Study Bulletin

THE BROADCAST FUNCTION VALUE $B(23)$ IS 33 OR 34*

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1. Introduction

Let G be a connected network of order n. Broadcasting is the process of distributing information from an originator to all other nodes of a communication network. The problem addressed in this paper is under the assumption that only one piece information is to be distributed, each communication involves exactly two adjacent nodes and takes one unit of time, and no node is involved into two or more simultaneous communications. Given a node x as originator, we define the broadcast time of x to be the minimum number $b(x)$ of time units required to complete broadcasting from node x. An obvious lower bound of $b(x)$ is $\lceil \log_2 n \rceil$. The broadcast time $b(G)$ of the graph G is max $\{b(x)|x \in G\}$. We say that G is a broadcast graph if $b(G) = \lceil \log_2 n \rceil$. The broadcast function $B(n)$ is the minimum number of edges in any broadcast graphs of order n. A minimum broadcast graph is a broadcast graph of size $B(n)$, mbg for short.

Minimum broadcast graphs are difficult to construct. In fact, even determining the value of $b(u)$ for an arbitrary node u in an arbitrary graph is NP-complete (see [1]). Since people construct an mbg with large order by means of constructing several mbgs with small order in general, the mbgs with small order are important and the values of $B(n)$ for $n \leq 22$ have been determined (see [2]). Thus the value of $B(23)$ is the first number unable to be determined. In [3], M. Mahéo and J-F. Saclé have proved $B(23) \leq 34$. In this paper, we indicate that $B(23) = 33$ or 34.

2. Main Result

Some terms and notations on Graph Theory are used and can be found in [4].

In the following suppose G is a minimum broadcast graph on 23 nodes with $B(23) = 32$.

Proposition 2.1. $\delta(G) = 2$ and $\Delta(G) \geq 4$.

Let $V_i = \{v \in V(G)|d(v) = i\}$ and $n_i = |V_i|$ for $i = 2, 3, \cdots$. We have

Proposition 2.2.

$$n_2 = 5 + \sum_{i \geq 4}(i-3)n_i, \qquad n_3 = 18 - \sum_{i \geq 4}(i-2)n_i.$$

Proposition 2.3. V_2 is an independent set.

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A broadcast originated by a node u determines a spanning tree rooted at u called a broadcast tree of u, denoted by $T(u)$.

Propositon 2.4. Let x be a 2-degree node. If x is adjacent to a 3-degree node and a 4-degree node, then $T(x)$ is uniquely determined as follows (see Fig.1).

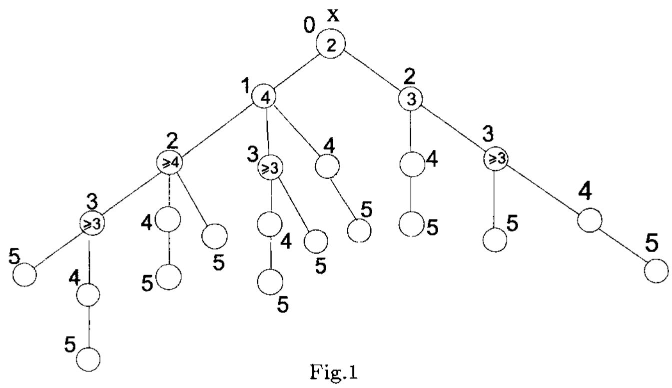

Fig.1

Proposition 2.5. (a) If a 2-degree node x is adjacent to a 3-degree node and a 4-degree node, then n_2 is no more than 11.

(b) If a 2-degree node x has a neighbour of degree 3 and one neighbor of degree i $(i \geq 5)$, then n_2 is no more than 12.

(c) If a 2-degree node x has two neighbours of degree 4, then n_2 is no more than 12.

Proposition 2.6. (a) No 2-degree node has all its neighbours of degree 3.

(b) No 3-degree node has two neighbours of degree 2 and one neighbour of degree 3.

(c) No i-degree node has all its neighbours of degree 2.

Proposition 2.7. (a) If a 4-degree node has three neighbours of degree 2, then each neighbour of degree 2 can't be adjacent to a 3-degree node.

(b) If an i-degree node has $i-1$ neighbours of degree 2 and one neighbour of degree 3, then each neighbour of degree 2 can't be adjacent to a 3-degree node.

Let $V' = \bigcup_{i\geq 4} V_i$. x_1 denotes the number of edges between V_2 and V'; x_2 denotes the number of edges between V_2 and V_3; x_3 denotes the number of edges between V_3 and V'.

Proposition 2.8. $x_1 \geq n_2$.

Now, by Proposition 2.1, we first assume $\Delta(G) = 4$. Thus by Proposition 2.2, we have $1 \leq n_4 \leq 9$. If $n_4 = 1$, then $n_2 = 6$ and $x_1 \geq 6$, which is impossible. If $n_4 = 2$, then $n_2 = 7$ and $x_1 \geq 7$. There must exist a 4-degree node x with $|N(x) \cap V_2| = 4$, a contradiction to Propositin 2.6 (c). Thus, $n_4 \geq 3$. By Proposition 2.5 and Proposition 2.2, we have $n_4 \leq 7$. If $n_4 = 7$, then $n_2 = 12$ and $n_3 = 4$. When $x_2 \neq 0$, by Proposition 2.5 (a), $n_2 \leq 11$, a contradiction. When $x_2 = 0$, i.e. $x_1 = 2n_2 = 24$, there must exist a 4-degree node x with $|N(x) \cap V_2| = 4$, a contradition to Proposition 2.6 (c). Thus $n_4 \leq 6$. If $n_4 = 3$, then $n_2 = 8$; we easily know $8 \leq x_1 \leq 9$, there exist at least two 4-degree nodes such that each has three neighbors of degree 2, and one of the three 2-degree nodes is adjacent to a 3-degree node, a contradiction by Proposition 2.7. If $n_4 = 4$, we also arrive at a contradiction like $n_4 = 3$. So $5 \leq n_4 \leq 6$. Using the above propositions, we can also show it is impossible for $n_4 = 5$ or 6.

Now, we consider $\Delta(G) \geq 5$.

Let $x \in V(G)$ and $\alpha(x) = |N(x) \cap V_2|$. We have $\alpha(x) = 0$ if $x \in V_2$ and $\alpha(x) \leq i-1$ if $x \in V_i$ $(i = 3, 4, \cdots)$. By Proposition 2.2, we have $\sum_{i\geq 4}(i-3)n_i \leq 8$. Hence we have that $n_{12} = n_{13} = \cdots = 0$ and $n_2 \leq 13$. By $n_2 = 5 + \sum_{i\geq 4}(i-3)n_i$ and Proposition 2.6, an obvious fact is $|V'| \geq 3$. In the following table , we raise all possible cases by $n_2 \leq 13$ and $|V'| \geq 3$.

In this table, we give the related propositions which are sufficent to prove these cases. "Pro" is an abridged notation of "Proposition".

$n_9 = 1$ $n_4 = 2$ $n_2 = 13$	Pro 2.5
$n_8 = 1$ $n_5 = 1$ $n_4 = 1$ $n_2 = 13$	Pro 2.5
$n_8 = 1$ $n_4 = 3$ $n_2 = 13$	Pro 2.5
$n_8 = 1$ $n_4 = 2$ $n_2 = 12$	Pro 2.5 and Pro 2.6
$n_7 = 1$ $n_6 = 1$ $n_4 = 1$ $n_2 = 13$	Pro 2.5
$n_7 = 1$ $n_5 = 2$ $n_2 = 13$	Pro 2.5
$n_7 = 1$ $n_5 = 1$ $n_4 = 2$ $n_2 = 13$	Pro 2.5
$n_7 = 1$ $n_5 = 1$ $n_4 = 1$ $n_2 = 12$	Pro 2.5 and Pro 2.6
$n_7 = 1$ $n_4 = 4$ $n_2 = 13$	Pro 2.5
$n_7 = 1$ $n_4 = 3$ $n_2 = 12$	Pro 2.5
$n_7 = 1$ $n_4 = 2$ $n_2 = 11$	Pro 2.7
$n_6 = 2$ $n_5 = 1$ $n_2 = 13$	Pro 2.5
$n_6 = 2$ $n_4 = 2$ $n_2 = 13$	Pro 2.5 and Pro 2.7
$n_6 = 2$ $n_4 = 1$ $n_2 = 12$	Pro 2.5 and Pro 2.7
$n_6 = 1$ $n_5 = 2$ $n_2 = 12$	Pro 2.7
$n_6 = 1$ $n_5 = 1$ $n_4 = 3$ $n_2 = 13$	Pro 2.5
$n_6 = 1$ $n_5 = 1$ $n_4 = 2$ $n_2 = 12$	Pro 2.5
$n_6 = 1$ $n_5 = 1$ $n_4 = 1$ $n_2 = 11$ $n_3 = 9$	like $n_4 = 5$ $n_3 = 8$ $n_2 = 10$
$n_6 = 1$ $n_4 = 5$ $n_2 = 13$	Pro 2.5
$n_6 = 1$ $n_4 = 4$ $n_2 = 12$	Pro 2.5
$n_6 = 1$ $n_4 = 3$ $n_2 = 11$ $n_3 = 8$	like $n_4 = 5$ $n_3 = 8$ $n_2 = 10$
$n_6 = 1$ $n_4 = 2$ $n_2 = 10$	Pro 2.7 and Pro 2.6
$n_5 = 4$ $n_2 = 13$	Pro 2.5
$n_5 = 3$ $n_4 = 2$ $n_2 = 13$	Pro 2.5
$n_5 = 3$ $n_4 = 1$ $n_2 = 12$	Pro 2.5
$n_5 = 3$ $n_2 = 11$ $n_3 = 9$	like $n_4 = 5$ $n_3 = 8$ $n_2 = 10$
$n_5 = 3$ $n_4 = 4$ $n_2 = 13$	Pro 2.5
$n_5 = 2$ $n_4 = 3$ $n_2 = 12$	Pro 2.5
$n_5 = 2$ $n_4 = 2$ $n_2 = 11$ $n_3 = 8$	like $n_4 = 5$ $n_3 = 8$ $n_2 = 10$
$n_5 = 2$ $n_4 = 1$ $n_2 = 10$ $n_3 = 10$	like $n_4 = 5$ $n_3 = 8$ $n_2 = 10$
$n_5 = 1$ $n_4 = 6$ $n_2 = 13$	Pro 2.5
$n_5 = 1$ $n_4 = 5$ $n_2 = 12$	Pro 2.5
$n_5 = 1$ $n_4 = 4$ $n_2 = 11$ $n_3 = 7$	like $n_4 = 5$ $n_3 = 8$ $n_2 = 10$
$n_5 = 1$ $n_4 = 3$ $n_2 = 10$ $n_3 = 9$	like $n_4 = 5$ $n_3 = 8$ $n_2 = 10$
$n_5 = 1$ $n_4 = 2$ $n_2 = 9$ $n_3 = 11$	Pro 2.7

Therefore, the main result is

Theorem. $B(23) = 33$ or 34.

References

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