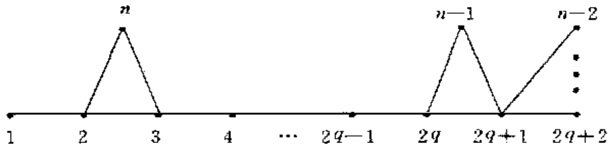


Figure 5

We assume that G is a primitive graph of order n with one cycle C exactly. Clearly, $\sigma A(G) = 2n$. Let $m = \max\{d(u, C) : u \in V(G)\}$ and c be the length of C. We denote G by $T(c, m)$. By the above Lemma $\gamma(G) = 2m + c - 1$ if c is odd. So $EG(2q, n) \supset \{T(c, m) : 2m + c - 1 = 2q\}$. On the other hand, if $G \in EG(2q, n)$, then G is a graph with no loop and $\gamma(G) = 2q$, $\sigma(A(G)) = 2n$. So G is a connected undirected graph with exactly one odd cycle C. Let $c(>1)$ be the length of C, and m the maximum length of the path from a vertex to C in G. Thus $\gamma(G) = 2m + c - 1 = 2q$. So $G = T(c, m) \in \{T(c, m) : 2m + c - 1 = 2q\}$. Hence $EG(2q, n) = \{T(c, m) : 2m + c - 1 = 2q\}$. So we have

Main Result *A is the minimal symmetric solution with zero trace of $A^k = J$ if and only if $G(A) \in \bigcup_{4 \le 2q \le k,\, 2q \in E_n} EG(2q, n)$ for $k \ge 4$. A is that of $A^2 = J$ if and only if $G(A) \cong G_1$ as n is odd and $G(A) \cong G_2$ as n is even. A is that of $A^3 = J$ if and only if $G(A) \cong G_1$ or G_3 as n is odd and $G(A) \cong G_4$ as n is even.*

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