OUTPATHS OF ARCS IN MULTIPARTITE TOURNAMENTS*

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Abstract

A k-outpath of an arc xy in a multipartite tournament is a directed path with length k starting from xy such that x does not dominate the end vertex of the directed path. This concept is a generalization of a directed cycle. We show that if T is an almost regular n-partite ($n\geq 8$) tournament with each partite set having at least two vertices, then every arc of T has a k-outpath for all k, $3\leq k\leq n-1$.

Key words. Outpaths, multipartite tournaments

1. Introduction

Throughout the paper, we use the terminology and notation of [1] and [2]. Let $D = (V(D), A(D))$ be a digraph. If xy is an arc of a digraph D, then we say that x dominates y, denoted by $x \to y$. More generally, if A and B are two disjoint vertex sets of D such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \Rightarrow B$. The outset $N^+(x)$ of a vertex x is the set of vertices dominated by x in D, and the inset $N^-(x)$ is the set of vertices dominating x in D. We define the outdegree $d^+(v) = |N^+(v)|$ and the indegree $d^-(v) = |N^-(v)|$. The maximum outdegree of D is denoted by Δ^+ and the minimum outdegree is denoted by δ^+. The irregularity $i(D)$ is $\mathrm{Max}\,|d^+(x) - d^-(y)|$ over all vertices x and y of D ($x = y$ is admissible). If $i(D) = 0$, we say D is regular; if $i(D) = 1$, we say D is almost regular. A digraph obtained by replacing each edge of a complete n-partite graph with exactly one arc is called an n-partite tournament or a multipartite tournament. If T is a multipartite tournament and $x \in V(T)$, we denote by $V(x)$ the partite set of T to which x belongs. If $U \subseteq V(T)$, we denote by $T[U]$ the subdigraph induced by U. A k-outpath of an arc xy in T is a directed path with length k starting from xy such that x does not dominate the end vertex of the directed path. Note that if T is a tournament, a k-outpath of an arc xy is in fact a $(k+1)$-cycle through xy, so the concept of an outpath is a generalization of a directed cycle. In this paper, $\lfloor \alpha \rfloor$ denotes the largest integer not more than α, and $\lceil \alpha \rceil$ denotes the least integer not less than α.

It is well known that if T is a regular tournament with n vertices, then each arc is contained in cycles of all lengths m, $3 \leq m \leq n$ (see [3]); and if T is an almost regular tournament with n vertices ($n \geq 8$), the each arc is contained in cycles of all lengths m, $4 \leq m \leq n$ (see [4]). Guo[5] proved that if T is a regular n-partite tournament, then every arc of T has an outpath of length k for all k, $2 \leq k \leq n-1$.

The main result of this paper is the following

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Theorem. Let T be an almost regular n-partite ($n \geq 8$) tournament. If each partite set of T has at least two vertices, then every arc of T has a k-outpath for all k, $3 \leq k \leq n-1$.

In order to prove the theorem, we need the following lemma:

Lemma. Let T be an almost regular n-partite tournament with partite sets $V_1, V_2, \cdots, V_n$. Then (a) $\Delta^+ - \delta^+ \leq 2$. (b) If $\Delta^+ - \delta^+ = 2$, then $d^-(x) = \delta^+ + 1$ for each $x \in V(T)$. (c) $\big\| |V_i| - |V_j| \big\| \leq 2$ for all $i \neq j$. (d) If $d^+(x) - d^+(y) = 2$, then $d^+(x) = \Delta^+$, $d^+(y) = \delta^+$ and $|V(y)| = |V(x)| + 2$.

Proof. (a) Suppose there exist $u, v \in V(T)$ such that $d^+(u) - d^+(v) \geq 3$. By the almost regularity of T, we have $d^-(u) \geq d^+(u) - 1 \geq d^+(v) + 3 - 1 = d^+(v) + 2$, a contradiction.

(b) Suppose $\Delta^+ - \delta^+ = 2$. Let $u, v \in V(T)$ with $d^+(u) - d^+(v) = 2$, where $d^+(u) = \Delta^+$ and $d^+(v) = \delta^+$. Let $z \in V(T)$. If $d^-(z) \geq \delta^+ + 2$, then $d^-(z) - d^+(v) \geq 2$, a contradiction; if $d^-(z) \leq \delta^+$, then $d^+(u) - d^-(z) \geq 2$, a contradiction too. So we have $d^-(x) = \delta^+ + 1$ for each $x \in V(T)$.

(c) Note that $d^+(x) + d^-(x) = |V(T)| - |V(x)|$ for each $x \in V(T)$. Let $x, y \in V(T)$ so that $V(x) = V_i$ and $V(y) = V_j$. Then $\big| |V_i| - |V_j| \big| = \big| |V(x)| - |V(y)| \big| = |d^+(y) + d^-(y) - d^+(x) - d^-(x)| = \big|(d^+(x) - d^-(y)) + (d^-(x) - d^+(y))\big| \leq |d^+(x) - d^-(y)| + |d^-(x) - d^+(y)| \leq 1 + 1 = 2$.

(d) By (a), it is easy to see that $d^+(x) = \Delta^+$ and $d^+(y) = \delta^+$. By (b), we know that $d^-(x) = d^-(y)$. Hence we have $|V(y)| - |V(x)| = \big(|V(T)| - d^+(y) - d^-(y)\big) - \big(|V(T)| - d^+(x) - d^-(x)\big) = d^+(x) - d^+(y) = 2$.

2. Proof of the Theorem

Let $V_1, V_2, \cdots, V_n$ be the partite sets of T, and let $s = \min\{|V_i|\}$. By Lemma (c) and the initial hypothesis, we have $s \geq 2$ and $|V_i| \leq s+2$. Further, since $d(x) = d^+(x) + d^-(x) \geq (n-1)s$ and $d^+(x) \geq d^-(x) - 1$, we have $\delta^+ \geq \frac{(n-1)s}{2}$. Let $e = (a_0, a_1) \in T$. There are at least three vertices (say, x, y, z) in $N^+(a_1)$ such that $V(x), V(y)$ and $V(z)$ are pairwise distinct. Otherwise we must have $d^+(a_1) \leq 2(s+2)$ and $d^-(a_1) \geq (n-3)s$. Noting that $n \geq 8$ and $s \geq 2$, we have $d^-(a_1) - d^+(a_1) \geq (n-3)s - 2(s+2) \geq 3s - 4 \geq 2$, which contradicts the almost regularity of T. Without loss of generality, we assume $x \to y \to z$.

We shall first show that e has a 3-outpath and a 4-outpath.

Suppose that e has no 3-outpath. If a_0 does not dominate y (or z), then a_0a_1xy (or a_0a_1yz) is a 3-outpath of e, a contradiction. So we have that $a_0 \Rightarrow \{y, z\}$. Similarly, we have $a_0 \Rightarrow N^+(z)$. Thus $d^+(a_0) \geq d^+(z) + |\{a_1, y, z\}|$, which contradicts Lemma (a). This proves that e has a 3-outpath.

Suppose that e has no 4-outpath. If $a_0 \not\to z$, then a_0a_1xyz is a 4-outpath of e, a contradiction. So $a_0 \to z$. Similarly, we have $a_0 \Rightarrow N^+(z)$. If $|N^+(z)| \geq \delta^+ + 1$, then since $z, a_1 \notin N^+(z)$ and $z, a_1 \in N^+(a_0)$, $d^+(a_0) \geq |N^+(z)| + |\{z, a_1\}| \geq \delta^+ + 3$, a contradiction. So we assume that $|N^+(z)| = \delta^+$. Since there exists $u \in N^+(z)$ such that $d^+_{T[N^+(z)]}(u) \leq \lfloor (|N^+(z)| - 1)/2 \rfloor$, we have $|N^+(u) \backslash N^+(z)| \geq \delta^+ - \lfloor (|N^+(z)| - 1)/2 \rfloor = \lceil (\delta^+ - 1)/2 \rceil + 1 \geq 2$. Hence we have $d^+(a_0) \geq |N^+(z)| + |N^+(u) \backslash (N^+(z) \cup \{a_1\})| + |\{a_1, z\}| \geq \delta^+ + 3$, a contradiction. This proves that e has a 4-outpath.

Let $P = a_0a_1 \cdots a_p$ be a p-outpath of e ($4 \leq p \leq n-2$). Suppose that e has no $(p+1)$-outpath.

Let $A = \{x \,|\, x \in V_i,\ V_i \cap V(P) = \emptyset,\ x \to a_0,\ 1 \leq i \leq n\}$, $B = \{y \,|\, y \in V_i,\ V_i \cap V(P) = \emptyset,\ a_0 \to y,\ 1 \leq i \leq n\}$.

It is obvious that $A \cup B \neq \emptyset$, since otherwise we must have $|V(P)| \geq n$ and then $p \geq n-1$, which contradicts that $p \leq n-2$. And for each vertex x in $A \cup B$ and for each vertex y in $V(P)$, either $x \to y$ or $y \to x$. Moreover, for each $x \in V(P)$, since

$d(x) = d^+(x) + d^-(x) \geq (n-1)s$ and $d^+(x) \geq d^-(x) - 1$, we have $\delta^+ \geq \frac{(n-1)s}{2}$. And so $N^+(x)\backslash V(P) \neq \emptyset$, since otherwise we have $p = |V(P)| - 1 \geq d^+(x) + |\{x\}| - 1 \geq \delta^+ \geq \lfloor (n-1)s/2 \rfloor \geq n-1$, a contradiction.

Suppose $A \neq \emptyset$. If there exists $x \in A$ such that $a_p \to x$, then $a_0a_1 \cdots a_px$ is a $(p+1)$-outpath of e. Hence $A \Rightarrow a_p$, and then we have that $A \Rightarrow V(P)$ since otherwise there must exist s such that $a_s \to x$ and $x \to a_{s+1}$, hence $a_0a_1 \cdots a_sxa_{s+1} \cdots a_p$ is a $(p+1)$-outpath of e, a contradiction. Let $a \in A$ and $u \in N^+(a_{p-3})\backslash V(P)$. If $u \to a$, then $a_0a_1 \cdots a_{p-3}uaa_{p-1}a_p$ is a $(p+1)$-outpath of e, a contradiction. If $V(u) = V(a)$, then u is not in $V(z)$ for every $z \in V(P)$ and $\{a_{p-2}, a_{p-1}, a_p\} \Rightarrow u$. When $a \Rightarrow N^+(u)\backslash V(P)$, then $a \Rightarrow N^+(u)$. Thus $d^+(a) \geq d^+(u) + |\{a_{p-3}, a_{p-2}, a_{p-1}, a_p\}| \geq \delta^+ + 4$, a contradiction to Lemma (a). When there exists $v \in N^+(u)\backslash V(P) \neq \emptyset$such that $v \to a$, then $a_0a_1 \cdots a_{p-3}uvaa_p$ is a $(p+1)$-outpath of e, a contradiction too. So, $a \to u$ for all such u, implying that $a \Rightarrow N^+(a_{p-3})\backslash V(P)$. And then $a \Rightarrow N^+(a_{p-3})$. It follows that $d^+(a) \geq d^+(a_{p-3}) + |\{a_{p-4}, a_{p-3}\}|$. By Lemma (a), we have that $d^+(a) = \delta^+ + 2$, and $d^+(a_{p-3}) = \delta^+$. So $a_{p-3} \Rightarrow \{a_0, a_1, \cdots, a_{p-5}, a_{p-2}, a_{p-1}, a_p\}$ if $p \geq 5$, and $a_{p-3} \Rightarrow \{a_2, a_3, a_4\}$ if $p = 4$. By analogous computations we obtain that $a_{p-2} \Rightarrow \{a_0, a_1, \cdots, a_{p-4}, a_{p-1}, a_p\}$.

Let $x \in N^+(a_{p-1})\backslash V(P)$. If $x \to a$, then $a_0a_1 \cdots a_{p-1}xa$ is a $(p+1)$-outpath of e, a contradiction. If $V(x) = V(a)$, then x is not in $V(z)$ for every $z \in V(P)$. If $x \to a_{p-2}$, then $a_0a_1 \cdots a_{p-3}a_{p-1}xa_{p-2}a_p$ is a $(p+1)$-outpath of e, a contradiction. So $a_{p-2} \to x$ and $a_p \to x$. Note that $N^+(x)\backslash V(P) \neq \emptyset$, hence if there exists $y \in N^+(x)\backslash V(P)$ such that $y \to a$, then $a_0a_1 \cdots a_{p-3}a_{p-1}xya$ is a $(p+1)$-outpath of e, a contradiction. So we have $a \Rightarrow N^+(x)\backslash V(P)$, and then $a \Rightarrow N^+(x)$. Now $d^+(a) \geq d^+(x) + |\{a_{p-2}, a_{p-1}, a_p\}|$, a contradiction to Lemma (a). Now we have $a \to x$ for all $x \in N^+(a_{p-1})\backslash V(P)$, that is, $a \Rightarrow N^+(a_{p-1})\backslash V(P)$. And then we have $a \Rightarrow N^+(a_{p-1})$. Thus $d^+(a) \geq d^+(a_{p-1}) + |\{a_{p-3}, a_{p-2}, a_{p-1}\}|$, a contradiction too.

Therefore $A = \emptyset$. Since $A \cup B \neq \emptyset$, $B \neq \emptyset$. Let b be an arbitrary vertex in B, note that $V(b) \subseteq B$. Suppose that $a_i \to b$ for $i = 1$ or 2. Then it is easy to check that $a_j \to b$ for all $j \geq 2$. If there exists $x \in N^+(b)\backslash V(P)$ such that $a_0 \not\to x$, then $a_0a_1 \cdots a_{p-1}bx$ is a $(p+1)$-outpath of e, a contradiction. Hence $a_0 \Rightarrow N^+(b)\backslash V(P)$, and then we have that $d^+(a_0) \geq d^+(b) + |V(b)| \geq d^+(b) + 2$. By Lemma (d), we have that $|B| \geq |V(b)| = |V(a_0)| + 2 \geq 4$. Thus we have $d^+(a_0) \geq d^+(b) + |V(b)| \geq \delta^+ + 4$, a contradiction to Lemma (a). So we have $b \Rightarrow \{a_1, a_2\}$, i.e., $B \Rightarrow \{a_1, a_2\}$.

Case 1. $p = 4$, then $|B| \geq (n-5)s \geq 6$.

Case 1.1. There exists $x \in B$ such that $a_3 \to x$.

We have $a_4 \to x$ and $a_0 \Rightarrow N^+(x)\backslash V(P)$. Hence $d^+(a_0) \geq d^+(x) + |V(x)| - |\{a_2\}| \geq d^+(x)+1$. If $d^+(a_0) \geq d^+(x)+2$, then by Lemma (a) and (d), we have $|V(x)| = |V(a_0)|+2 \geq 4$, thus $d^+(a_0) \geq d^+(x) + 3$, a contradiction. So $d^+(a_0) = d^+(x) + 1$, which implies that $\delta^+ \leq d^+(x) \leq \delta^+ + 1$. If $d^+(x) = \delta^+ + 1$, then $d^+(a_0) = \delta^+ + 2$. By Lemma (b), $|V(x)| = |V(a_0)| + 1 \geq 3$ and then $d^+(a_0) \geq d^+(x) + |V(x)| - |\{a_2\}| \geq d^+(x) + 2$. This contradicts $d^+(a_0) = d^+(x) + 1$. Hence we have $d^+(x) = \delta^+$. If $a_0 \to a_2$ or $a_0 \to a_3$, then $d^+(a_0) \geq d^+(x)+|V(x)| \geq d^+(x)+2$, a contradiction. So we have $a_0 \not\to a_2$ and $a_0 \not\to a_3$. Note that if $a_1 \to a_i$ $(i \geq 3)$, then $a_0 \to a_{i-1}$. Otherwise $a_0a_1a_i \cdots a_4xa_2 \cdots a_{i-1}$ is a 5-outpath of e. Hence $a_1 \not\to a_3$ and $a_1 \not\to a_4$, otherwise we must have $a_0 \to a_2$ or $a_0 \to a_3$, a contradiction. Now clearly $V(x) \cap (N^+(a_1)\backslash V(P)) = \emptyset$ since $B \Rightarrow a_1$. Hence $x \Rightarrow N^+(a_1)\backslash V(P)$, since otherwise let $y \in N^+(a_1)\backslash V(P)$ be chosen such that $y \to x$ and observe that $a_0a_1yxa_2a_3$ is a 5-outpath of e since $a_0 \not\to a_3$. So $d^+(x) \geq |N^+(a_1)\backslash V(P)| + |\{a_1, a_2\}| \geq \delta^+ - 1 + 2$, which contradicts $d^+(x) = \delta^+$.

Case 1.2. $B \Rightarrow a_3$.

Without loss of generality, we assume $B = V_1 \cup V_2 \cup \cdots \cup V_l$ with $|V_{l-1}| \geq |V_l|$, where

$l \geq 3$. If for each $y \in B$, $d^+_{T[B]}(y) \leq 1$, then $|A(T[B])| \leq \sum_{i=1}^{l} |V_i|$. On the other hand, $|A(T[B])| = \sum_{1 \leq i<j \leq l} |V_i||V_j| \geq 2\sum_{i=1}^{l-1} |V_i| \geq \sum_{i=1}^{l} |V_i| + 1$, a contradiction. Hence there exists $x \in B$ such that $d^+_{T[B]}(x) \geq 2$. Note that $x \Rightarrow N^+(a_1)\backslash V(P)$ and $(N^+(a_1)\backslash V(P)) \cap B = \emptyset$. If $x \to a_4$, then $d^+(x) \geq d^+(a_1) + d^+_{T[B]}(x) + |\{a_1\}| \geq d^+(a_1) + 3$, a contradiction. If $a_4 \to x$, then $a_1 \to a_4$, otherwise $d^+(x) \geq d^+(a_1) + 3$, a contradiction. Hence $a_0 \to a_3$ and $|N^+(a_3)\backslash V(P)| \geq \delta^+ - 2$. Suppose there exists $a \in N^+(a_3)\backslash V(P)$ such that $a_0 \not\to a$; then $a_0a_1a_4xa_3a$ is a 5-outpath of e, a contradiction. Hence $a_0 \Rightarrow N^+(a_3)\backslash V(P)$, which implies that $d^+(a_0) \geq |N^+(a_3)\backslash V(P)| + |B| + 1 \geq \delta^+ + 5$, a contradiction.

Case 2. $p \geq 5$ and there is $b \in B$ with $b \to a_p$.

It is easy to check that $b \Rightarrow \{a_1, a_2, \cdots, a_p\}$. Note that $V(b) \Rightarrow a_1$, so b is not in $V(z)$ for every $z \in N^+(a_1)\backslash V(P)$. Hence we have that $b \Rightarrow N^+(a_1)\backslash V(P)$. Otherwise there exists $x \in N^+(a_1)\backslash V(P)$ such that $x \to b$, then $a_0a_1xba_3 \cdots a_p$ is a $(p+1)$-outpath of e, a contradiction. An analogous argument to that above will deduce that $b \Rightarrow N^+(a_2)\backslash V(P)$.

Case 2.1. $\big(N^+(a_1)\backslash V(P)\big) \cap \big(N^+(a_2)\backslash V(P)\big) \neq \emptyset$.

Let $u \in \big(N^+(a_1)\backslash V(P)\big) \cap \big(N^+(a_2)\backslash V(P)\big)$. Then obviously $u \not\to a_3$. Note that $N^+(u)\backslash V(P) \neq \emptyset$. If $b \Rightarrow N^+(u)\backslash V(P)$, then $d^+(b) \geq d^+(u) + |\{u, a_1, a_2, a_3\}| - |\{a_0\}| = d^+(u) + 3$, a contradiction; if there exists $x \in (N^+(u)\backslash V(P))$ such that $V(b) = V(x)$, then it is easy to see that $\{a_3, a_4, \cdots, a_p\} \Rightarrow x$ and hence there exists $y \in N^+(x)\backslash V(P)$ such that $y \to b$, otherwise $d^+(b) \geq d^+(x) + |\{u, a_3, a_4, \cdots, a_p\}| \geq d^+(x) + 4$, a contradiction. But $a_0a_1uxyba_5 \cdots a_p$ is a $(p+1)$-outpath of e, a contradiction. So there exists $v \in N^+(u)\backslash V(P)$ such that $v \to b$, then $a_0a_1uvba_4 \cdots a_p$ is a $(p+1)$-outpath of e, a contradiction.

Case 2.2. $\big(N^+(a_1)\backslash V(P)\big) \cap \big(N^+(a_2)\backslash V(P)\big) = \emptyset$.

Note that $N^+(a_2)\backslash V(P) \neq \emptyset$. If $|N^+(a_2)\backslash V(P)| = 1$, then $|V(P)| \geq |N^+(a_2)| - 1 + |\{a_1, a_2\}| \geq \delta^+ + 1 \geq n$, a contradiction. So we have $|N^+(a_2)\backslash V(P)| \geq 2$ and $d^+(b) \geq d^+(a_1) + |\{a_1\}| + |N^+(a_2)\backslash V(P)| \geq \delta^+ + 3$, a contradiction.

Case 3. $p \geq 5$ and $a_p \to b$ for any $b \in B$.

Case 3.1. $b \to a_3$.

If there exists $u \in N^+(a_1)\backslash V(P)$ with $u \to b$, then $a_0a_1uba_3 \cdots a_p$ is a $(p+1)$-outpath of e, a contradiction. So we have $b \Rightarrow N^+(a_1)\backslash V(P)$. For each $a_i \in N^+(a_1) \cap V(P)$, we must have $a_0 \to a_{i-1}$, otherwise $a_0a_1a_i \cdots a_pba_2 \cdots a_{i-1}$ will be a $(p+1)$-outpath of e, a contradiction. This means that $|N^+(a_0) \cap V(P)| \geq |N^+(a_1) \cap V(P)|$.

Case 3.1.1. $a_{p-1} \to b$.

If there exists $u \in N^+(a_1)\backslash V(P)$ such that $a_0 \not\to u$, then $a_0a_1 \cdots a_{p-1}bu$ is a $(p+1)$-outpath of e, a contradiction. Hence we have $a_0 \Rightarrow N^+(a_1)\backslash V(P)$ and then $d^+(a_0) \geq d^+(a_1)+|B|$. It follows that $B = V(b)$ and $|V(b)| = 2$. Now it is easy to see that $T[V(P)]$ is a tournament since otherwise we obtain that $|V(P)| \geq n$, and then $p \geq n-1$, which contradicts that $p \leq n-2$. Note that $a_0 \Rightarrow \big(N^+(a_1)\backslash V(P)\big) \cup \big(N^+(b)\backslash V(P)\big)$, thus $N^+(b)\backslash V(P) = N^+(a_1)\backslash V(P) = N^+(a_0)\backslash(V(P) \cup B)$, otherwise we will get $d^+(a_0) \geq \delta^+ + 3$. Hence there exists $a \in N^-(a_0)\backslash V(P)$ such that $V(a) = V(a_1)$ since $s \geq 2$. So we have $a \to a_p$, otherwise $a_0a_1 \cdots a_pa$ is a $(p+1)$-outpath of e, a contradiction. This implies $a \Rightarrow \{a_2, a_3, \cdots, a_{p-1}\}$. Since $a_{p-1} \to b, a \to b$. Note that a is not in $V(z)$ for every $z \in N^+(a_1)\backslash V(P)$, and therefore we have $a \Rightarrow N^+(a_1)\backslash V(P)$, otherwise there exists $x \in N^+(a_1)\backslash V(P)$ such that $x \to a$, then $a_0a_1xaa_3 \cdots a_p$ is a $(p+1)$-outpath of e, a contradiction. Since $N^+(b)\backslash V(P) = N^+(a_1)\backslash V(P)$, we have $a \Rightarrow N^+(b)\backslash V(P)$. Hence $d^+(a) \geq d^+(b) + |\{a_0, b, a_{p-1}, a_p\}| - |\{a_1\}| = d^+(b) + 3$, a contradiction.

Case 3.1.2. $b \to a_{p-1}$.

It is easy to see that $b \Rightarrow \{a_1, a_2, \cdots, a_{p-1}\}$, and $b \Rightarrow N^+(a_i)\backslash V(P)$ $(i = 1, 2)$.

Suppose $\left(N^+(a_1)\backslash V(P)\right) \cap \left(N^+(a_2)\backslash V(P)\right) = \emptyset$. Note that $|N^+(a_2)\backslash V(P)| \geq 2$, otherwise $|V(P)| \geq |N^+(a_2)| - 1 + |\{a_1, a_2\}| \geq n$, a contradiction. Hence $d^+(b) \geq d^+(a_1) + |N^+(a_2)\backslash V(P)| + |\{a_1\}| - |\{a_p\}| \geq d^+(a_1) + 2$. This implies that $|N^+(a_2)\backslash V(P)| = 2$, and by Lemma (d) we have $|V(a_1)| = |V(b)| + 2 \geq 4$. Observe that $a_2 \notin V(a_1)$ and $d^+(a_2) + d^-(a_2) \geq 2(n-2) + |V(a_1)|$; by almost regularity of T, we have $d^+(a_2) \geq \lfloor (2(n-2)+4)/2 \rfloor \geq n$. So $|V(P)| \geq d^+(a_2) - 2 + |\{a_1, a_2\}| \geq n$, a contradiction.

Thus $\left(N^+(a_1)\backslash V(P)\right) \cap \left(N^+(a_2)\backslash V(P)\right) \neq \emptyset$. Now, let $a \in \left(N^+(a_1)\backslash V(P)\right) \cap \left(N^+(a_2)\backslash V(P)\right)$.

Case 3.1.2.1. There exists $b' \in N^+(a)\backslash V(P)$ with $V(b') = V(b)$.

Then $a_3 \to b'$, otherwise $a_0 a_1 a b' a_3 \cdots a_p$ will be a $(p+1)$-outpath of e, a contradiction. And so $\{a_3, a_4, \cdots, a_p\} \Rightarrow b'$. If $b \Rightarrow N^+(b')\backslash V(P)$, then $d^+(b) \geq d^+(b') + |\{a_3, a_4, \cdots, a_{p-1}\}| \geq d^+(b') + 2$. Thus by Lemma (d) $|V(b')| \geq |V(b)| + 2$, which contradicts $V(b) = V(b')$. Hence there exists $x \in N^+(b')\backslash V(P)$ such that $x \to b$. If $a_0 \not\to a_{p-1}$, then $a_0 a_1 a b' x b a_4 \cdots a_{p-1}$ is a $(p+1)$-outpath of e, a contradiction. Hence we have $a_0 \to a_{p-1}$. On the other hand, since $a_{p-1} \to b'$, it is easy to check that $a_0 \Rightarrow N^+(b')\backslash V(P)$. So $d^+(a_0) \geq d^+(b') + |V(b')| + |\{a_{p-1}\}| - |\{a_2\}| = d^+(b') + |V(b')|$. Because of $s \geq 2$ and Lemma (d), $|V(b')| = |V(a_0)| + 2$. Hence $d^+(a_0) \geq d^+(b') + 4$, a contradiction.

Case 3.1.2.2. $b \Rightarrow N^+(a)\backslash V(P)$.

Since $a \not\to a_3$ and $b \to a$, $d^+(b) \geq d^+(a) + |\{a_1, a_2, a_3, a\}| - |\{a_0, a_p\}| = d^+(a) + 2$. By Lemma (a) we have $d^+(b) = \delta^+ + 2$ and $a \Rightarrow \{a_0, a_p\}$. Since $a \to a_0$, we must have $a_2 \not\to a_4$, $a_1 \not\to a_3$, $a_1 \not\to a_4$ and $a_1 \not\to a_5$, since otherwise we will obtain $(p+1)$-outpaths of e ending in a. Now since $b \Rightarrow N^+(a_1)\backslash V(P)$, $d^+(b) \geq d^+(a_1) + |\{a_1, a_3, a_4, a_5\}| - |\{a_p\}| = d^+(a_1) + 3$, a contradiction.

Case 3.1.2.3. There exists $x \in N^+(a)\backslash V(P)$ such that $x \to b$.

In this case $a_0 a_1 a x b a_4 \cdots a_p$ is a $(p+1)$-outpath of e, a contradiction.

Case 3.2. $a_3 \to b$, and then $\{a_3, a_4, \cdots, a_p\} \Rightarrow b$.

Since $a_0 \Rightarrow N^+(b)\backslash V(P)$, $d^+(a_0) \geq d^+(b) + |V(b)| - |\{a_2\}| \geq d^+(b) + 1$. Supposing $a_0 \to a_{p-1}$, then $d^+(a_0) \geq d^+(b) + |V(b)| - |\{a_2\}| + |\{a_{p-1}\}| = d^+(b) + |V(b)|$. Due to $|V(b)| \geq 2$ and Lemma (d), $|V(b)| = |V(a_0)| + 2 \geq 4$. So we have $d^+(a_0) \geq d^+(b) + 4$, a contradiction. Hence we can always assume that $a_0 \not\to a_{p-1}$. Now since $B \Rightarrow a_1$, $b \Rightarrow N^+(a_1)\backslash V(P)$ and then $a_0 \Rightarrow N^+(a_1)\backslash V(P)$. If there is a vertex a_i with $a_1 \to a_i$ and $a_0 \not\to a_{i-1}$, then $a_0 a_1 a_i \cdots a_p b a_2 \cdots a_{i-1}$ is a $(p+1)$-outpath of e, a contradiction. This means that $|N^+(a_0) \cap V(P)| \geq |N^+(a_1) \cap V(P)|$. Hence $d^+(a_0) \geq d^+(a_1) + |V(b)|$. And then by $|V(b)| \geq 2$ and Lemma (b), we get that for each $x \in V(T)$, $d^-(x) = \delta^+ + 1$. Since $d^+(a_0) \geq d^+(b) + 1$, $|V(b)| \geq |V(a_0)| + 1 \geq 3$. So $d^+(a_0) \geq d^+(a_1) + |V(b)| \geq d^+(a_1) + 3$, a contradiction.

This completes the proof of the theorem.

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